26. Fundamental Concepts of 3-Dimensional Geometry

Exercise 26

1. Question

Find the direction cosines of a line segment whose direction ratios are:

(i) 2. – 6, 3

(ii) 2, - 1, - 2,

(iii) – 9, 6, -2

Answer

(i) direction ratios are:- (2, -6, 3)

So, the direction cosines are- (l, m, n), where, $l^2 + m^2 + n^2 = 1$,

So, l, m, and n are:-

$$l = \frac{2}{\sqrt{2^2 + (-6)^2 + 3^2}}$$

$$m = -\frac{6}{\sqrt{2^2 + (-6)^2 + 3^2}}$$

$$n = \frac{3}{\sqrt{2^2 + (-6)^2 + 3^2}}$$

$$(l, m, n) = (\frac{2}{7}, -\frac{6}{7}, \frac{3}{7})$$

The direction cosines are:- $(\frac{2}{7}, -\frac{6}{7}, \frac{3}{7})$

(ii) direction ratios are:- (2, -1, -2)

So, the direction cosines are:- (l, m, n), where, $l^2 + m^2 + n^2 = 1$,

So, l, m, and n are:-

$$l = \frac{2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$m = -\frac{-1}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$n = \frac{-2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$(l, m, n) = (\frac{2}{3}, -\frac{1}{3}, \frac{-2}{3})$$

The direction cosines are:- $(\frac{2}{3}, -\frac{1}{3}, \frac{-2}{3})$

(iii) direction ratios are:- (-9, 6, -2)

So, the direction cosines are- (l, m, n), where, $l^2 + m^2 + n^2 = 1$,

So, l, m, and n are:-

$$l = -\frac{9}{\sqrt{(-9)^2 + 6^2 + (-2)^2}}$$

$$m = \frac{6}{\sqrt{(-9)^2 + 6^2 + (-2)^2}}$$

$$n = \frac{-2}{\sqrt{(-9)^2 + 6^2 + (-2)^2}}$$

$$(l, m, n) = (\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11})$$

The direction cosines are:- $(\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11})$

2. Question

Find the direction ratios and the direction cosines of the line segment joining the points:

(i) A (1, 0, 0) and B(0, 1, 1)

(ii) A(5, 6, -3) and B (1, -6, 3)

(iii) A (-5, 7, -9) and B (-3, 4, -6)

Answer

Given two line segments , we have the direction ratios,

Of the line joining these 2 points as,

AB = $-\hat{\imath} + \hat{\jmath} + k$, (direction ratio)

The unit vector in this direction will be the direction cosines, i.e.,

Unit vector in this direction is:- $(-\hat{\imath} + \hat{\jmath} + k)/\sqrt{3}$

The direction cosines are $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

(ii) Given two line segments , we have the direction ratios,

Of the line joining these 2 points as,

AB = $-4\hat{\imath} + (-12)\hat{\jmath} + 6k$

The direction ratio in the simplest form will be, (2, 6, -3)

The unit vector in this direction will be the direction cosines, i.e.,

Unit vector in this direction is:- $(2\hat{\imath} + 6\hat{\jmath} - 3k)/\sqrt{2^2 + 6^2 + (-3)^2}$

The direction cosines are $(\frac{2}{7}, \frac{6}{7}, -\frac{3}{7})$

(iii) Given two line segments , we have the direction ratios,

Of the line joining these 2 points as,

AB = $2\hat{\imath} - 3\hat{\jmath} + 3k$, (direction ratio)

The unit vector in this direction will be the direction cosines, i.e.,

Unit vector in this direction is:- $(2\hat{\imath} - 3\hat{\jmath} + 3k)/\sqrt{2^2 + (-3)^2 + 3^2}$

The direction cosines are $(\frac{2}{\sqrt{22}}, -\frac{3}{\sqrt{22}}, \frac{3}{\sqrt{22}})$

3. Question

Show that the line joining the points A(1,-1, 2) and B(3, 4, -2) is perpendicular to the line joining the points C(0, 3, 2) and D (3, 5, 6).

Answer

Given: A(1, -1, 2) and B(3, 4, -2)

The line joining these two points is given by,

AB = 2i + 5j -4k

C(0, 3, 2) and D(3, 5, 6),

The line joining these two points,

CD = 3i + 2j +4k

To prove that the two lines are perpendicular we need to show that the angle between these two lines is $\frac{\pi}{2}$

So, AB.CD = 0 (dot product)

Thus, (2i + 5j -4k). (3i + 2j +4k) = 6 + 10 – 16 = 0.

Thus, the two lines are perpendicular.

4. Question

Show that the line segment joining the origin to the point A(2, 1, 1) is perpendicular to the line segment joining the points B(3, 5, -1) and C(4,3, -1).

Answer

Given: O(0, 0, 0) and A(2, 1, 1)

The line joining these two points is given by,

OA = 2i + j +k

B(3, 5, -1) and D(4, 3, -1),

The line joining these two points,

BC = i - 2j +0k

To prove that the two lines are perpendicular we need to show that the angle between these two lines is $\frac{\pi}{2}$

So, OA.BC = 0 (dot product)

Thus, (2i + j +k). (i - 2j +0k) = 2 - 2 + 0 = 0.

Thus, the two lines are perpendicular.

5. Question

Find the value of p for which the line through the points A(3, 5, -1) and B(5, p, 0) 9 is perpendicular to the line through the points C(2, 1, 1) and D (3, 3, -1).

Answer

Given: A(3, 5, -1) and B(5, p, 0)

The line joining these two points is given by,

AB = 2i + (p-5)j + k

C(2, 1, 1) and D(3, 3, -1),

The line joining these two points,

CD = i + 2j - 2k

As the two lines are perpendicular, we know that the angle between these two lines is $\frac{\pi}{2}$

So, AB.CD = 0 (dot product)

Thus, (2i + (p-5)j + k) .(i + 2j - 2k) = 0.

ð $2 + 2(p - 5) - 2 = 0$

ð p = 5

Thus, p = 5.

6. Question

If O is the origin and P (2, 3,4) and Q (1, -2, 1) be any two points show that OP$\perp$OQ.

Answer

Given O(0, 0, 0), P(2, 3, 4) So, OP = 2i + 3j + 4k

Q(1, -2, 1), So, OQ = i – 2j + k

To prove that OP$\perp$OQ we have,

OP.OQ = 0, i.e. the angle between the line segments is $\frac{\Pi}{2}$

So, the dot product i.e. |OP||OQ|cosθ = 0,cosθ = 0,

OP.OQ = 0

Thus, (2i + 3j + 4k).(i – 2j + k) = 2 – 6 + 4 = 0

Hence, proved.

7. Question

Show that the line segment joining the points A(1, 2, 3) and B (4, 5, 7) is parallel to the segment joining the points C(-4, 3, -6) and D (2, 9, 2).

Answer

Given A(1, 2, 3), B(4, 5, 7), the line joining these two points will be

AB = 3i + 3j + 4k

And the line segment joining, C(-4, 3, -6) and D(2, 9, 2) will be

CD = 6i + 6j + 8k

If CD = r(AB), where r is a scalar constant then,

The two lines are parallel.

Here, CD = 2(AB),

Thus, the two lines are parallel.

8. Question

If the line segment joining the points A(7, p, 2) and B(q, -2, 5) be parallel to the line segment joining the points C(2, -3, 5) and D(-6, -15,11), find the values of p and q.

Answer

Given: A(7, p, 2) and B(q, -2, 5), line segment joining these two points will be, AB = (q-7)i + (-2-p)j + 3k

And the line segment joining C(2, -3, 5) and D(-6, -15, 11) will be, CD = -8i – 12j + 6k

Then, the angle between these two line segments will be 0 degree. So, the cross product will be 0.

AB$\times$CD = 0

ð ((q-7)i + (-2-p)j + 3k)×(-8i – 12j + 6k) = 0

Thus, solving this we get,

p = 4 and q = 3

9. Question

Show that the points A(2, 3, 4), B(-1, -2, 1) and C (5, 8, 7) are collinear.

Answer

We have to show that the three points are colinear , i.e. they all lie on the same line,

If we define a line which is having a parallel line to AB and the points A and B lie on it, if point C also satisfies the line then, the three points are colinear,

Given A(2, 3, 4) and B(-1, -2, 1), AB = -3i – 5j -3k

The points on the line AB with A on the line can be written as,

R = (2, 3, 4) +a(-3, -5, -3)

Let C = (2-3a, 3-5a, 4-3a)

ð (5, 8, 7) = (2-3a, 3-5a, 4-3a)

ð If a = -1, then L.H.S = R.H.S, thus

The point C lies on the line joining AB,

Hence, the three points are colinear.

10. Question

Show that the points A(-2, 4.7), B(3, -6. -8) and C(1, -2, -2) are collinear.

Answer

We have to show that the three points are colinear , i.e. they all lie on the same line,

If we define a line which is having a parallel line to AB and the points A and B lie on it, if point C also satisfies the line then, the three points are colinear,

Given A(-2, 4, 7) and B(3, -6, -8), AB = 5i – 10j -15k

The points on the line AB with A on the line can be written as,

R = (-2, 4, 7) +a(5, -10, -15)

Let C = (-2+5a, 4-10a, 7-15a)

ð (1, -2, -2) = (-2+5a, 4-10a, 7-15a)

ð If a = 3/5, then L.H.S = R.H.S, thus

The point C lies on the line joining AB,

Hence, the three points are colinear.

11. Question

Find the value of p for which the points A(-1, 3, 2), B(-4, 2, -2), and C(5, 5, p) are collinear.

Answer

We have to show that the three points are colinear , i.e. they all lie on the same line,

If we define a line which is having a parallel line to AB and the points A and B lie on it, as the points are colinear so C must satisfy the line,

Given A(-1, 3, 2) and B(-4, 2, -2), AB = -3i – j -4k

The points on the line AB with A on the line can be written as, R = (-1, 3, 2) +a(-3, -1, -4)

Let C = (-1-3a, 3-1a, 2-4a)

ð (5, 5, p) = (-1-3a, 3-1a, 2-4a)

ð As L.H.S = R.H.S, thus

ð 5 = -1 – 3a, a = -2

Substituting a = -2 we get, p = 2-4(-2) = 10

Hence, p = 10.

12. Question

Find the angle between the two lines whose direction cosines are:

$\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$ and $\frac{3}{7}, \frac{2}{7}, \frac{6}{7}$

Answer

Let

$R_1 = \frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k$

And $R_2 = \frac{3}{7}i + \frac{2}{7}j + \frac{6}{7}k$

$R_1.R_2 = |R_1||R_2|\cos\theta$

Here, as R1 and R2 are the unit vectors with a direction given by the direction cosines hence, |R1| and |R2| are 1.

So, $\cos\theta = R_1.R_2 / 1$

ð $\cos\theta = \frac{6}{21} - \frac{2}{21} - \frac{12}{21} = \frac{8}{21}$

ð $\theta = \cos^{-1} - \frac{8}{21}$

The angle between the lines is $\cos^{-1} - \frac{8}{21}$

13. Question

Find the angle between the two lines whose direction ratios are:

a, b, c and (b – c), (c – a), (a – b).

Answer

The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where R_1 an R_2 denote the vectors with the direction ratios,

So, here we have,

R_1 = ai + bj + ck and R_2 = (b-c)i +(c-a)j + (a-b)k

$\cos\theta = \frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}} = 0$

$\cos\theta = 0$

Hence, $\theta = \frac{\pi}{2}$

14. Question

Find the angle between the lines whose direction ratios are:

2, -3, 4 and 1, 2, 1.

Answer

The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where R_1 and R_2 denote the vectors with the direction ratios,

So, here we have,

R_1 = 2i - 3j + 4k and R_2 = i +2j + k

$\cos\theta = \frac{2-6+4}{\sqrt{2^2+(-3)^2+4^2}\sqrt{1^2+2^2+1^2}} = 0$

$\cos\theta = 0$

Hence, $\theta = \frac{\pi}{2}$

15. Question

Find the angle between the lines whose direction ratios are:

1, 1, 2 and $(\sqrt{3-1}),(-\sqrt{3-1})$,4

Answer

The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where R_1 and R_2 denote the vectors with the direction ratios,

So, here we have,

R1 = i + j + 2k and R2 = $(\sqrt{3}-1)$i -$(\sqrt{3}+1)$j + (4)k

$\cos\theta = \frac{\sqrt{3}-1-\sqrt{3}-1+8}{\sqrt{1^2+1^2+2^2}\sqrt{(\sqrt{3}-1)^2+(-(\sqrt{3}+1))^2+4^2}} = \frac{6}{\sqrt{6}.\sqrt{24}}$

$\cos\theta = \frac{1}{2}$

Hence, $\theta = \frac{\pi}{3}$

16. Question

Find the angle between the vectors $\vec{r_1} = (3\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r_2} = (4\hat{i} + 5\hat{j} + 7\hat{k})$

Answer

The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where R_1 and R_2 denote the vectors with the direction ratios,

So, here we have,

R1 = 3i - 2j + k and R2 = 4i + 5j + 7k

$\cos\theta = \frac{12-10+7}{\sqrt{3^2+(-2)^2+1^2}\sqrt{4^2+5^2+7^2}} = \frac{9}{\sqrt{14}.\sqrt{90}}$

$\cos\theta = \frac{3}{2\sqrt{35}}$

Hence, $\theta = \cos^{-1}\frac{3}{2\sqrt{35}}$

17. Question

Find the angles made by the following vectors with the coordinate axes:

(i) $(\hat{i} - \hat{j} + \hat{k})$

(ii) $(\hat{j} - \hat{k})$

(iii) $(\hat{i} - 4\hat{j} + 8\hat{k})$

Answer

(i) The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where R_1 and R_2 denote the vectors with the direction ratios,

So, here we have,

R1 = i - j + k and R2 = i for x- axis

$\cos\theta = \frac{1-0+0}{\sqrt{1^2+(-1)^2+1^2}\sqrt{1^2}} = \frac{1}{\sqrt{3}}$

$\cos\theta = \frac{1}{\sqrt{3}}$

Hence, $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$

With y- axis, i. e. R2 = j

$\cos\theta = \frac{0-1+0}{\sqrt{1^2+(-1)^2+1^2}\sqrt{1^2}} = -\frac{1}{\sqrt{3}}$

$\cos\theta = -\frac{1}{\sqrt{3}}$

Hence, $\theta = \cos^{-1}(-\frac{1}{\sqrt{3}})$

With z- axis, i. e. R2 = k

$\cos\theta = \frac{0-0+1}{\sqrt{1^2+(-1)^2+1^2}\sqrt{1^2}} = \frac{1}{\sqrt{3}}$

$\cos\theta = \frac{1}{\sqrt{3}}$

Hence, $\theta = \cos^{-1}(\frac{1}{\sqrt{3}})$

(ii) The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where R_1 and R_2 denote the vectors with the direction ratios,

So, here we have,

R1 = j - k and R2 = i for x- axis

$\cos\theta = \frac{0-0+0}{\sqrt{0^2+1^2+(-1)^2}\sqrt{1^2}} = 0$

$\cos\theta = 0$

Hence, $\theta = \frac{\pi}{2}$

With y- axis, i. e. R2 = j

$\cos\theta = \frac{0+1+0}{\sqrt{0^2+1^2+(-1)^2}\sqrt{1^2}} = \frac{1}{\sqrt{2}}$

$\cos\theta = \frac{1}{\sqrt{2}}$

Hence, $\theta = \frac{\pi}{4}$

With z- axis, i. e. R2 = k

$\cos\theta = \frac{0+0-1}{\sqrt{0^2+1^2+-(1)^2}\sqrt{1^2}} = -\frac{1}{\sqrt{2}}$

$\cos\theta = -\frac{1}{\sqrt{2}}$

Hence, $\theta = \frac{3\pi}{4}$

(iii) The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where R_1 and R_2 denote the vectors with the direction ratios,

So, here we have,

R1 = i - 4j + 8k and R2 = i for x- axis

$\cos\theta = \frac{1-0+0}{\sqrt{1^2+(-4)^2+8^2}\sqrt{1^2}} = \frac{1}{\sqrt{81}}$

$\cos\theta = \frac{1}{9}$

Hence, $\theta = \cos^{-1}\frac{1}{9}$

With y- axis, i. e. R2 = j

$\cos\theta = \frac{0-4+0}{\sqrt{1^2+(-4)^2+8^2}\sqrt{1^2}} = -\frac{4}{9}$

$\cos\theta = -\frac{1}{9}$

Hence, $\theta = \cos^{-1}(-\frac{1}{9})$

With z- axis, i. e. R2 = k

$\cos\theta = \frac{0-0+8}{\sqrt{1^2+(-4)^2+8^2}\sqrt{1^2}} = \frac{8}{9}$

$\cos\theta = \frac{8}{9}$

Hence, $\theta = \cos^{-1}(\frac{8}{9})$

18. Question

Find the coordinates of the foot of the perpendicular drawn from the point A(1, 8, 4) to the line joining the points B(0, -1, 3) and C(2, -3, -1).

Answer

Given: A(1, 8, 4)

Line segment joining B(0, -1, 3) and C(2, -3, -1) is

BC = 2i – 2j – 4k

Let the foot of the perpendicular be R then,

As R lies on the line having point B and parallel to BC,

So, R = (0, -1, 3) + a(2, -2, -4)

R(2a, -1-2a, 3-4a)

The line segment AR is

AR = (2a-1)i + (-1-2a-8)j + (3-4x-4)k

As the lines AR and BC are perpendicular thus, (as R is the foot of the perpendicular on BC)

AR.BC = 0

ð 2(2a-1) + (-2)(-9-2a) + (-4)(-1-4a) = 0

ð 24a + 20 = 0

ð a = $-\frac{5}{6}$

Substituting a in R we get,

R($-\frac{5}{3}, \frac{2}{3}, \frac{19}{3}$)