MATHEMATICS

(Maximum Marks: 100)

(Time allowed: Three hours)

(Candidates are allowed additional 15 minutes for only reading the paper.

They must NOT start writing during this time.)

The Question paper consists of three section *A*, *B* and *C*.

Candidates are required to attempt all questions form ***Section A*** and all questions

EITHER from ***Section B*** **OR** ***Section C***

Section A: Internal choice has been provided in three questions of four marks each and two Questions of six marks each.

Section B: Internal choice has been provided in two question of four marks each.

Section C: Internal choice has been provided in two questions of four marks each.

All working including rough work , should be done on the same sheet as, and adjacent to the rest of the answer.

The intended marks for questions or parts of questions are given in brackets [ ].

Mathematical tables and graph papers are provide.

SECTION A (80 Marks)

Question 1

(i) If $\begin{bmatrix} a+b & 4 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}$**, find the values of a and b.**

Sol:

The corresponding elements of two equal matrices are equal.

$$\begin{bmatrix} a+b & 4 \\ 5 & \text{ab} \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 5 & 6 \end{bmatrix}$$

$\Rightarrow a+b=8$

and,

$ab=6$

$\Rightarrow a+\frac{8}{a}=6$

$\Rightarrow a^2+8=6a$

$\Rightarrow a^2-6a+8=0$

$\Rightarrow (a-2)(a-4)=0$

$\Rightarrow a=2,4$

Now,

$a=2,4$ *and* ab = 6

$\Rightarrow \text{b} = 3, \frac{3}{2}$

Hence, the value is $a=2,4$ and $\text{b} = 3, \frac{3}{2}$.

(ii) Simplify the equation $\tan^{-1}\left\{\sqrt{\frac{a-x}{a+x}}\right\}, -a<x<a$ **.**

Sol:

We have,

$\tan^{-1}\sqrt{\frac{a-x}{a+x}}$

Putting $x=a\cos\theta$

,we obtain,

$$\tan^{-1}\sqrt{\frac{a-x}{a+x}}=\tan^{-1}\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}$$
$$=\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$
$$=\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}$$
$$=\tan^{-1}\left(\left|\tan\frac{\theta}{2}\right|\right)$$
$$=\tan^{-1}\left(\left|\tan\frac{\theta}{2}\right|\right)\quad\left[\begin{array}{l}\because -a<x<a\Rightarrow 0<\theta<\pi\Rightarrow 0<\frac{\theta}{2}\\ \therefore \left|\tan\frac{\theta}{2}\right|=\tan\frac{\theta}{2}\end{array}\right]$$
$$=\frac{\theta}{2}=\frac{1}{2}\cos^{-1}\frac{x}{a}\quad\left[\because x=a\cos\theta\Rightarrow\cos\theta=\frac{x}{a}\Rightarrow\theta=\cos^{-1}\frac{x}{a}\right]$$

(iii) Find the function *f(x)* given by $f(x)=\begin{cases}x\cos\frac{1}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$ **is continuous at** $x=0$**.**

Sol:

$$(LHL\text{ at }x=0)=\lim_{x\to0^-}f(x)=\lim_{h\to0}f(0-h)=\lim_{h\to0}f(-h)=\lim_{h\to0}-h\cos\left(\frac{1}{-h}\right)$$
$$=\lim_{h\to0}h\cos\left(\frac{1}{h}\right)=0\times\text{ (an oscillating number between -1 and 1)=0}$$
$$=(RHL\text{ at }x=0)=\lim_{x\to0^+}f(x)=\lim_{h\to0}f(0+h)=\lim_{h\to0}f(h)=\lim_{h\to0}-h\cos\left(\frac{1}{h}\right)$$
$$=0\times\text{ (an oscillating number between -1 and 1)=0}$$
and, $f(0)=0$

Thus, obtain

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^+}f(x)=f(0).$$

Hence, *f(x)* is continuous at $x=0$.

(iv) if $x^2+2xy+y^3=22$ **find,** $\frac{dy}{dx}$

Sol:

We have,

$x^2 + 2xy + y^3 = 22$

Differentiating both sides of this with respect to *x*, we get

$$\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) + \frac{d}{dx}y^3 = 22$$
$$\Rightarrow 2x + 2\left(x\frac{dy}{dx} + y\right) + 3y^2 = \frac{d}{dx}(22)$$
$$\Rightarrow 2x + 2y + \frac{dy}{dx}\left(2x + 3y^2\right) = 0$$
$$\Rightarrow \frac{dy}{dx}\left(2x + 3y^2\right) = -2(x + y)$$
$$\Rightarrow \frac{dy}{dx} = -\frac{-2(x + y)}{\left(2x + 3y^2\right)}.$$

(v) if $x^3 + y^3 = 4axy$ find, $\frac{dy}{dx}$

Sol:

We have,

$x^3 + y^3 = 4axy$

Differentiating both sides of this with respect to *x*, we get

$$\frac{d}{dx}(x^3) + \frac{d}{dx}y^3 = 4a\frac{d}{dx}(xy)$$
$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 4a\left\{x\frac{dy}{dx} + y\right\}$$
$$\Rightarrow \left(3y^2 - 3ax\right)\frac{dy}{dx} = 4ay - 3x^2$$
$$\Rightarrow 3\left(y^2 - ax\right)\frac{dy}{dx} = \left(4ay - 3x^2\right)$$
$$\Rightarrow \frac{dy}{dx} = \frac{4ay - 3x^2}{3\left(y^2 - ax\right)}$$

(vi) Mother, father and son line up at random for a family picture. Find P (AB), if A and B are defined as follows:

A = Father on one end, B = Mother in the middle.

Sol:

Total number of ways in which Mother (M), Father (F) and Son (S) can be lined up at random in one of the following ways:

MFS, MSF, FMS, FSM, SFM, SMF

We have,

$A = \{SMF, FMS, MSF, FSM\}$ and B = $\{FMS, SMF\}$
$\therefore A \cap B = \{\text{FMS}, SMF\}$

Clearly,

$n(A \cap B) = 2$ and $n(B) = 2$

$\therefore$ Required probability = $P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{2}{2} = 1$

(vii) Evaluate: $\int_0^1 \frac{2x}{6x^2+1}dx$

Sol:

Let $6x^2 + 1 = t$

Then,

$(6x^2 + 1) = dt$
$\Rightarrow 12xdx = dt$

When,

$x = 1, t = 6x^2 + 1 \Rightarrow t = 7$

And,

$x = 2, t = 6x^2 + 1 \Rightarrow t = 25$

$$\therefore \int_0^1 \frac{2x}{6x^2+1}dx = \int_7^{25} \frac{2x}{t} \times \frac{dt}{12dx}$$
$$\frac{1}{6}\int_1^7 \frac{1}{t}dt = \frac{1}{6}[\log t]_7^{25} = \frac{1}{6}(\log 25 - \log 7) = \frac{1}{6}\log 18$$

(viii)

Find the two regression coefficient. If you are given the following data $\bar{X} = 36, \bar{Y} = 85, \sigma_X = 11$ and $\sigma_Y = 8$**. In a bivariate distribution and the correlation between X and Y is 0.66.**

We have,

$\bar{X} = 36, \bar{Y} = 85, \sigma_X = 11$, $\sigma_Y = 8$ and r=0.66

$\Rightarrow \mathrm{b}_{YX} = r\dfrac{\sigma_Y}{\sigma_X}$

$= 0.66 \times \dfrac{8}{11}$

$= 0.48,$

$b_{XY} = r\dfrac{\sigma_X}{\sigma_Y}$

$= 0.66 \times \dfrac{11}{8}$

$= 0.9075$

(ix) Prove that $(AB)^T = B^T A^T$ **. If** $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ **and** $B = [-2 -1 - 4]$

Sol:

We have $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = [-2 -1 - 4]$

$AB = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} [-2 -1 - 4]$

So, $= \begin{bmatrix} 2 & 1 & 4 \\ -4 & -2 & -8 \\ -6 & -3 & -12 \end{bmatrix}$

$\Rightarrow (AB)^T = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} ....(1)$

And,

$$B^T A^T = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} -2 -1 -4 \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix}$$

Hence, $(AB)^T = B^T A^T$ proved.

(x) Prove that: $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Sol:

We have,

$$\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\left\{\frac{4}{5} \times \frac{12}{13} - \sqrt{1-\left(\frac{4}{5}\right)^2}\sqrt{1-\left(\frac{12}{13}\right)^2}\right\}$$

$$\Rightarrow \cos^{-1}\left\{\frac{4}{3} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}\right\} = \cos^{-1}\left\{\frac{48}{65} - \frac{15}{65}\right\}$$

$$= \cos^{-1}\frac{33}{65} RHS$$

Question 2

(a) Find non-zero values of x **satisfying the matrix** $x\begin{bmatrix} 4x & 4 \\ 3x & x \end{bmatrix} + 5\begin{bmatrix} 4 & 5x \\ 8 & 4x \end{bmatrix} = 2\begin{bmatrix} x^2 + 8 & 24 \\ 10 & 6x \end{bmatrix}$**.**

Sol:

We have,

$$x\begin{bmatrix} 4x & 4 \\ 3x & x \end{bmatrix} + 5\begin{bmatrix} 4 & 5x \\ 8 & 4x \end{bmatrix} = 2\begin{bmatrix} x^2+8 & 24 \\ 10 & 6x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4x^2 & 4x \\ 3x & x^2 \end{bmatrix} + \begin{bmatrix} 20 & 25x \\ 40 & 20x \end{bmatrix} = \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4x^2+20 & 4x+25x \\ 3x+40 & x^2+20x \end{bmatrix} = \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$$

$\Rightarrow 4x^2+20 = 2x^2+16$

$4x^2 - 2x^2 = 16 - 20$

$2x^2 = -4$

$x = \pm\sqrt{2}$

And,

4x+25x= 48

29x = 48

x = $\frac{48}{29}$

And,

3x+40= 20

3x = -20

x = -$\frac{20}{3}$

And finally,

$x^2 + 20x = 12x$

$x^2 + 8x = 0$

$x(x+8) = 0$

$x = 0$

$or, x = -8$

(b) let $f(x) = \begin{bmatrix} \cos a & -\sin a & 0 \\ \sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix}$**. Show that** $f(a)f(b) = f(a+b)$ **.**

Sol:

We have,

$$f(a)\cdot f(b)=\begin{bmatrix}\sin a & \cos a & 0\\ \cos a & -\sin a & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sin b & \cos b & 0\\ \cos b & -\sin b & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\sin a\sin b+\cos a\sin b & \cos a\sin b+\cos a\cos b & 0\\ \cos a\cos b-\sin a\sin b & -\sin a\sin b+\cos a\cos b & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\sin(a+b) & \cos(a-b) & 0\\ \cos(a+b) & -\sin(a+b) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow f(a)\cdot f(a)=f(a+b)$$

Question 3

Prove that: $\frac{\alpha^3}{2}\cos ec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)=\left(\alpha+\beta\right)\left(\alpha^2+\beta^2\right)$

Sol:

We have,

LHS,

$$\frac{\alpha^3}{2}\cos ec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$$

$$=\frac{\alpha^3}{2\sin^2\theta}+\frac{\beta^3}{2\cos^2\phi}$$

Where, $\theta=\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta} and \phi=\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}$

$$\frac{\alpha^3}{1-\cos 2\theta}+\frac{\beta^3}{1+\cos 2\phi}$$

$$=\frac{\alpha^3}{1-\cos\left(\tan^{-1}\frac{\alpha}{\beta}\right)}+\frac{\beta^3}{1+\cos\left(\tan^{-1}\frac{\beta}{\alpha}\right)}$$

$$=\frac{\alpha^3}{1-\cos\left(\cos^{-1}\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$=\frac{\alpha^3}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}+\frac{\beta^3}{1+\sqrt{\alpha^2+\beta^2}}$$

$$=\left\{\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2-\beta}}+\frac{\beta^3}{\sqrt{\alpha^2+\beta^2+\alpha}}\right\}\sqrt{\alpha^2+\beta^2}$$

$$=\left[\frac{\alpha^3\left\{\sqrt{\alpha^2+\beta^2}+\beta\right\}}{\alpha^2+\beta^2-\beta}+\frac{\beta^3\left\{\sqrt{\alpha^2+\beta^2}-\alpha\right\}}{\alpha^2+\beta^2-\alpha^2}\right]\sqrt{\alpha^2+\beta^2}$$

$$=\left\{\alpha\left(\sqrt{\alpha^2+\beta^2}+\beta\right)+\beta\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right\}\sqrt{\alpha^2+\beta^2}$$

$$=\alpha\left(\alpha^2+\beta^2\right)+\beta\left(\alpha^2+\beta^2\right)$$

$$=\left(\alpha+\beta\right)\left(\alpha^2+\beta^2\right)$$

Hence, RHS proved.

Question 4

Simplify: $2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}$

Sol:

$$2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(\frac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)}{1-\tan^2\frac{\alpha}{2}\left(\frac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)^2}\right\}$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}\right)^2-\tan^2\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}+\tan^2\frac{\beta}{2}\right)-\tan^2\frac{\alpha}{2}\left(1-2\tan\frac{\beta}{2}+\tan^2\frac{\beta}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\alpha}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)+\tan^2\frac{\beta}{2}\left(1-\tan^2\frac{\alpha}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\times\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}\right\}$$

$$=\tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right)$$

Question 5.

Let * be the binary operation on N given by $a*b = LCM(a,b)$ for all $a,b \in N$.

Then, find the

(i) Find $a,b \in N$

(ii) Find the identity element in N

Sol: (i) We have,

$$a*\text{b} = \text{LCM of a and b}$$
$$\therefore 5*7 = (LCM \text{ of 5 and 7}) = 35$$
$$and,$$
$$20*16 = (LCM \text{ of 20 and 16}) = 80$$

(ii) Let e be the identity element.

Then,

$$a*\text{e} = a = e*a$$

For all $a \in N$

$$a*\text{e} = a$$

For all $a \in N$

$\Rightarrow LCM(a,e) = a$ for all $a \in N$

$\Rightarrow e = 1$

So, 1 is the identity element in N,

Question: 6

Show that $f(x) = \sqrt{|x| - x}$ **is continuous for all** $x \geq 0$

Sol:

Let $g(x) = |x| - x$ and $\text{h}(x) = \sqrt{x}$

We can saw that domain (g)=R and domain(h)=$[0,\omega)$.

Also, g(x) and h(x) are continuous in theirs domains.

$$Domain\ (hog) = \{x \in Domain(g : g(x) \in domain(h)\}$$
$$\Rightarrow Domain\ (hog) = \{x \in R : |x| - x \in [0, \infty)\}$$
$$\Rightarrow \{x \in R : x \geq 0\} = [0, \infty)$$

Question 7.

Evaluate: $x\dfrac{dy}{dx} = y - x\tan\left(\dfrac{y}{x}\right)$ **.**

Sol:

We know that,

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$
$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$$

Clearly, the given differential equation is homogeneous. Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (i), we get,

$$v + x\frac{dv}{dx} = v - \tan v$$
$$\Rightarrow x\frac{dv}{dx} = -\tan v$$

By separating the variables,

$$\Rightarrow \cot v dv = \frac{-dx}{x}, if\ \ x \neq 0$$

Integrating both sides,

$$\int \cot v dv = -\int \frac{dx}{x}$$
$$\Rightarrow \log|\sin v| = -\log|x| + \log c$$
$$\Rightarrow |\sin\mathrm{v}| = \left|\frac{c}{x}\right|$$
$$\Rightarrow \left|\sin\frac{y}{x}\right| = \left|\frac{c}{x}\right|$$

Hence, $\left|\sin\dfrac{y}{x}\right| = \left|\dfrac{c}{x}\right|$ given the required solution.

Question 8.

(a) Find The $P(A)$ and $P(B)$. If A and B be two independent events. The probability of its simultaneous occurrence is $\frac{2}{8}$ and the probability that never occur $\frac{4}{8}$.

Sol:

Let $P(A)=x$
$P(B)=y$

We have $P(A\cap B)=\frac{2}{8}$ and $P(\bar{A}\cap\bar{B})=\frac{4}{8}$.

$$P(A\cap B)=\frac{2}{8}$$

Now, $\Rightarrow P(A)P(B)$

$$\Rightarrow xy=\frac{2}{8}$$

Since, A and B are the independent events.

Therefore,

$$P(\bar{A}\cap\bar{B})=\frac{4}{8}$$

$$\Rightarrow P(\bar{A})P(\bar{B})=\frac{4}{8}$$

$$\Rightarrow (1-x)(1-y)=\frac{4}{8}$$

$$\Rightarrow 1-x-y+xy=\frac{4}{8}$$

$$\Rightarrow x+y-\frac{2}{8}=\frac{1}{2}$$

$$\Rightarrow x+y=\frac{2}{8}+\frac{1}{2}=\frac{3}{4}$$

Now,

$$(x-y)^2 = (x+y)^2 - 4xy$$

$$\Rightarrow (x-y)^2 = \frac{9}{16} - 4\times\frac{2}{8} = -\frac{7}{16}$$

$$\Rightarrow x - y = \pm\frac{\sqrt{7}}{4}$$

Here, we get two cases

(i) When $x - y = \frac{\sqrt{7}}{4}$. In this case we get,

$$x - y = \frac{\sqrt{7}}{4} \text{ and } x+y=\frac{3}{4}$$

$$\Rightarrow x = \frac{3+\sqrt{7}}{8} \text{ and } y= \frac{6-3+\sqrt{7}}{8}$$

$$\Rightarrow P(A) = \frac{3+\sqrt{7}}{8} \text{and } P(B) = \frac{6-3+\sqrt{7}}{8}$$

(ii) When $x - y = -\frac{\sqrt{7}}{4}$ In this case we get,

$$x - y = -\frac{\sqrt{7}}{4} \text{ and } x+y=\frac{3}{4}$$

$$\Rightarrow x = \frac{3-\sqrt{7}}{8} \text{ and } y= \frac{6-3+\sqrt{7}}{8}$$

$$\Rightarrow P(A) = \frac{3-\sqrt{7}}{8} \text{ and } P(B) = \frac{6-3+\sqrt{7}}{8}$$

(b) Determine the probability.

(i) The three products are successful

(ii) None of the product are successful

If a company has estimated that the probabilities of success for three products introduced in the market are $\frac{2}{3}, \frac{4}{7}$ *and* $\frac{3}{5}$ **respectively.**

Sol: (i)

Let

$A =$ First product is successful and B= Second product is successfu, C= Third product is successful.

Therefore, $P(A) = \frac{2}{3}.P(B) = \frac{4}{7}$ *and* $P(C) = \frac{3}{5}$

So, the required probability= P (All three products are successful)

$$= P\left(A \cap B \cap C\right) = P\left(A\right)P\left(B\right)P\left(C\right)$$
$$= \frac{2}{3} \times \frac{4}{7} \times \frac{3}{5}$$
$$= \frac{8}{35}$$

(ii)

$A =$ First product is successful and B= Second product is successfu, C= Third product is successful.

Therefore, $P\left(A\right) = \frac{2}{3}.P\left(B\right) = \frac{4}{7} \text{ and } \mathrm{P}\left(C\right) = \frac{3}{5}$

So, the required probability= P (None of the products are successful)

$$= P\left(\bar{A} \cap \bar{B} \cap \bar{C}\right) = P\left(\bar{A}\right)P\left(\bar{B}\right)P\left(\bar{C}\right)$$
$$= \frac{1}{3} \times \frac{3}{7} \times \frac{2}{5}$$
$$= \frac{2}{35}$$

Question 9.

Show that the differential equation of all circle having radius *r* is

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = r^2\left(\frac{d^2x}{dy^2}\right)^2.$$

Sol:

We know that the equation of the family of the circle is

$$\left(x-a\right)^2 + \left(y-b\right)^2 = r^2 \text{................}(1)$$

Where *a* and *b* are the parameters.

We have seen that the equation (1) contains two arbitrary constant.

Therefore, we have to differentiate it two times.

$$2\left(x-a\right) + 2(y-b)\frac{dy}{dx} = 0$$
$$\Rightarrow \left(x-a\right) + \left(y-b\right)\frac{dy}{dx} = 0 \text{..................}(2)$$

Differentiating eq (2) with respect to x ,we get

$$(x-a)+(y-b)\frac{dy}{dx}=0$$

$$\Rightarrow 1+(y-b)\frac{d^2x}{dy^2}+\left(\frac{dy}{dx}\right)^2=0$$

$$\Rightarrow y-b=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2x}{dy^2}}$$

Put this value in eq (2), we get

$$(x-a)+\left(\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2x}{dy^2}}\right)\frac{dy}{dx}=0.$$

$$\Rightarrow (x-a)=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2x}{dy^2}}$$

Now, put these value in eq (1), we get

$$(x-a)^2+(y-b)^2=r^2.$$

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2x}{dy^2}}+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2x}{dy^2}}=r^2$$

$$\Rightarrow \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3=r^2\left(\frac{d^2x}{dy^2}\right)^2$$

Hence, the differential equation of all circle having radius r is $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3=r^2\left(\frac{d^2x}{dy^2}\right)^2$ proved**.**

SECTION B(20 Marks)

Question10.

a) **Find the equation of the line passing through the point $(2,2,3)$ having $(4,5,4)$ as direction ratios normal to the plane.**

Sol:

We know that the required plane passes through the point having position vector $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ and it is normal to the vector $\vec{n} = 4\hat{i} + 5\hat{j} + 4\hat{k}$ **.**

Therefore, the vector equation of the plane is

$$\left(\vec{r} - \vec{a}\right) \cdot \vec{n} = 0$$
$$\Rightarrow \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} = 0$$
$$\Rightarrow \vec{r} \cdot \left(4\hat{i} + 5\hat{j} + 4\hat{k}\right) = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) \cdot \left(4\hat{i} + 5\hat{j} + 4\hat{k}\right)$$
$$\Rightarrow \vec{r} \cdot \left(4\hat{i} + 5\hat{j} + 4\hat{k}\right) = 8 + 10 + 12$$
$$\Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \cdot \left(4\hat{i} + 5\hat{j} + 4\hat{k}\right) = 30$$
$$\Rightarrow 4x + 5y + 4z = 30 \qquad \left[\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\right]$$

Hence, the required equation is $4x + 5y + 4z = 30$.

b) Prove that the vectors $\vec{a} = -4\hat{i} - 4\hat{j} + 8\hat{k}, \vec{b} = -4\hat{i} + 8\hat{j} - 2\hat{k}$ and $\vec{c} = 8\hat{i} - 4\hat{j} - 4\hat{k}$ **are coplanar.**

Sol:

We know that three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if its scalar triple product is zero. i.e. $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.

So, we have,

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} -4 & -4 & 8 \\ -4 & 8 & -4 \\ 8 & -4 & -4 \end{vmatrix}$$
$$= -4\left(-32 - 16\right) + 4\left(16 + 32\right) - 8\left(16 - 64\right)$$
$$= 192 + 192 - 384$$
$$= 0$$

Hence, the given vectors $\vec{a} = -4\hat{i} - 4\hat{j} + 8\hat{k}, \vec{b} = -4\hat{i} + 8\hat{j} - 2\hat{k}$ and $\vec{c} = 8\hat{i} - 4\hat{j} - 4\hat{k}$ are coplanar.

Question 11.

Simplify: $\int_0^{\frac{\pi}{2}} \frac{\cos^2}{\cos^2 x + 4\sin^2 x} dx$

Sol:

We have,

Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2}{\cos^2 x + 4\sin^2 x} dx$ Then,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\left(1-\cos^2 x\right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{4-3\cos^2 x} dx$$

$$= \frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{-3\cos^2 x}{4-3\cos^2 x} dx$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{(4-3\cos^2 x)-4}{4-3\cos^2 x} dx$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} \left\{1-\frac{4}{4-3\cos^2 x}\right\} dx$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} 1\cdot dx + \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{1}{4-3\cos^2 x} dx$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} 1\cdot dx + \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{4(1+\tan^2 x)-3} dx$$ [Diviing N^r and D^r by $\cos^2$x]

$$= -\frac{1}{3}\left[x\right]_0^{\frac{\pi}{2}} + \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{4(1+\tan^2 x)-3} dx$$

$$= \frac{-1}{3}\left(\frac{\pi}{2}-0\right) + \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{1}{1+4t^2} dt, \text{ where t = } \tan x$$

$$= \frac{-\pi}{6} + \frac{4}{3}\times\frac{1}{2}\left[\tan^{-1} 2t\right]_0^{\alpha}$$

$$= \frac{-\pi}{6} + \frac{2}{3}\times\left(\frac{\pi}{2}-0\right) = \frac{\pi}{6}$$

Question 12.

Determine

(a) $\vec{a} \cdot \vec{b}$ **If** $\vec{a} = 2\hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$

Sol:

We have,

$$\vec{a} = 2\hat{i} + 2\hat{j} - 2\hat{k} \text{ and } \vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$$
$$\therefore \vec{a} \cdot \vec{b} = \left(2\hat{i} + 2\hat{j} - 2\hat{k}\right) \cdot \left(6\hat{i} - 3\hat{j} + 2\hat{k}\right)$$
$$= (2)(6) + (2)(-3) + (-2)(2)$$
$$= 12 + -6 - 4$$
$$= 2$$

Hence, the value of $\vec{a} \cdot \vec{b}$ is 2.

(b) Find $|\vec{a}|\, and\, |\vec{b}|$ **if** $\left(\vec{a} + \vec{b}\right) \cdot \left(\vec{a} - \vec{b}\right) = 12$ and $|\vec{a}| = 2|\vec{b}|$

Sol:

We have,

$$\left(\vec{a} + \vec{b}\right) \cdot \left(\vec{a} - \vec{b}\right) = 12$$
$$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 12$$
$$\Rightarrow 4|\vec{b}|^2 - |\vec{b}|^2 = 12$$
$$\Rightarrow 3|\vec{b}|^2 = 12$$
$$\Rightarrow |\vec{b}|^2 = 4$$
$$\Rightarrow |\vec{b}| = 2$$
$$\therefore |\vec{a}| = 4 \quad |\vec{b}| = 2$$

Hence, the value of $|\vec{a}|\, and\, |\vec{b}|$ is 4 and 2.

SECTION C (20 Marks)

Question 13.

(a) Determine the $\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC)$**.If the total cost function is given by** $C = a + bx + cx^2$**.**

Sol:

We have

$$C = a + bx + cx^2$$

$$AC = \frac{C}{x} = \frac{a}{x} + b + cx,$$

And

$$MC = \frac{dC}{dx} = \frac{d}{dx}\left(a + bx + cx^2\right) = b + 2cx$$

Now,

$$\frac{1}{x}\left(MC - AC\right)$$

$$= \frac{1}{x}\left\{\left(b + 2cx\right) - \left(\frac{a}{x} + b + cx\right)\right\}$$

$$= \frac{1}{x}\left\{cx - \frac{a}{x}\right\}$$

$$= c - \frac{a}{x^2} ....(i)$$

And,

$$\frac{d}{dx}\left(\frac{a}{x} + b + cx\right)$$

$$= \frac{-a}{x^2} + 0$$

$$= c - \frac{-a}{x^2} ....(ii)$$

From eq (i) and (ii) we get,

$$\frac{d}{dx}\left(AC\right) = \frac{1}{x}\left(MC - AC\right)$$

(b) Prove that the slope of the average cost curve is $a - \frac{c}{x^2}$ **. If** $C = ax^2 + bx + c$ **represent the total cost function.**

Sol:

The average cost function AC is given by,

$$Ac = \frac{C}{x} = \frac{ax^2 + bx + c}{x}$$

$$= ax + b + \frac{c}{x}$$

The slope of the tangent to cost curve is $\frac{d}{dx}\left(AC\right)$

$$\frac{d}{dx}\left(ax+b+\frac{c}{x}\right)$$

$$=a-\frac{c}{x^2}$$

Hence, the slope of the average cost curve is $a-\frac{c}{x^2}$.

Question 14

(1) If you are given the following data $\bar{X}=30, \bar{Y}=75, \sigma_X=10$, $\sigma_Y=6$ and r=0.66**. In a bivariate distribution and the correlation between X and Y is 0.66.**

(a) Find the two regression coefficient

(b) The two regression equation

Sol: (a)

We have

$\bar{X}=30, \bar{Y}=75, \sigma_X=10$, $\sigma_Y=6$ and r=0.66

$$\Rightarrow b_{YX}=r\frac{\sigma_Y}{\sigma_X}$$

$$=0.66\times\frac{6}{10}$$

$$=0.40,$$

$$b_{XY}=r\frac{\sigma_X}{\sigma_Y}$$

$$=0.66\times\frac{10}{6}$$

$$=1.1$$

Sol: (b)

The line of regression Y on X is given by,

$$y-\bar{Y}=b_{YX}\left(x-\bar{X}\right)$$

$$\Rightarrow y-85=0.40(x-36)$$

$$\Rightarrow y=0.40x+63$$

The line of regression X on Y is given by,

$$x-\bar{X}=b_{XY}\left(y-\bar{Y}\right)$$

$$\Rightarrow x-30=1.1(y-75)$$

$$\Rightarrow x=1.1y-52.5$$

(2) Determine the regression lines of P on S and S on P. If there are two series on index number. P for price index and S for stock of a commodity. The mean and standard deviation of P are 104 and 6 and that of S are 102 and 6 respectively. And the correlation coefficient between the two series is 0.6.

Sol:

We have

$\bar{P} = 104, \bar{S} = 102, \sigma_p = 6, \sigma_s = 6$ and $r(P,S) = 0.6$

Therefore, the line of regression of P on S is

$$P - \bar{P} = r(P,S)\frac{\sigma_p}{\sigma_s}(S - \bar{S})$$

$$\Rightarrow P - 104 = 0.6 \times \frac{6}{6}(S - 102)$$

$$\Rightarrow P - 0.6S - 42.8 = 0$$

Therefore, the line of regression of S on P is

$$S - \bar{S} = r(P,S)\frac{\sigma_S}{\sigma_P}(P - \bar{P})$$

$$\Rightarrow S - 102 = 0.6 \times \frac{6}{6}(P - 104)$$

$$\Rightarrow 0.6P - S + 39.6 = 0$$

Question 15.

(a) Calculate the number of toys of type A and type B produced per day. If a toy company manufacture two types of dolls; a basic version doll A and a deluxe version doll B. Each doll of type B takes twice as long as to produce as one type A, and the company would have time to make a maximum of 1,000 per day if it produces only the basic version. The supply of plastic is sufficient to produce 100 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only300 per day available. If the company makes profit of ₹ 4 ₹ 6 per doll respectively on doll A and doll B.

Sol:

Let *x* doll of type A and *y* dolls of type B. Then,

Therefore, total profit is $4x + 6y$.

Since, each doll of type B takes twice as long to produce as one of type A, So, total time taken to produce *x* dolls of types A and *y* dolls of type B is $x + 2y$.

$\therefore x + 2y \leq 1{,}000$

Since, the plastic is available to produce dolls only. So, $x + y \leq 900$.

Also fancy dress is available for 600 dolls per day only so, $y \le 300$.

We know that the number of doll cannot be negative. Therefore, $x \ge 0, y \ge 0$.

Hence, the linear programming problem for the given problem is as follows:

Maximize $Z = 4x + 6y$.

Subject to constraints

$x + 2y \ge 1,000$
$x + y \le 900$
$y \le 300$
and $x \ge 0, y \ge 0$,

(b) A furniture firm manufactures chairs and tables, each requiring the use of three machines A,B and C. Production of one chair requires 2 hours on machine A, 1 hour on machine B, and 1 hour on machine C. Each table requires 1 hour each on machines A and B 3 hours on machine C. The profit realized by selling one chair is ₹ 30 while for a table the figure is ₹60.The total time available per week on machine A is 70hours, on machine B is 40 hours, and on machine C is 90 hours. How many chairs and tables should be made per week so as to maximize profit? Develop a mathematical formulation.

Sol:

The given data may be put in the following tabular form:

Machine	Chair	Table	Available time per week (in hours)
A B C	2 1 1	1 1 3	70 40 30
Profit per unit	₹30	₹60	

Let x chairs and y tables be produced per week to maximize the profit. Then, the total profit for x chairs and y tables is $30x + 60y$.

It is given that a chair requires 2 hours on machine A and a table requires 1 hour on machine A. Therefore, total time taken by machine A to produce x chairs and y tables is $(2x + y)$ hours. This must be less than or equal to total hours available on machine A.

$\therefore 2x + y \le 70$

Similarly, the total time taken by machine B to produce x chairs and y tables is $(x+y)$ hours. But, the total time available per week on machine B is 40 hours.

$\therefore x+y \leq 40$

Finally, the total time taken by machine C to produce x chains and y tables is $(x+3y)$ hours and the total time available per week on machine C is 90 hours.

$\therefore x+3y \leq 90$

Since, the number of chairs and tables cannot be negative.

$\therefore x \geq 0$ and $y \geq 0$

Let Z denote the total profit. Then,

$Z = 30x + 60y$.

Hence, the mathematical form of the given LPP is as follows:

Maximize $Z = 30x + 60y$

Subject to

$$\begin{aligned} &2x + y \leq 70 \\ &x + y \leq 40 \\ &x + 3y \leq 90 \\ &and, x \geq 0, y \geq 0 \end{aligned}$$