MATHEMATICS

(Maximum Marks: 100)

(Time allowed: Three hours)

(Candidates are allowed additional 15 minutes for only reading the paper.

They must NOT start writing during this time.)

Section A – Answer **Question** 1(compulsory) and five other questions.

Section B and Section C- Answer **two** questions from **either** section B **or** section C.

All working including rough work, should be done on the same sheet as, and adjacent to

The rest of the answer.

The intended marks for questions or parts of questions are given in brackets [ ].

Mathematical tables and graph papers are provide.

SECTION A (80 Marks)

Question 1

(i) If $\begin{bmatrix} a+b & 4 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}$**, find the values of a and b.**

Sol:

The corresponding elements of two equal matrices are equal.

$$\begin{bmatrix} a+b & 4 \\ 5 & \text{ab} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}$$

$\Rightarrow a+b=4$

$and,$

$ab=4$

$\Rightarrow a+\frac{4}{a}=4$

$\Rightarrow a^2+4=4a$

$\Rightarrow a^2-4a+4=0$

$\Rightarrow (a-2)(a-2)=0$

$\Rightarrow a=2,2$

Now,

$a=2$ *and* ab = 2

$\Rightarrow$ b = 2

Hence, the value is $a=2$ and $\text{b}=2$.

(ii) Determine the equation $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$ **.**

Sol:

We have,

$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

$\sin^{-1}(1-x)=\frac{\pi}{2}+2\sin^{-1}x$

$\sin\left\{\sin^{-1}(1-x)\right\}=\sin\left(\frac{\pi}{2}+2\sin^{-1}x\right)$

$1-x=\cos(2\sin^{-1}x)$

$1-x=\cos\left\{\cos^{-1}\left(1-2x^2\right)\right\}$ $\quad [\because 2\sin^{-1}x=\cos^{-1}(1-2x^2)]$

$1-x=\left(1-2x^2\right)$

$x=2x^2$

$x\left(2x-1\right)=0$

$x=0,\frac{1}{2}$

For,

$x=\frac{1}{2}$

Hence,

$$LHS=\sin^{-1}(1-x)-2\sin^{-1}x=\sin^{-1}\frac{1}{2}-2\sin^{-1}\frac{1}{2}=\sin^{-1}\frac{1}{2}=\frac{\pi}{6}\neq RHS$$

(iii) find the function *f(x)* given by $f(x)=\begin{cases} x\sin\frac{1}{x} & , x\neq 0 \\ 0 & , x=0 \end{cases}$ **is continuous at** $x=0$**.**

Sol:

$$(LHL \text{ at } x=0)=\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)=\lim_{h\to 0} -h\sin\left(\frac{1}{-h}\right)$$

$=\lim_{h\to 0} h\sin\left(\frac{1}{h}\right)=0\times$ (an oscillating number between -1 and 1)=0

$$=(RHL \text{ at } x=0)=\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0} -h\sin\left(\frac{1}{h}\right)$$

$=0\times$ (an oscillating number between -1 and 1)=0

and, $f(0)=0$

Thus, obtain

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0).$$

Hence, *f(x)* is continuous at $x=0$.

(iv) if $x^2+2xy+y^3=32$ **find,** $\frac{dy}{dx}$

Sol:

We have,

$$x^2+2xy+y^3=32$$

Differentiating both sides of this with respect to *x*, we get

$$\frac{d}{dx}(x^2)+2\frac{d}{dx}(xy)+\frac{d}{dx}y^3=32$$

$$\Rightarrow 2x+2\left(x\frac{dy}{dx}+y\right)+3y^2=\frac{d}{dx}(32)$$

$$\Rightarrow 2x+2y+\frac{dy}{dx}\left(2x+3y^2\right)=0$$

$$\Rightarrow \frac{dy}{dx}\left(2x+3y^2\right)=-2\left(x+y\right)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{-2\left(x+y\right)}{\left(2x+3y^2\right)}.$$

(v) if $x^3+y^3 3axy$ **find,** $\frac{dy}{dx}$

Sol:

We have,

$x^3+y^3 3axy$

Differentiating both sides of this with respect to *x*, we get

$$\frac{d}{dx}(x^3)+\frac{d}{dx}y^3=3a\frac{d}{dx}(xy)$$

$$\Rightarrow 3x^2+3y^2\frac{dy}{dx}=3a\left\{x\frac{dy}{dx}+y\right\}$$

$$\Rightarrow \left(3y^2-3ax\right)\frac{dy}{dx}=3ay-3x^2$$

$$\Rightarrow 3\left(y^2-ax\right)\frac{dy}{dx}=3\left(ay-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$$

(vi) Mother, father and son line up at random for a family picture. Find P (AB), if A and B are defined as follows:

A = Son on one end, B = Father in the middle.

Sol:

Total number of ways in which Mother (M), Father (F) and Son (S) can be lined up at random in one of the following ways:

MFS, MSF, FMS, FSM, SFM, SMF

We have,

$A = \{SMF, SFM, MFS, FMS\}$ and B = $\{MFS, SFM\}$

$\therefore A \cap B = \{MFS, SFM\}$

Clearly,

$n(A \cap B) = 2$ and $n(B) = 2$

$\therefore$ Required probability = $P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{2}{2} = 1$

(vii) Evaluate: $\int_0^1 \frac{2x}{6x^2+1} dx$

Sol:

Let $6x^2 + 1 = t$

Then,

$(6x^2 + 1) = dt$

$\Rightarrow 12xdx = dt$

When,

$x = 0, t = 6x^2 + 1 \Rightarrow t = 1$

And,

$x = 1, t = 6x^2 + 1 \Rightarrow t = 7$

$\therefore \int_0^1 \frac{2x}{6x^2+1} dx = \int_1^7 \frac{2x}{t} \times \frac{dt}{12dx}$

$\frac{1}{6}\int_1^7 \frac{1}{t} dt = \frac{1}{6}[\log t]_1^7 = \frac{1}{6}(\log 7 - \log 1) = \frac{1}{6}\log 7$

(viii) Find the two regression coefficient. If you are given the following data $\bar{X} = 30, \bar{Y} = 75, \sigma_X = 10$, $\sigma_Y = 6$ and r=0.66**. In a bivariate distribution and the correlation between X and Y is 0.66.**

Sol:

We have,

$\overline{X} = 30, \overline{Y} = 75, \sigma_X = 10$, $\sigma_Y = 6$ and r=0.66

$\Rightarrow b_{YX} = r\frac{\sigma_Y}{\sigma_X}$

$= 0.66 \times \frac{6}{10}$

$= 0.40,$

$b_{XY} = r\frac{\sigma_X}{\sigma_Y}$

$= 0.66 \times \frac{10}{6}$

$= 1.1$

(ix) Prove that $(AB)^T = B^T A^T$ **. If** $A = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$

Sol:

We have $A = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$

$AB = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$

So, $= \begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 10 \\ 8 & 12 & 20 \end{bmatrix}$

$\Rightarrow (AB)^T = \begin{bmatrix} 2 & 4 & 8 \\ 3 & 6 & 12 \\ 5 & 10 & 20 \end{bmatrix} \text{...........}(1)$

And,

$$B^T A^T = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}^T \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 8 \\ 3 & 6 & 12 \\ 5 & 10 & 20 \end{bmatrix}$$

Hence, $(AB)^T = B^T A^T$ proved.

(x) Evaluate: $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}$

Sol:

We have,

$$\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\left\{\frac{4}{5} \times \frac{12}{13} - \sqrt{1 - \left(\frac{4}{5}\right)^2}\sqrt{1 - \left(\frac{12}{13}\right)^2}\right\}$$

$$\Rightarrow \cos^{-1}\left\{\frac{4}{3} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}\right\} = \cos^{-1}\left\{\frac{48}{65} - \frac{15}{65}\right\}$$

$$= \cos^{-1}\frac{33}{65}$$

Question 2

(a) Find non-zero values of x satisfying the matrix $x\begin{bmatrix} 4x & 4 \\ 3x & x \end{bmatrix} + 2\begin{bmatrix} 4 & 5x \\ 8 & 4x \end{bmatrix} = 2\begin{bmatrix} x^2 + 8 & 24 \\ 10 & 6x \end{bmatrix}$**.**

Sol:

We have,

$$x\begin{bmatrix}4x & 4\\ 3x & x\end{bmatrix}+2\begin{bmatrix}4 & 5x\\ 8 & 4x\end{bmatrix}=2\begin{bmatrix}x^2+8 & 24\\ 10 & 6x\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}4x^2 & 4x\\ 3x & x^2\end{bmatrix}+\begin{bmatrix}8 & 10x\\ 16 & 8x\end{bmatrix}=\begin{bmatrix}2x^2+16 & 48\\ 20 & 12x\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}4x^2+8 & 4x+10x\\ 3x+16 & x^2+8x\end{bmatrix}=\begin{bmatrix}2x^2+16 & 48\\ 20 & 12x\end{bmatrix}$$

$\Rightarrow 4x^2+8=2x^2+16$

$4x^2-2x^2=16-8$

$2x^2=8$

$x=2$

And,

4x+10x= 48

14x = 48

x = $\frac{24}{7}$

And,

3x+16= 20

3x = 4

x = $\frac{4}{3}$

And finally,

$x^2+8x=12x$

$x^2-4x=0$

$x(x-4)=0$

$x=0$

$or, x=4$

(b) let $f(x)=\begin{bmatrix}\cos a & -\sin a & 0\\ \sin a & \cos a & 0\\ 0 & 0 & 1\end{bmatrix}$**. Show that** $f(a)f(b)=f(a+b)$**.**

Sol:

We have,

$$f(a)\cdot f(b)=\begin{bmatrix}\cos a & -\sin a & 0\\ \sin a & \cos a & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos b & -\sin b & 0\\ \sin b & \cos b & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\cos a\cos b-\sin a\sin b & -\sin b\cos a-\sin a\cos b & 0\\ \sin a\sin b+\cos a\sin b & -\sin a\sin b+\cos a\cos b & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\cos(a+b) & -\sin(a+b) & 0\\ \sin(a+b) & \cos(\mathrm{a}+b) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow f(a)\cdot f(a)=f(a+b)$$

Question 3

Evaluate: $\frac{\alpha^3}{2}\cos ec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$

Sol:

We have,

$$\frac{\alpha^3}{2}\cos ec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$$

$$=\frac{\alpha^3}{2\sin^2\theta}+\frac{\beta^3}{2\cos^2\phi}$$

Where, $\theta=\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta} and \phi=\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}$

$$\frac{\alpha^3}{1-\cos 2\theta}+\frac{\beta^3}{1+\cos 2\phi}$$

$$=\frac{\alpha^3}{1-\cos\left(\tan^{-1}\frac{\alpha}{\beta}\right)}+\frac{\beta^3}{1+\cos\left(\tan^{-1}\frac{\beta}{\alpha}\right)}$$

$$=\frac{\alpha^3}{1-\cos\left(\cos^{-1}\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$=\frac{\alpha^3}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}+\frac{\beta^3}{1+\sqrt{\alpha^2+\beta^2}}$$

$$=\left\{\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2-\beta}}+\frac{\beta^3}{\sqrt{\alpha^2+\beta^2+\alpha}}\right\}\sqrt{\alpha^2+\beta^2}$$

$$=\left[\frac{\alpha^3\left\{\sqrt{\alpha^2+\beta^2}+\beta\right\}}{\alpha^2+\beta^2-\beta}+\frac{\beta^3\left\{\sqrt{\alpha^2+\beta^2}-\alpha\right\}}{\alpha^2+\beta^2-\alpha^2}\right]\sqrt{\alpha^2+\beta^2}$$

$$=\left\{\alpha\left(\sqrt{\alpha^2+\beta^2}+\beta\right)+\beta\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right\}\sqrt{\alpha^2+\beta^2}$$

$$=\alpha\left(\alpha^2+\beta^2\right)+\beta\left(\alpha^2+\beta^2\right)$$

$$=\left(\alpha+\beta\right)\left(\alpha^2+\beta^2\right)$$

Question 4

Prove that: $2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}=\tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right)$

Sol:

$$LHS = 2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)}{1-\tan^2\frac{\alpha}{2}\left(\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)^2}\right\}$$

$$= \tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}\right)^2-\tan^2\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}+\tan^2\frac{\beta}{2}\right)-\tan^2\frac{\alpha}{2}\left(1-2\tan\frac{\beta}{2}+\tan^2\frac{\beta}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\alpha}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)+\tan^2\frac{\beta}{2}\left(1-\tan^2\frac{\alpha}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\times\dfrac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\dfrac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}\right\}$$

$$= \tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right) = RHS$$

Question 5.

Let * be the binary operation Q_0 (Set of all non-zero rational number) defined by
$a*b=\frac{ab}{4}$ for all $a,b\in Q_0$.

Then, find the

(i) Identify element in Q_0.

(ii) Inverse of an element in Q_0

Sol: (i)

For identify the element, Let e be the identify element in Q_0. Then,

$a*e=a=e*a$ for all a $\in Q_0$
$\Rightarrow a*e=a$ for all a $\in Q_0$
$\Rightarrow \frac{ae}{4}=a$ for all a $\in Q_0$
$\Rightarrow e=4$

Thus, 4 is the identify element in Q_0 for the binary operation '*'.

(ii) For inverse of an element, Let a be the invertible element in and let b be the inverse.

$$a*b=e=b*a$$
$$\Rightarrow a*\text{b}=4$$
$$\Rightarrow \frac{ab}{4}=4$$
$$\Rightarrow b=\frac{16}{a}$$

Therefore, $\frac{16}{a}\in Q_0$ for all $a\in Q_0$. So, every element of Q_0 is invertible and the inverse of an element is

$\frac{16}{a}$.

Question: 6

If the function $f(\mathrm{x})$ defined by $\int \tan x \tan 2x \tan 3x$, if $x \neq 0, if\ \mathrm{x=0}$ is continuous at $\mathrm{x=0}$.find k.

Sol:

Since $f(\mathrm{x})$ is continuous at $\mathrm{x=0}$.

$$\lim_{x\to 0} f(x) = f(0)$$
$$\Rightarrow \lim_{x\to 0} \frac{\log(1+\mathrm{ax}) - \log(1-bx)}{x} = k$$
$$\Rightarrow \lim_{x\to 0} \left\{ \frac{\log(1+\mathrm{ax})}{x} - \frac{\log(1-bx)}{x} \right\} = k$$
$$\Rightarrow \lim_{x\to 0} \frac{\log(1+\mathrm{ax})}{x} - \lim_{x\to 0} \frac{\log(1-bx)}{x} = k$$
$$\Rightarrow a \lim_{x\to 0} \frac{\log(1+\mathrm{ax})}{ax} - (-\mathrm{b}) \frac{\log(1-bx)}{(-\mathrm{b})x} = k$$
$$\Rightarrow a(1) - (-\mathrm{b})(1) = \mathrm{k}$$
$$\Rightarrow \mathrm{a+b=k}$$

Thus, $f(\mathrm{x})$ is continuous at $\mathrm{x=0}$, if $\mathrm{k=a+b}$.

Question 7.

(i) $\int \tan x \tan 2x \tan 3x dx$

Sol:

We know that

$$\tan 3x = \tan(2x+x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$
$$\Rightarrow \tan 3x(1 - \tan 2x \tan x) = \tan 2x + \tan x$$
$$\Rightarrow \tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x$$
$$\Rightarrow \tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$
$$\therefore I = \int \tan x \tan 2x \tan 3\mathrm{x}d\mathrm{x}$$
$$\Rightarrow \int \tan 3x - \tan 2x - \tan x = \frac{-1}{3}\log_e|\cos 3x| + \frac{1}{2}\log_e|\cos 2x| + \log_e|\cos x| + C$$

(ii) $\int \tan(x-\theta)\tan(x+\theta)\tan 2x dx$

Sol:

We know that

$$2x = (x-\theta) + (x+\theta)$$

$$\tan 2x = \tan\{(x-\theta) + (x+\theta)\}$$

$$\Rightarrow \tan 2x = \frac{\tan(x-\theta) + (x+\theta)}{1-\tan(x-\theta)\tan(x+\theta)}$$

$$\Rightarrow \tan 2x - \tan(x-\theta)\tan(x+\theta)\tan 2x = \tan(x-\theta) + (x+\theta)$$

$$\Rightarrow \tan(x-\theta)\tan(x+\theta)\tan 2x = \tan 2x - \tan(x-\theta) - (x+\theta)$$

$$I = \int \tan(x-\theta)\tan(x+\theta)\tan 2x dx = \int\{\tan 2x - \tan(x-\theta) - (x+\theta)\}dx$$

$$I = \frac{-1}{2}\log|\cos 2x| + \log|\cos x-\theta| + \log|\cos x+\theta| + C$$

Question 8.

(a) Find The $P(A)$ and $P(B)$. If A and B be two independent events. The probability of its simultaneous occurrence is $\frac{1}{8}$ and the probability that never occur $\frac{3}{8}$.

Sol:

Let $P(A)=x$

$P(B)=y$

We have $P(A\cap B) = \frac{1}{8}$ and $P(\bar{A}\cap\bar{B}) = \frac{3}{8}$.

Now,

$$P(A\cap B) = \frac{1}{8}$$

$$\Rightarrow P(A)P(B)$$

$$\Rightarrow xy = \frac{1}{8}$$

Since, A and B are the independent events.

Therefore,

$$P\left(\bar{A}\cap\bar{B}\right)=\frac{3}{8}$$
$$\Rightarrow P\left(\bar{A}\right)P\left(\bar{B}\right)=\frac{3}{8}$$
$$\Rightarrow (1-x)(1-y)=\frac{3}{8}$$
$$\Rightarrow 1-x-y+xy=\frac{3}{8}$$
$$\Rightarrow x+y-\frac{1}{8}=\frac{5}{8}$$
$$\Rightarrow x+y=\frac{5}{8}+\frac{1}{8}=\frac{3}{4}$$

Now,

$$(x-y)^2=(x+y)^2-4xy$$
$$\Rightarrow (x-y)^2=\frac{9}{16}-4\times\frac{1}{8}=\frac{1}{16}$$
$$\Rightarrow x-y=\pm\frac{1}{4}$$

Here, we get two cases

(i) When $x-y=\frac{1}{4}$. In this case we get,

$x-y=\frac{1}{4}$ and x+y= $\frac{3}{4}$

$\Rightarrow x=\frac{1}{2}$ and y= $\frac{1}{4}$

$\Rightarrow P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{4}$

(ii) When $x-y=-\frac{1}{4}$ In this case we get,

$x-y=-\frac{1}{4}$ and x+y= $\frac{3}{4}$

$\Rightarrow x=\frac{1}{4}$ and y= $\frac{1}{2}$

$\Rightarrow P(A)=\frac{1}{4}$ and $P(B)=\frac{1}{2}$

(b) Determine the probability.

(i) The three products are successful

(ii) None of the product are successful

If a company has estimated that the probabilities of success for three products introduced in the market are $\frac{1}{3}, \frac{2}{5}$ *and* $\frac{2}{3}$ respectively.

Sol: (i)

Let

$A =$ First product is successful and B= Second product is successfu, C= Third product is successful.

Therefore, $P(A) = \frac{1}{3}.P(B) = \frac{2}{5}$ *and* $P(C) = \frac{2}{3}$

So, the required probability= P (All three products are successful)

$$= P(A \cap B \cap C) = P(A)P(B)P(C)$$

$$= \frac{1}{3} \times \frac{2}{5} \times \frac{2}{3}$$

$$= \frac{4}{45}$$

(ii)

$A =$ First product is successful and B= Second product is successfu, C= Third product is successful.

Therefore, $P(A) = \frac{1}{3}.P(B) = \frac{2}{5}$ *and* $P(C) = \frac{2}{3}$

So, the required probability= P (None of the products are successful)

$$= P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A})P(\bar{B})P(\bar{C})$$

$$= \frac{2}{3} \times \frac{3}{5} \times \frac{1}{5}$$

$$= \frac{2}{15}$$

Question 9.

Find the differential equation of all circle having radius *r*.

Sol:

We know that the equation of the family of the circle is

$$(x-a)^2 + (y-b)^2 = r^2 ....(1)$$

Where *a* and *b* are the parameters.

We have seen that the eq (1) contains two arbitrary constant.

Therefore, we have to differentiate it two times.

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow (x-a) + (y-b)\frac{dy}{dx} = 0..................(2)$$

Differentiating eq (2) with respect to *x* ,we get

$$(x-a)+(y-b)\frac{dy}{dx}=0$$

$$\Rightarrow 1+(y-b)\frac{d^2x}{dy^2}+\left(\frac{dy}{dx}\right)^2=0$$

$$\Rightarrow y-b=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2x}{dy^2}}$$

Put this value in eq (2), we get

$$(x-a)+\left(\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2x}{dy^2}}\right)\frac{dy}{dx}=0.$$

$$\Rightarrow (x-a)=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2x}{dy^2}}$$

Now, put these value in eq (1), we get

$$(x-a)^2+(y-b)^2=r^2.$$

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2x}{dy^2}}+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2x}{dy^2}}=r^2$$

$$\Rightarrow \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3=r^2\left(\frac{d^2x}{dy^2}\right)^2$$

Hence, the differential equation of all circle having radius r is $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3=r^2\left(\frac{d^2x}{dy^2}\right)^2$.

SECTION B(20 Marks)

Question10.

a) **Find the equation of the line passing through the point $(1,1,2)$ having $(2,3,2)$ as direction ratios normal to the plane.**

Sol:

We know that the required plane passes through the point having position vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and it is normal to the vector $\vec{n} = 2\hat{i} + 3\hat{j} + 2\hat{k}$.

Therefore, the vector equation of the plane is

$$\left(\vec{r} - \vec{a}\right) \cdot \vec{n} = 0$$
$$\Rightarrow \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} = 0$$
$$\Rightarrow \vec{r} \cdot \left(2\hat{i} + 3\hat{j} + 2\hat{k}\right) = \left(\hat{i} + \hat{j} + 2\hat{k}\right) \cdot \left(2\hat{i} + 3\hat{j} + 2\hat{k}\right)$$
$$\Rightarrow \vec{r} \cdot \left(2\hat{i} + 3\hat{j} + 2\hat{k}\right) = 2 + 3 + 4$$
$$\Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \cdot \left(2\hat{i} + 3\hat{j} + 2\hat{k}\right) = 9$$
$$\Rightarrow 2x + 3y + 2z = 9 \qquad \left[\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\right]$$

Hence, the required equation is $2x + 3y + 2z = 9$.

b) Prove that the vectors $\vec{a} = -2\hat{i} - 2\hat{j} + 4\hat{k}, \vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{c} = 4\hat{i} - 2\hat{j} - 2\hat{k}$ **are coplanar.**

Sol:

We know that three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if its scalar triple product is zero. i.e. $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.

So, we have,

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} -2 & -2 & -4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix}$$
$$= -2\left(-8 - 4\right) + 2\left(4 + 8\right) + 4\left(4 - 16\right)$$
$$= 24 + 24 - 48$$
$$= 0$$

Hence, the given vectors $\vec{a} = -2\hat{i} - 2\hat{j} + 4\hat{k}, \vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{c} = 4\hat{i} - 2\hat{j} - 2\hat{k}$ are coplanar.

Question 11.

Determine:

(i) $\int_0^{\frac{\pi}{2}} \sin^2 x dx$

Sol:

Let $I = \int_0^{\frac{\pi}{2}} \sin^2 x dx$ Then,

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2x) dx$$

$$= -\frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left\{\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right\} = \frac{\pi}{4}$$

(ii) $\int_0^{\frac{\pi}{4}} \sin 3x \sin 2x dx$

Sol:

Let $I = \int_0^{\frac{\pi}{4}} \sin 3x \sin 2x dx$ Then,

$$I=\frac{1}{2}\int_{0}^{\frac{\pi}{4}}\left(2\sin 3x\sin 2x\right)dx$$

$$=\frac{1}{2}\int_{0}^{\frac{\pi}{4}}\left(\cos x-\cos 5x\right)dx$$

$$=\frac{1}{2}\left[\sin x-\frac{1}{5}\sin 5x\right]_{0}^{\frac{\pi}{4}}$$

$$\Rightarrow I=\left(\sin\frac{\pi}{4}-\frac{1}{5}\sin\frac{5\pi}{4}\right)-\left(\sin 0-\frac{1}{5}\sin 0\right)$$

$$=\frac{1}{2}\left\{\frac{1}{\sqrt{2}}+\frac{1}{5\sqrt{2}}\right\}$$

$$=\frac{6}{2\left(5\sqrt{2}\right)}$$

$$=\frac{3\sqrt{2}}{10}$$

Question 12.

Determine

(a) $\vec{a}\cdot\vec{b}$ **If** $\vec{a}=2\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{b}=6\hat{i}-3\hat{j}+2\hat{k}$

Sol:

We have,

$\vec{a}=2\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{b}=6\hat{i}-3\hat{j}+2\hat{k}$

$\therefore\vec{a}\cdot\vec{b}=\left(2\hat{i}+2\hat{j}-2\hat{k}\right)\cdot\left(6\hat{i}-3\hat{j}+2\hat{k}\right)$

$=(2)(6)+(2)(-3)+(-2)(2)$

$=12+-6-4$

$=2$

Hence, the value of $\vec{a}\cdot\vec{b}$ is 2.

(b) Find $|\vec{a}|\,and\,|\vec{b}|$ **if** $\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)=27$ and $|\vec{a}|=2|\vec{b}|$

Sol:

We have,

$$\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)=27$$

$$\Rightarrow \left|\vec{a}\right|^2-\left|\vec{b}\right|^2=27$$

$$\Rightarrow 4\left|\vec{b}\right|^2-\left|\vec{b}\right|^2=27$$

$$\Rightarrow 3\left|\vec{b}\right|^2=27$$

$$\Rightarrow \left|\vec{b}\right|^2=9$$

$$\Rightarrow \left|\vec{b}\right|=3$$

$$\therefore \left|\vec{a}\right|=6 \quad \left|\vec{b}\right|=3$$

Hence, the value of $\left|\vec{a}\right| \, and \, \left|\vec{b}\right|$ is 6 and 3.

SECTION C (20 Marks)

Question 13.

(a) Show that $\frac{d}{dx}(AC)=\frac{1}{x}(MC-AC)$ **.If the total cost function is given by** $C=a+bx+cx^2$ **.**

Sol:

We have

$$C=a+bx+cx^2$$

$$AC=\frac{C}{x}=\frac{a}{x}+b+cx,$$

And

$$MC=\frac{dC}{dx}=\frac{d}{dx}\left(a+bx+cx^2\right)=b+2cx$$

Now,

$$\frac{1}{x}(MC-AC)$$

$$=\frac{1}{x}\left\{(b+2cx)-\left(\frac{a}{x}+b+cx\right)\right\}$$

$$=\frac{1}{x}\left\{cx-\frac{a}{x}\right\}$$

$$=c-\frac{a}{x^2}\text{.................}(i)$$

And,

$$\frac{d}{dx}\left(\frac{a}{x}+b+cx\right)$$

$$=\frac{-a}{x^2}+0$$

$$=c-\frac{-a}{x^2}...........(ii)$$

From eq (i) and (ii) we get,

$$\frac{d}{dx}(AC)=\frac{1}{x}(MC-AC)$$

(b) Find the slope of the average cost curve. If $C=ax^2+bx+c$ represent the total cost function.

Sol:

The average cost function AC is given by,

$$Ac=\frac{C}{x}=\frac{ax^2+bx+c}{x}$$

$$=ax+b+\frac{c}{x}$$

The slope of the tangent to cost curve is $\frac{d}{dx}(AC)$

$$\frac{d}{dx}\left(ax+b+\frac{c}{x}\right)$$

$$=a-\frac{c}{x^2}$$

Hence, the slope of the average cost curve is $a-\frac{c}{x^2}$.

Question 14

(1) If you are given the following data $\overline{X}=36, \overline{Y}=85, \sigma_X=11$ and $\sigma_Y=8$. In a bivariate distribution and the correlation between X and Y is 0.66.

(a) Find the two regression coefficient

(b) The two regression equation

Sol: (a)

We have

$\bar{X} = 36, \bar{Y} = 85, \sigma_X = 11$, $\sigma_Y = 8$ and r=0.66

$\Rightarrow b_{YX} = r\dfrac{\sigma_Y}{\sigma_X}$

$= 0.66 \times \dfrac{8}{11}$

$= 0.48,$

$b_{XY} = r\dfrac{\sigma_X}{\sigma_Y}$

$= 0.66 \times \dfrac{11}{8}$

$= 0.9075$

Sol: (b)

The line of regression Y on X is given by,

$y - \bar{Y} = b_{YX}\left(x - \bar{X}\right)$

$\Rightarrow y - 85 = 0.48(x - 36)$

$\Rightarrow y = 0.48x + 67.72$

The line of regression X on Y is given by,

$x - \bar{X} = b_{XY}\left(y - \bar{Y}\right)$

$\Rightarrow x - 36 = 0.9075(y - 85)$

$\Rightarrow y = 0.9075y + 41.41$

(2) Determine the regression lines of P on S and S on P. If there are two series on index number. P for price index and S for stock of a commodity. The mean and standard deviation of P are 100 and 8 and that of S are 103 and 4 respectively. And the correlation coefficient between the two series is 0.4.

Sol:

We have

$\bar{P} = 100, \bar{S} = 103, \sigma_p = 8, \sigma_s = 4$ and $r(P, S) = 0.4$

Therefore, the line of regression of P on S is

$P - \bar{P} = r(P, S)\dfrac{\sigma_p}{\sigma_s}\left(S - \bar{S}\right)$

$= P - 100 = 0.4 \times \dfrac{8}{4}(S - 103)$

$= 5P - 4S - 88 = 0$

Therefore, the line of regression of S on P is

$$S-\overline{S}=r(P,S)\frac{\sigma_S}{\sigma_P}(P-\overline{P})$$
$$=S-103=0.4\times\frac{4}{8}(P-100)$$
$$=P-5S+415=0$$

Question 15.

(a) Calculate the number of toys of type A and type B produced per day. If a toy company manufacture two types of dolls; a basic version doll A and a deluxe version doll B. Each doll of type B takes twice as long as to produce as one type A, and the company would have time to make a maximum of 2,000 per day if it produces only the basic version. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only600 per day available. If the company makes profit of Rs 3 Rs 5 per doll respectively on doll A and doll B.

Sol:

Let *x* doll of type A and *y* dolls of type B. Then,

Therefore, total profit is $3x+5y$.

Since, each doll of type B takes twice as long to produce as one of type A, So, total time taken to produce *x* dolls of types A and *y* dolls of type B is $x+2y$.

$\therefore x+2y\le 2,000$

Since, the plastic is available to produce dolls only. So, $x+y\le 1500$.

Also fancy dress is available for 600 dolls per day only so, $y\le 600$.

We know that the number of doll cannot be negative. Therefore, $x\ge 0, \mathrm{y}\ge 0$.

Hence, the linear programming problem for the given problem is as follows:

Maximize $Z=3x+5y$.

Subject to constraints

$$x+2y\ge 2,000$$
$$x+y\le 1500$$
$$y\le 600$$
$$and\ x\ge 0, \mathrm{y}\ge 0,$$

(b) Determine the linear programming problem to maximize the income. If a firm can produce three types of cloth, say C_1,C_2 and C_3 .Three kinds of wood are required for it, say red wool, green wood , 2 meters of green wool and 2 meters of blue wool; and

one unit of cloth C_3 needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 16 meters of cloth C_1 is ₹ 6 of cloth C_2 is ₹ 10 rupees and of cloth C_3 is ₹ 8.

Sol:

The given information can be put in the following tabular form:

	Cloth C_1 Cloth C_2 Cloth C_3	Total quality of wool available
Red Wool Green Wool Blue Wool	2 3 0 0 2 5 3 2 4	16 20 30
Income (In ₹)	6 10 8	

Let X_1, X_2 and X_3 be the quantity produced in meters of the cloth of type C_1, C_2 and C_3 respectively.

Since 2 meters of red wool are required for one meter of cloth C_1 and X_1 meters of cloth C_1 are produced, therefore $2x_1$ meters of red wool will be required for cloth C_1. Similarly, cloth C_2 requires $3x_2$ meters of red wool and cloth C_3 does not require red wool. Thus, the total quantity of red wool required is $2x_1 + 3x_2 + 0x_3$.

But, the maximum available quantity of red wool is 16 meters. Therefore, $2x_1 + 3x_2 + 0x_3 \le 16$

Similarly,

The total quantities of green and blue wool required are $0x_1 + 2x_2 + 5x_3$ and $3x_1 + 2x_2 + 4x_3$ respectively.

But, the total quantities of green and blue wool available are 20 meters and 30 meters respectively.

$\therefore 0x_1 + 2x_2 + 5x_3 \le 20$ and $3x_1 + 2x_2 + 4x_3 \le 30$

Also, we cannot produce negative quantities, therefore $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$

The total income is $Z = 6x_1 + 10x_2 + 8x_3$

Hence, the linear programming problem for the given problem is

Maximize $Z = 6x_1 + 10x_2 + 8x_3$

Subject to the constraints,

$$
\begin{aligned}
&2x_1 + 3x_2 + 0x_3 \le 16 \\
&0x_1 + 2x_2 + 5x_3 \le 20 \\
&3x_1 + 2x_2 + 4x_3 \le 30 \\
&and, x_1 \ge 0, x_2 \ge 0, x_3 \ge 0
\end{aligned}
$$