17. Area of Bounded Regions

Exercise 17

1. Question

Find the area of the region bounded by the curve $y=x^2$, the x-axis, and the lines x=1 and x=3.

Answer

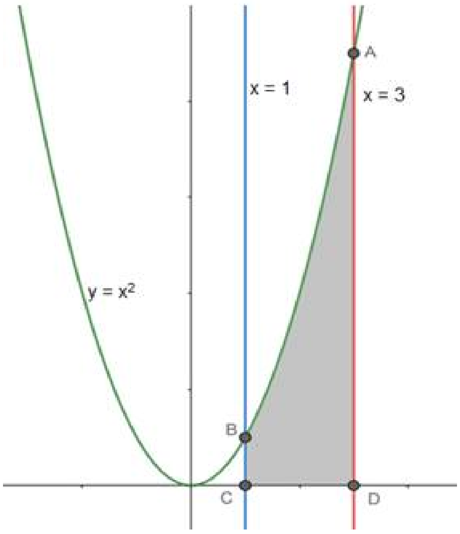
Given the boundaries of the area to be found are,

• The curve $y = x^2$

• The x-axis

• x = 1 (a line parallel toy-axis)

• x = 3 (a line parallel toy-axis)



As per the given boundaries,

• The curve $y = x^2$, has only the positive numbers as x has even power, so it is about the y-axis equally distributed on both sides.

• x= 1 and x=3 are parallel toy-axis at of 1 and 3 units respectively from the y-axis.

• The four boundaries of the region to be found are,

•Point A, where the curve $y = x^2$ and x=3 meet

•Point B, where the curve $y = x^2$ and x=1 meet

•Point C, where the x-axis and x=1 meet i.e. C(1,0).

•Point D, where the x-axis and x=3 meet i.e. D(3,0).

Area of the required region = Area of ABCD.

$$\text{Area of ABCD} = \int_1^3 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_1^3 = \left(\frac{3^3}{3} - \frac{1^3}{3}\right)$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \left(\frac{27}{3} - \frac{1}{3}\right) = \frac{26}{3}$$

The Area of the required region $= \frac{26}{3}$ sq. units

2. Question

Find the area of the region bounded by the parabola $y^2=4x$, the x-axis, and the lines x=1 and x=4.

Answer

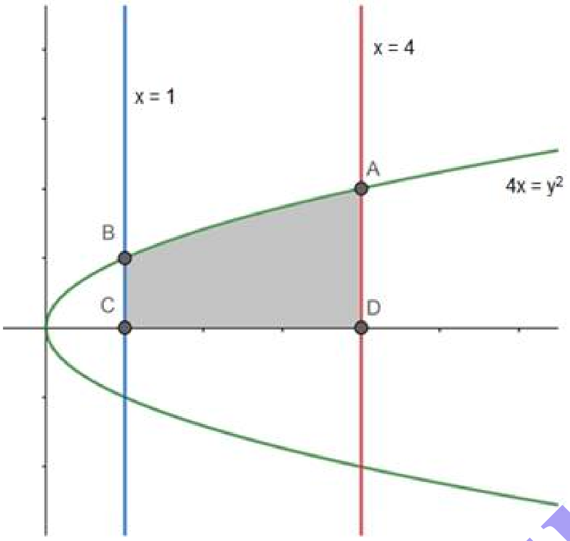
Given the boundaries of the area to be found are,

• The parabola $y^2 = 4x$

• The x-axis

• x = 1 (a line parallel toy-axis)

• x = 4 (a line parallel toy-axis)



As per the given boundaries,

• The curve y^2 =4x, has only the positive numbers as y has even power, so it is about the x-axis equally distributed on both sides.

• x= 1 and x=4 are parallel toy-axis at of 1 and 4 units respectively from the y-axis.

• The four boundaries of the region to be found are,

•Point A, where the curve $y^2 = 4x$ and x=4 meet

•Point B, where the curve $y^2 = 4x$ and x=1 meet

•Point C, where the x-axis and x=1 meet i.e. C(1,0).

•Point D, where the x-axis and x=4 meet i.e. D(4,0).

Area of the required region = Area of ABCD.

$$\text{Area of ABCD} = \int_1^4 y\,dx = \int_1^4 \sqrt{4x}\,dx$$

$$= 2\int_1^4 \sqrt{x}\,dx = 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = 2\left[\frac{2\,x^{\frac{3}{2}}}{3}\right]_1^4$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= 2\left(\frac{2\,(4)^{\frac{3}{2}}}{3} - \frac{2\,(1)^{\frac{3}{2}}}{3}\right) = 4\left(\frac{8}{3} - \frac{1}{3}\right) = 4\left(\frac{7}{3}\right)$$

$$= \frac{28}{3}$$

The Area of the required region $= \frac{28}{3}$ sq. units

3. Question

Find the area under the curve y=$\sqrt{6x+4}$ (above the x-axis) from x=0 to x=2

Answer

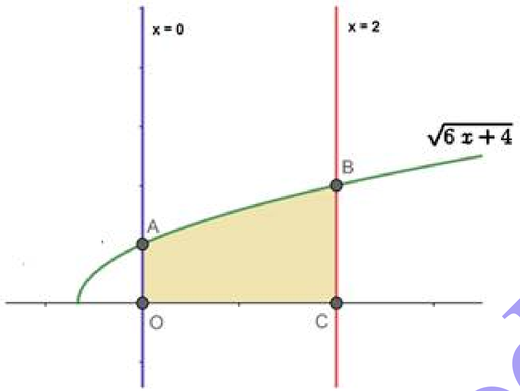
Given the boundaries of the area to be found are,

• The curve $y = \sqrt{6x+4}$

• The x-axis

• x = 0 (y-axis)

• x = 4 (a line parallel toy-axis)



As per the given boundaries,

• The curve $y = \sqrt{6x+4}$,is a curve with vertex at $(0, -\frac{2}{3})$.

• x=2 is parallel toy-axis at 2 units away from the y-axis.

• x=0 is the y-axis.

• The four boundaries of the region to be found are,

•Point A, where the curve $y^2 = 6x + 4$ and x=0 meet.

•Point B, where the curve $y^2 = 6x + 4$ and x=2 meet.

•Point C, where the x-axis and x=2 meet i.e. C(2,0).

•Point O, or the origin i.e. O(0,0).

Area of the required region = Area of OABC.

$$\text{Area of OBCD} = \int_0^2 y\,dx = \int_0^2 \sqrt{6x+4}\,dx$$

$$= \int_{0}^{2} \sqrt{6x+4}\,dx = \left[\frac{(6x+4)^{\frac{3}{2}}}{\frac{3}{2}(6)}\right]_{0}^{2} = \frac{1}{9}\left[(6x+4)^{\frac{3}{2}}\right]_{0}^{2}$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \frac{1}{9}\left(((6\times 2)+4)^{\frac{3}{2}} - ((6\times 0)+4)^{\frac{3}{2}}\right) = \frac{1}{9}(64-8) = \frac{1}{9}(56)$$

$$= \frac{56}{9}$$

The Area of the required region $= \frac{56}{9}$ sq. units

4. Question

Determine the area enclosed by curve y=x^3, and the lines y=0, x=2 and x=4.

Answer

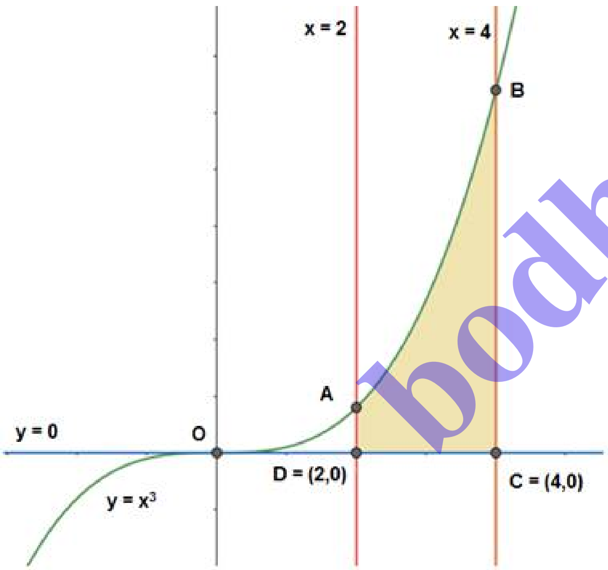
Given the boundaries of the area to be found are,

• The curve y = x^3

• The y= 0, x-axis

• x = 2 (a line parallel toy-axis)

• x = 4 (a line parallel toy-axis)



As per the given boundaries,

• The curve y = x^3 is a curve with vertex at (0,0).

• x=2 is parallel toy-axis at 2 units away from the y-axis.

• x=4 is parallel toy-axis at 4 units away from the y-axis.

• The four boundaries of the region to be found are,

•Point A, where the curve y = x^3 and x=2 meet.

•Point B, where the curve y = x^3 and x=4 meet.

•Point C, where the x-axis and x=4 meet i.e. C(4,0).

•Point D, where the x-axis and x=2 meet i.e. D(2,0).

Area of the required region = Area of ABCD.

$$\text{Area of ABCD} = \int_2^4 y\,dx = \int_2^4 x^3\,dx$$

$$= \int_2^4 x^3\,dx = \left[\frac{x^4}{4}\right]_2^4 = \frac{1}{4}[(x)^4]_2^4$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \frac{1}{4}(4^4 - 2^4) = \frac{1}{4}(256 - 16) = \frac{1}{4}(240)$$

= 60 sq. units

The Area of the required region = 60 sq. units.

5. Question

Determine the area under the curve y= $\sqrt{a^2 - x^2}$, included between the lines x=0 and x=4.

Answer

Given the boundaries of the area to be found are,

• The curve $y = \sqrt{a^2 - x^2}$

• x = 0 (y-axis)

• x = 4 (a line parallel toy-axis)

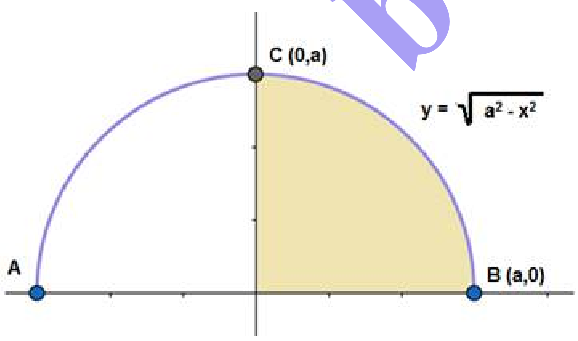
Here the curve, $y = \sqrt{a^2 - x^2}$, can be re-written as

$y^2 = a^2 - x^2$

$x^2 + y^2 = a^2$ ----- (1)

This equation (1) represents a circle equation with (0,0) as center and, a units as radius.

As x and y have even powers, the given curve will be about the x-axis and y-axis.



As per the given boundaries,

• The curve $y = \sqrt{a^2 - x^2}$, is a curve with vertex at (0,0).

• x=4 is parallel toy-axis at 4 units away from the y-axis. (but this might not really effect the boundaries as the value of ‘a’ in the equation is unknown.)

• x=0 is the y-axis.

Area of the required region = Area of OBC.

$$\text{Area of OBC} = \int_0^a y\,dx = \int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx = \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

[Using the formula, $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$]

$$= \left[\frac{a\sqrt{a^2 - a^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right] - \left[\frac{0\sqrt{a^2 - 0^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{0}{a}\right)\right]$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) - (0 + 0) = \frac{\pi a^2}{4}$$

[$\sin^{-1}(1) = 90°$ and $\sin^{-1}(0) = 0°$]

The Area of the required region $= \frac{\pi a^2}{4}$ sq. units

6. Question

Using integration, find the area of the region bounded by the lines 2y=5x+7, the x-axis and the lines 2y=5x+7, the x-axis, and the lines x=2 and x=8.

Answer

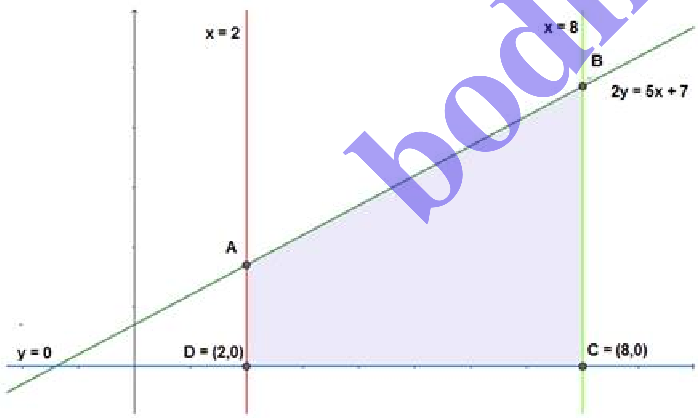
Given the boundaries of the area to be found are,

• The line equation is 2y = 5x + 7

• The y= 0, x-axis

• x = 2 (a line parallel toy-axis)

• x = 8 (a line parallel toy-axis)



As per the given boundaries,

• The line 2y = 5x + 7.

• x=2 is parallel toy-axis at 2 units away from the y-axis.

• x=8 is parallel toy-axis at 8 units away from the y-axis.

• y = 0, the x-axis.

• The four boundaries of the region to be found are,

•Point A, where the line 2y = 5x + 7 and x=2 meet.

•Point B, where the line 2y = 5x + 7and x=8 meet.

•Point C, where the x-axis and x=8 meet i.e. C(8,0).

•Point D, where the x-axis and x=2 meet i.e. D(2,0).

The line equation 2y = 5x + 7 can be written as,

$$y = \frac{5x + 7}{2}$$

Area of the required region = Area of ABCD.

$$\text{Area of ABCD} = \int_2^8 y\,dx = \int_2^8 \frac{5x+7}{2}\,dx$$

$$= \frac{1}{2}\int_2^8 (5x+7)\,dx = \frac{1}{2}\left[5\left(\frac{x^2}{2}\right) + 7x\right]_2^8$$

$$= \frac{1}{2}\left\{\left[5\left(\frac{8^2}{2}\right) + 7\,(8)\right] - \left[5\left(\frac{2^2}{2}\right) + 7\,(2)\right]\right\}$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

$$= \frac{1}{2}\left\{\left[5\left(\frac{64}{2}\right) + 56\right] - [10 + 14]\right\} = \frac{1}{2}\,[(5 \times 32) + 56 - 24] = \frac{1}{2}\,(160 + 32)$$

$$= \frac{1}{2}\,(192) = 96$$

The Area of the required region = 96 sq. units.

7. Question

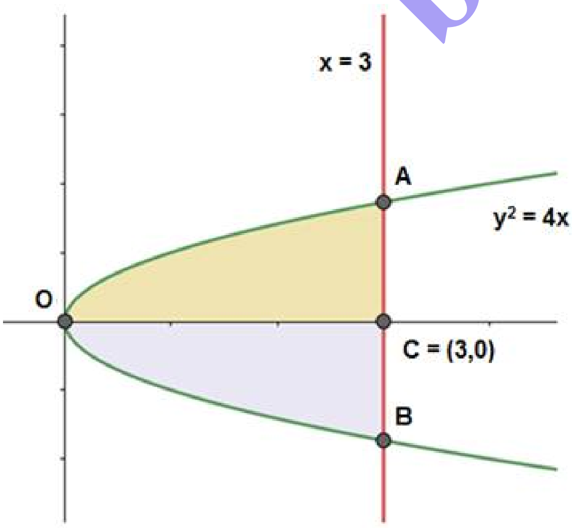
Find the area of the region bounded by the curve y^2=4x and the lines x=3.

Answer

Given the boundaries of the area to be found are,

• The parabola $y^2 = 4x$

• x = 3 (a line parallel toy-axis)



As per the given boundaries,

• The curve y^2 =4x with vertex at (0,0), has only the positive numbers as y has even power, so it is about the x-axis equally distributed on both sides.

• x= 3 are parallel toy-axis at 3 units from the y-axis.

• The boundaries of the region to be found are,

•Point A, where the curve $y^2 = 4x$ and x=3 meet when y is positive.

•Point B, where the curve $y^2 = 4x$ and x=3 meet when y is negative.

•Point C, where the x-axis and x=3 meet i.e. C(3,0).

•Point O, the origin.

Area of the required region = Area of OAB

Area of OAB = Area of OAC + Area of OBC.

[area under OAC = area under OBC as the curve $y^2 = 4x$ is symmetric]

Area of OAB = 2 × Area of OAC

$$\text{Area of OAB} = 2\int_0^3 y\,dx = 2\int_0^3 \sqrt{4x}\,dx$$

$$= 4\int_0^3 \sqrt{x}\,dx = 4\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3 = 4\left[\frac{2\,x^{\frac{3}{2}}}{3}\right]_0^3$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= 4\left(\frac{2\,(3)^{\frac{3}{2}}}{3} - \frac{2\,(0)^{\frac{3}{2}}}{3}\right) = \frac{8}{3}\left(3\sqrt{3}\right) = 8\sqrt{3}$$

The Area of the required region $= 8\sqrt{3}$ sq. units

8. Question

Evaluate the area bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ above the x-axis.

Answer

Given the boundaries of the area to be found are,

• The ellipse, $\frac{x^2}{4} + \frac{y^2}{9} = 1$

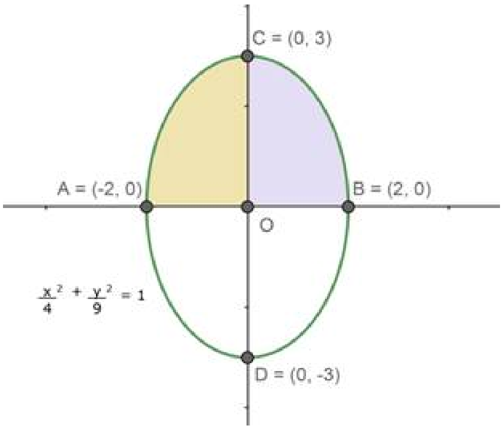
• y = 0 (x-axis)

From the equation, of the ellipse

• the vertex at (0,0) i.e. the origin,

• the minor axis is the x-axis and the ellipse intersects the x- axis at A(-2,0) and B(2,0).

• the major axis is the y-axis and the ellipse intersects the y- axis at C(3,0) and D(-3,0).



As x and y have even powers, the area of the ellipse will be symmetrical about the x-axis and y-axis.

Here the ellipse, $\frac{x^2}{4} + \frac{y^2}{9} = 1$, can be re-written as

$$y^2 = 9\left(1 - \frac{x^2}{4}\right)$$

• $y = \sqrt{\frac{9}{4}(4 - x^2)}$

• $y = \frac{3}{2}\sqrt{4 - x^2}$----- (1)

As given, the boundaries of the re to be found will be

• The ellipse, $y = \frac{3}{2}\sqrt{4 - x^2}$ with vertex at (0,0).

• The x-axis.

Now, the area to be found will be the area under the ellipse which is above the x-axis.

Area of the required region = Area of ABC.

Area of ABC = Area of AOC + Area of BOC

[area of AOC = area of BOC as the ellipse is symmetrical about the y-axis]

Area of ABC = 2 Area of BOC

$$\text{Area of ABC} = 2\int_0^2 y\,dx = 2\int_0^2 \frac{3}{2}\sqrt{4 - x^2}\,dx$$

$$= 3\int_0^2 \sqrt{(2)^2 - x^2}\,dx = 3\left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

[Using the formula, $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$]

$$= 3\left[\frac{2\sqrt{4 - 2^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right)\right] - 3\left[\frac{0\sqrt{a^2 - 0^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{0}{2}\right)\right]$$

$$= 3 \times 2\left(\frac{\pi}{2}\right) - 3(0 + 0) = 3\pi$$

[$\sin^{-1}(1)$ = 90° and $\sin^{-1}(0)$ = 0°]

The Area of the required region 3π sq. units

9. Question

Using integration, find the area of the region bounded by the lines y=1|x+1|, x=-2, x=3 and y=0.

Answer

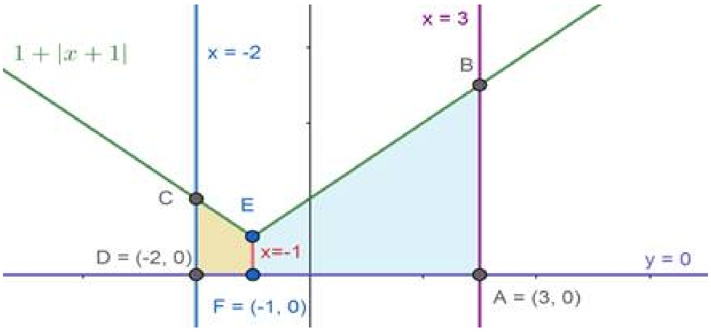
Given the boundaries of the area to be found are,

• The line equation is y = 1 + |x+1|

• The y= 0, x-axis

• x = -2 (a line parallel toy-axis)

• x = 3 (a line parallel toy-axis)



Consider the given line is

y = 1 + |x+1|

this can be written as

y = 1 +(x+1), when x+1≥0 (or) y = 1 –(x+1), when x+1<0

y = x+2, when x≥ -1 (or) y = y = -x, when x<-1 ----(1)

Thus the given boundaries are,

• The line y = 1 +|x+1|.

• x=-2 is parallel toy-axis at -2 units away from the y-axis.

• x=3 is parallel toy-axis at 3 units away from the y-axis.

• y = 0, the x-axis.

The four vertices of the region are,

•Point A, where the x-axis and x=3 meet i.e. A(3,0).

•Point B, where the line y = 1 +|x+1| and x=3 meet.

•Point C, where the line y = 1 +|x+1| and x=-2 meet.

•Point D, where the x-axis and x=-2 meet i.e. D(-2,0).

Area of the required region = Area of ABCD.

From (1) we can clearly say that, the area of ABCD has to be divided into twopieces i.e. area under CDFE and EFAB as the line equations changes at x = -1.

$$\text{Area of ABCD} = \int_{-2}^{-1} \mathrm{y\,dx} + \int_{-1}^{3} \mathrm{y\,dx} = \int_{-2}^{-1} (-\mathrm{x})\,\mathrm{dx} + \int_{-1}^{3} (\mathrm{x}+2)\,\mathrm{dx}$$

$$= -\int_{-2}^{-1} (\mathrm{x})\,\mathrm{dx} + \int_{-1}^{3} (\mathrm{x}+2)\,\mathrm{dx} =$$

$$= -\left[\frac{x^2}{2}\right]_{-2}^{-1} + \left[\left(\frac{x^2}{2}\right) + 2x\right]_{-1}^{3}$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

$$= -\left[\frac{(-1)^2}{2} - \frac{(-2)^2}{2}\right] + \left\{\left[\left(\frac{3^2}{2}\right) + 2(3)\right] - \left[\left(\frac{(-1)^2}{2}\right) + 2(-1)\right]\right\}$$

$$= -\left[\frac{1-4}{2}\right] + \left\{\left[\frac{9+12}{2}\right] - \left[\frac{1-4}{2}\right]\right\} = \frac{3}{2} + \left(\frac{21}{2} + \frac{3}{2}\right) = \frac{27}{2}$$

The Area of the required region $= \frac{27}{2}$ sq. units.

10. Question

Find the area bounded by the curve y=(4-x^2), the y-axis and the lines y=0,y=3.

Answer

Given the boundaries of the area to be found are,

• The curve y = 4-x^2

• The y-axis

• y = 0 (x - axis)

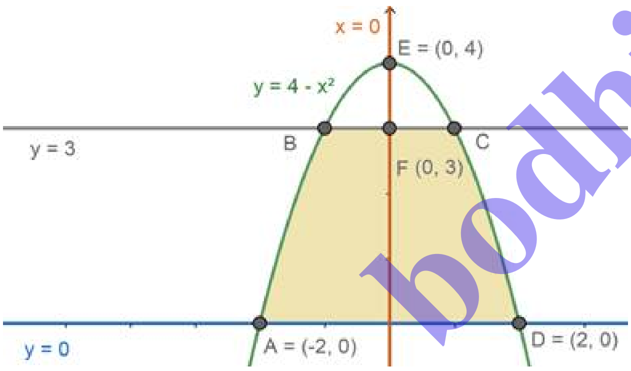
• y = 3 (a line parallel to x-axis)

Consider the curve,

y = 4-x^2

x^2 = 4-y

$x = \sqrt{4-y}$ ---- (1)



About the area to be found,

• The curve y = 4 - x^2, has only the positive numbers as x has even power, so it is about the y-axis equally distributed on both sides.

• From (1) as, $x = \sqrt{4-y}$, the curve has its vertex at (0,4) and cannot g•beyond y = 4 as the value of x cannot be negative and imaginary.

• y= 0 is the x – axis

• y =3 is parallel to x-axis which is 3 units away from the x-axis.

The four boundaries of the region to be found are,

•Point A, where the x-axis and $x = \sqrt{4-y}$ meet i.e.

C(-2,0).

•Point B, where the curve $x = \sqrt{4-y}$ and y=3 meet where x is negative.

•Point C, where the curve $x = \sqrt{4-y}$ and y=3 meet where x is positive.

•Point D, where the x-axis and $x = \sqrt{4-y}$ meet i.e. D(2,0).

Area of the required region = Area of ABCD.

$$\text{Area of ABCD} = \int_0^3 x\,\text{dy} = \int_0^3 \sqrt{4-y}\,\text{dy}$$

$$= \left[\frac{(4-y)^{\frac{3}{2}}}{\frac{3}{2}(-1)}\right]_1^3 = -\frac{2}{3}\left[(4-y)^{\frac{3}{2}}\right]_0^3$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= -\frac{2}{3}\left[(4-3)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}}\right] = -\frac{2}{3}\left[1-(2^2)^{\frac{3}{2}}\right] = -\frac{2}{3}(1-8)$$

$$= -\frac{2}{3}(-7) = \frac{14}{3}$$

The Area of the required region $= \frac{14}{3}$ sq. units

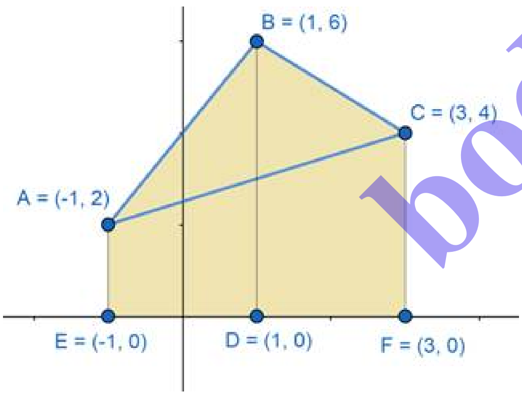
11. Question

Using integration, find the area of the region bounded by the triangle whose vertices are A(-1, 2), B(1,6) and C (3,4)

Answer

Given,

• A (-1,2), B (1,6) and C (3,4) are the 3 vertices of a triangle.



From above figure we can clearly say that, the area between ABC and DEF is the area to be found.

For finding this area, we can consider the lines AB, BC and CA which are the sides of the given triangle. By calculating the area under these lines we can find the area of the complete region.

Consider the line AB,

If (x_1,y_1) and (x_2, y_2) are two points, the equation of a line passing through these points can be given by

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

Using this formula, equation of the line A(-1,2) B =(1,5)

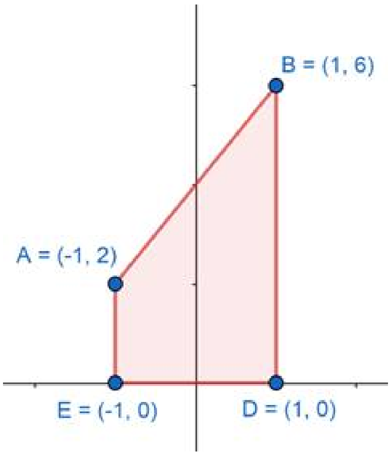
$$\frac{y-(2)}{5-2} = \frac{x-(-1)}{1-(-1)}$$

$$\frac{y-(2)}{3} = \frac{x+1}{2}$$

$$y = \frac{3}{2}(x+1) + 2 = \frac{3x+3+4}{2} = \frac{3x+7}{2}$$

$$y = \frac{3x+7}{2}$$

Consider the area under AB:



From the above figure, the area under the line AB will be given by,

$$Area\ of\ ABED = \int_{-1}^{1} y\,dx = \int_{-1}^{1}\left(\frac{3x+7}{2}\right) dx$$

$$= \int_{-1}^{1} \frac{1}{2}(3x+7)\,dx = \frac{1}{2}\left[\frac{3x^2}{2} + 7x\right]_{-1}^{1}$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

$$= \frac{1}{2}\left\{\left[\frac{3(1)^2}{2} + 7(1)\right] - \left[\frac{3(-1)^2}{2} + 7(-1)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{3}{2} + 7\right] - \left[\frac{3}{2} - 7\right]\right\} = \frac{1}{2}(14)$$

= 7

Area of ABDE = 7 sq. units. ---- (1)

Consider the line BC,

Using this 2-point formula for line, equation of the line B(1,5) and C (3,4)

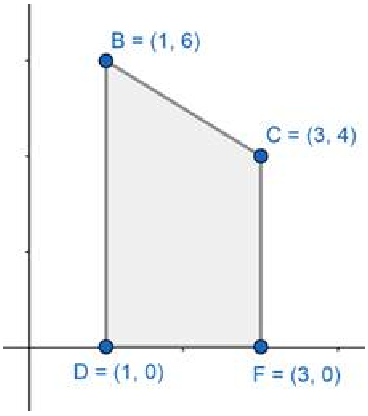
$$\frac{y-(5)}{4-5} = \frac{x-(1)}{3-(1)}$$

$$\frac{y-(5)}{-1} = \frac{x-1}{2}$$

$$y = \frac{1}{2}(1-x) + 5 = \frac{1-x+10}{2} = \frac{11-x}{2}$$

$$y = \frac{11-x}{2}$$

Consider the area under BC:



From the above figure, the area under the line BC will be given by,

$$Area\ of\ BCDF = \int_1^3 y\, dx = \int_1^3 \left(\frac{11-x}{2}\right) dx$$

$$= \int_1^3 \frac{1}{2}(11-x)\, dx = \frac{1}{2}\left[11x - \frac{x^2}{2}\right]_1^3$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\, dx = cx$]

$$= \frac{1}{2}\left\{\left[11(3) - \frac{(3)^2}{2}\right] - \left[11(1) - \frac{(1)^2}{2}\right]\right\}$$

$$= \frac{1}{2}\left\{\left[33 - \frac{9}{2}\right] - \left[11 - \frac{1}{2}\right]\right\} = \frac{1}{2}\left[\left(\frac{57}{2}\right) - \left(\frac{21}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{57-21}{2}\right] = \frac{36}{4} = 9$$

Area of BCFD = 9 sq. units. ---- (2)

Consider the line CA,

Using this 2-point formula for line, equation of the line C(3,4) and A(-1,2)

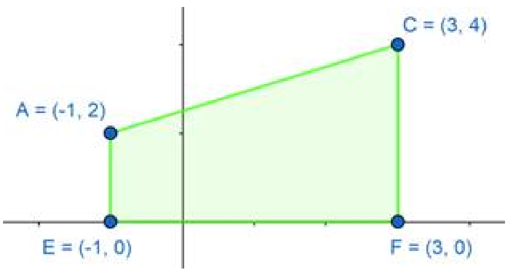
$$\frac{y-(4)}{2-4} = \frac{x-(3)}{-1-(3)}$$

$$\frac{y-(4)}{-2} = \frac{x-3}{-4}$$

$$y = \frac{1}{2}(x-3) + 4 = \frac{x-3+8}{2} = \frac{x+5}{2}$$

$$y = \frac{x+5}{2}$$

Consider the area under CA:



From the above figure, the area under the line CA will be given by,

$$Area\ of\ ACFE = \int_{-1}^{3} y\,dx = \int_{-1}^{3}\left(\frac{x+5}{2}\right)dx$$

$$= \int_{-1}^{3}\frac{1}{2}(x+5)\,dx = \frac{1}{2}\left[\left(\frac{x^2}{2}\right) + 5x\right]_{-1}^{3}$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

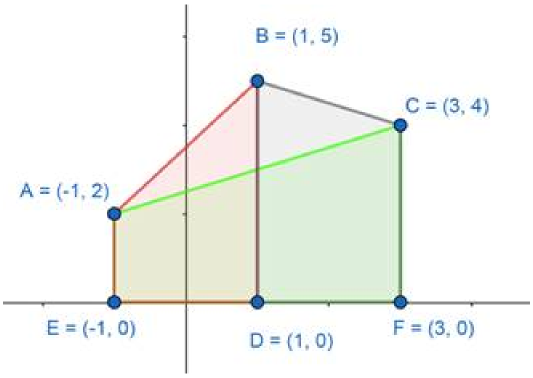
$$= \frac{1}{2}\left\{\left[\frac{(3)^2}{2} + 5(3)\right] - \left[\frac{(-1)^2}{2} + 5(-1)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{9}{2} + 15\right] - \left[\frac{1}{2} - 5\right]\right\} = \frac{1}{2}\left[\left(\frac{39}{2}\right) - \left(-\frac{9}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{39-9}{2}\right] = \frac{48}{4} = 12$$

Area of ACFE = 12 sq.units ---- (3)

If we combined, the areas under AB, BC and AC in the below graph, we can clearly say that the area under AC (3) is overlapping the previous twoareas under AB & BC.



Now, the combined area under the rABC is given by

Area under rABC

=Area under AB + Area under BC - Area under AC

From (1), (2) and (3), we get

Area under rABC = 7 + 9 -12

= 16 – 12 = 4 sq. units.

Therefore, area under rABC = 4 sq.units.

12. Question

Using integration, find the area of ΔABC, the equation of whose sides AB,BC and AC are given by

Y=4x+5,x+y=5 and 4y=x+5 respectively.

Answer

Given,

• ABC is a triangle

• Equation of side AB of y = 4x + 5

• Equation of side BC of x + y = 5

• Equation of side CA of 4y = x+5

By solving AB & BC we get the point B,

AB : y = 4x+5 , BC: y = 5-x

4x + 5 = 5-x

5x = 0

x = 0

by substituting x = 0 in AB we get y = 5

The point B = (0,5)

By solving BC & CA we get the point C,

AC : 4y = x+5 , BC: y = 5-x

4y - 5 = 5-y

5y = 10

y = 2

by substituting y = 2 in BC we get x = 3

The point C = (3,2)

By solving AB & AC we get the point A,

AB : y = 4x+5 , AC : 4y = x+5

16x + 20 = x+5

15x = -15

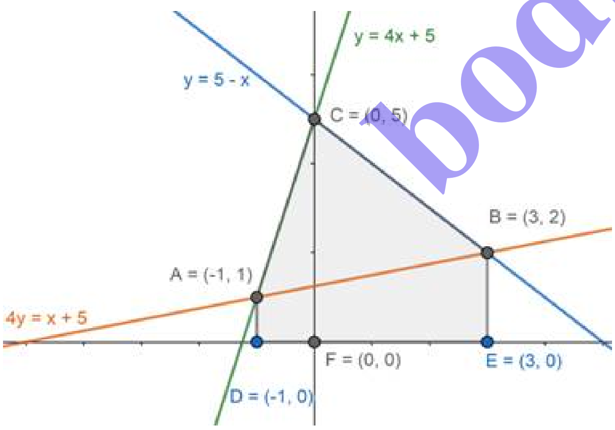
x = -1

by substituting x = -1 in AB we get y = 1

The point A = (-1,1)

These points are used for obtaining the upper and lower bounds of the integral.

From the given information, the area under the triangle (colored) can be given by the below figure.

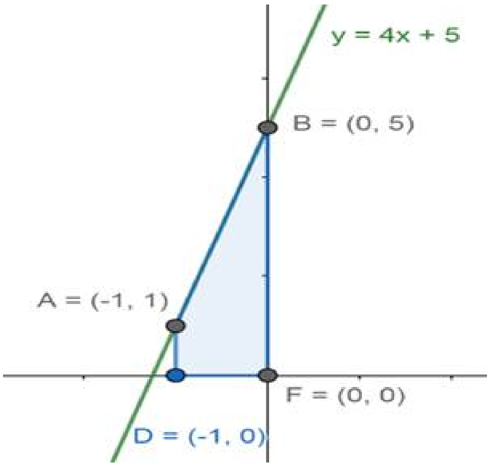


From above figure we can clearly say that, the area between ABC and DEF is the area to be found.

For finding this area, the line equations of the sides of the given triangle are considered. By calculating the area under these lines we can find the area of the complete region.

Consider the line AB, y = 4x+5

The area under line AB:



From the above figure, the area under the line AB will be given by,

$$Area\ of\ AB = \int_{-1}^{0} y_{AB}\, dx = \int_{-1}^{0} (4x+5)\, dx$$

$$= \int_{-1}^{0} (4x+5)\, dx = \left[\frac{4x^2}{2} + 5x\right]_{-1}^{0}$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\, dx = cx$]

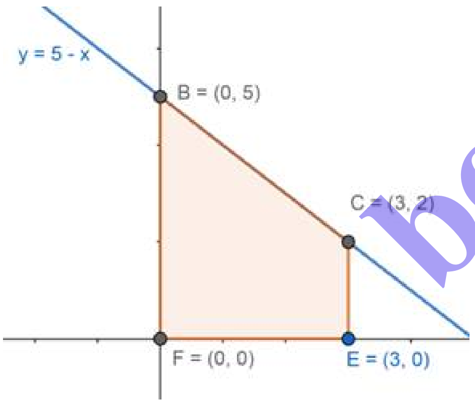
= {[2(0^2) + 5(0)] – [2(-1)^2 + 5 (-1)]}

= (0) – (2-5) = 0+3 = 3

Area under AB = 3 sq. units. ---- (1)

Consider the line BC, y = 5-x

Consider the area under BC:



From the above figure, the area under the line BC will be given by,

$$Area\ of\ BC = \int_{0}^{3} y_{BC}\, dx = \int_{0}^{3} (5-x)\, dx$$

$$= \int_{0}^{3} (5-x)\, dx = \left[5x - \frac{x^2}{2}\right]_{0}^{3}$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\, dx = cx$]

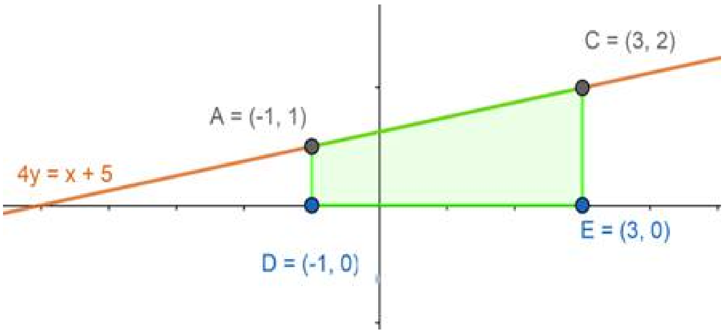
$$= \left\{\left[5(3) - \frac{(3)^2}{2}\right] - \left[5(0) - \frac{(0)^2}{2}\right]\right\}$$

$$= \left\{\left[15 - \frac{9}{2}\right] - 0\right\} = \frac{30-9}{2} = \frac{21}{2}$$

Area under BC $= \frac{21}{2}$ sq. units. ---- (2)

Consider the line AC, $y = \frac{1}{4}(x+5)$

Consider the area under AC:



From the above figure, the area under the line AC will be given by,

$$Area\ of\ ACFE = \int_{-1}^{3} y_{AC}\, dx = \int_{-1}^{3} \left(\frac{x+5}{4}\right) dx$$

$$= \int_{-1}^{3} \frac{1}{4}(x+5)\, dx = \frac{1}{4}\left[\left(\frac{x^2}{2}\right) + 5x\right]_{-1}^{3}$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\, dx = cx$]

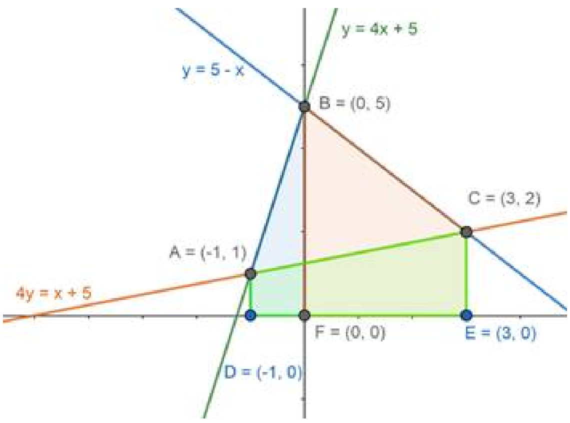
$$= \frac{1}{4}\left\{\left[\frac{(3)^2}{2} + 5(3)\right] - \left[\frac{(-1)^2}{2} + 5(-1)\right]\right\}$$

$$= \frac{1}{4}\left\{\left[\frac{9}{2} + 15\right] - \left[\frac{1}{2} - 5\right]\right\} = \frac{1}{4}\left[\left(\frac{39}{2}\right) - \left(-\frac{9}{2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{39-9}{2}\right] = \frac{48}{8} = 6$$

Area under AC = 6 sq.units ---- (3)

If we combined, the areas under AB, BC and AC in the below graph, we can clearly say that the area under AC (3) is overlapping the previous twoareas under AB & BC.



Now, the combined area under the rABC is given by

Area under rABC

=Area under AB + Area under BC - Area under AC

From (1), (2) and (3), we get

$$Ares\ under\ \Delta ABC = 3 + \frac{21}{2} - 6$$

$$= \frac{6 + 21 - 12}{2} = \frac{21 - 6}{2} = \frac{15}{2}$$

Therefore, area under rABC $= \frac{15}{2}$ sq.units.

13. Question

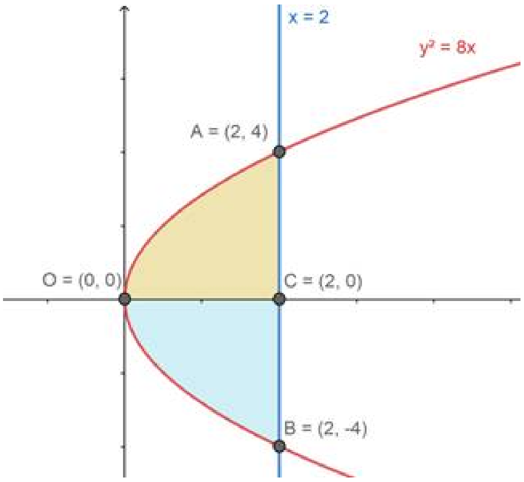
Using integration, find the area of the region bounded between the line x=2 and the parabola y^2=8x.

Answer

Given the boundaries of the area to be found are,

• The parabola y^2 = 8x

• x = 2 (a line parallel toy-axis)



As per the given boundaries,

• The curve y^2 =8x, has only the positive numbers as y has even power, so it is about the x-axis equally distributed on both sides as the vertex is at (0,0).

• x= 2 is parallel toy-axis which is 2 units away from the y-axis.

The boundaries of the region to be found are,

•Point A, where the curve y^2 = 8x and x=2 meet which has positive y.

•Point B, where the curve y^2 = 8x and x=2 meet which has negative y.

•Point C, where the x-axis and x=2 meet i.e. C(2,0).

Area of the required region = Area under OACB.

But,

Area under OACB = Area under OAC + Area under OBC

This can also be written as,

Area under OACB = 2 × Area under OAC

[area under OAC = area under OBC as AOB is symmetrical about the x-axis.]

$$\text{Area of OACB} = 2\int_0^2 y\,dx = 2\int_0^2 \sqrt{8x}\,dx$$

$$= 2 \times 2\sqrt{2}\int_0^2 \sqrt{x}\,dx = 4\sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = 4\sqrt{2}\left[\frac{2\,x^{\frac{3}{2}}}{3}\right]_0^2$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= 8\sqrt{2}\left(\frac{(2)^{\frac{3}{2}}}{3} - \frac{(0)^{\frac{3}{2}}}{3}\right) = 8\sqrt{2}\left(\frac{2\sqrt{2}}{3}\right) = \left(\frac{32}{3}\right)$$

$$= \frac{32}{3}$$

The Area of the required region $= \frac{32}{3}$ sq. units

14. Question

Using integration, find the area of region bounded by the line y-1=x, the x-axis, and the ordinates x=-2 and x=3.

Answer

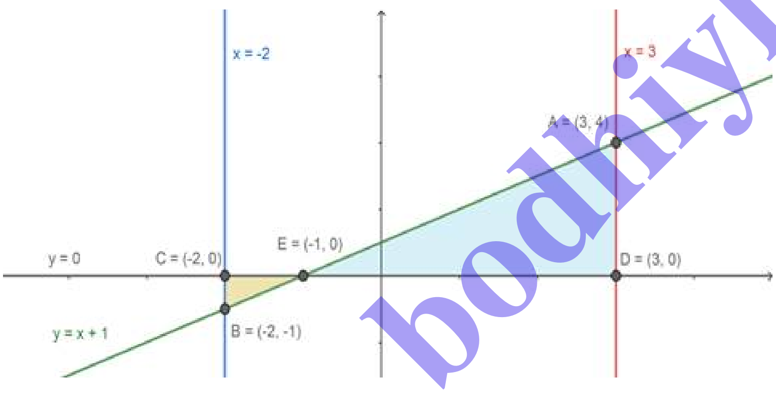
Given the boundaries of the area to be found are,

• The line equation is y = x +1

• The y= 0, x-axis

• x = -2 (a line parallel toy-axis)

• x = 3 (a line parallel toy-axis)



Thus the given boundaries are,

• The line y = x+1.

• x=-2 is parallel toy-axis at 2 units away from the y-axis.

• x=3 is parallel toy-axis at 3 units away from the y-axis.

• y = 0, the x-axis.

The four vertices of the region are,

•Point A, where the line y = x+ and x=3 meet i.e. A(3,4).

•Point B, where the line y = x +1 and x=-1 meet i.e.

B(-2,-1).

•Point C, where the x-axis and x=-2 meet i.e. C (-2,0).

•Point D, where the x-axis and x=3 meet i.e. D(3,0).

Area of the required region = Area of ABCD.

From (1) we can clearly say that, the area of ABCD has to be divided into twopieces i.e. area under CBE and ADE as the line equations changes the sign of x.

So the equation AB becomes negative between after it crosses the point E.

$$\text{Area of ABCD} = \int_{-2}^{-1} -y\,dx + \int_{-1}^{3} y\,dx = \int_{-1}^{3}(x+1)\,dx - \int_{-2}^{-1}(x+1)\,dx$$

$$= \int_{-1}^{3}(x+1)\,dx - \int_{-2}^{-1}(x+1)\,dx$$

$$= \left[\left(\frac{x^2}{2}\right)+x\right]_{-1}^{3} - \left[\left(\frac{x^2}{2}\right)+x\right]_{-2}^{-1}$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

$$= \left\{\left[\left(\frac{3^2}{2}\right)+(3)\right]-\left[\left(\frac{(-1)^2}{2}\right)+(-1)\right]\right\} - \left\{\left[\left(\frac{(-1)^2}{2}\right)+(-1)\right] - \left[\left(\frac{(-2)^2}{2}\right)+(-2)\right]\right\}$$

$$\left\{\left[\frac{9+6}{2}\right]-\left[\frac{1-2}{2}\right]\right\}-\left\{\left[\frac{1-2}{2}\right]-\left[\frac{4-4}{2}\right]\right\} = \left(\frac{15+1}{2}\right)-\left(-\frac{1}{2}\right)$$

$$= \frac{17}{2} = 8.5$$

The Area of the required region = 8.5 sq. units.

15. Question

Sketch the region lying in the first quadrant and bounded by $y=4x^2$, x=0,y=2 and y=4. Find the area of the region using integration.

Answer

Given the boundaries of the area to be found are,

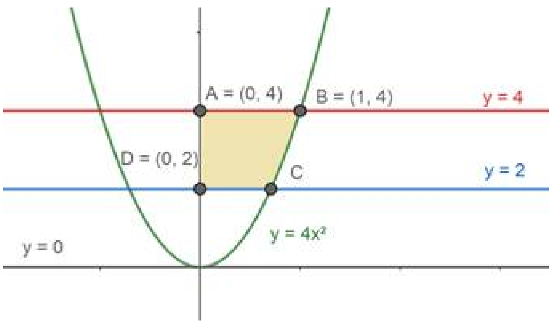
• The curve $y = 4x^2$

• y = 0, (x-axis)

• y = 2 (a line parallel to x-axis)

• y = 4 (a line parallel to x-axis)

• The area which is occurring in the 1st quadrant is required.



As per the given boundaries,

• The curve $y = 4x^2$, has only the positive numbers as x has even power, so it is about the y-axis equally distributed on both sides.

• y= 2 and y=4 are parallel to x-axis at of 2 and 4 units respectively from the x-axis.

As the area should be in the 1st quadrant, the four boundaries of the region to be found are,

•Point A, where the curve y = $4x^2$ and y-axis meet i.e. A(0,4)

•Point B, where the curve y = $4x^2$ and y=4 meet i.e. B(1,4)

•Point C, where the curve y = $4x^2$ and y=2 meet

•Point D, where the y-axis and y=2 meet i.e. D(0,2).

Consider the curve, y = $4x^2$

Now,

$$x = \frac{1}{2}\sqrt{y}$$

Area of the required region = Area of ABCD.

$$\text{Area of ABCD} = \int_2^4 \mathrm{x\,dy} = \frac{1}{2}\int_2^4 \sqrt{y}\,\mathrm{dy}$$

$$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4 = \frac{1}{2} \times \frac{2}{3}\left[y^{\frac{3}{2}}\right]_2^4$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \frac{1}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = \frac{1}{3}\,(8 - 2\sqrt{2})$$

The Area of the required region $= \frac{(8-2\sqrt{2})}{3}$ sq. units

16. Question

Sketch the region lying in the first quadrant and bounded by y=$9x^2$, x=0, y=1 and y=4. Find the area of the region, using integration.

Answer

Given the boundaries of the area to be found are,

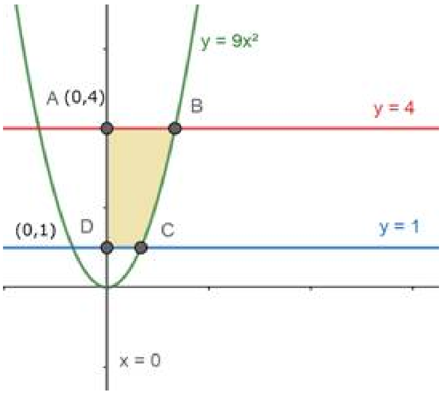
• The curve y = $9x^2$

• x = 0, (y-axis)

• y = 1 (a line parallel to x-axis)

• y = 4 (a line parallel to x-axis)

• The area which is occurring in the 1st quadrant is required.



As per the given boundaries,

• The curve y = $9x^2$, has only the positive numbers as x has even power, so it is about the y-axis equally distributed on both sides.

• y= 1 and y=4 are parallel to x-axis at of 1 and 4 units respectively from the x-axis.

As the area should be in the 1st quadrant, the four boundaries of the region to be found are,

•Point A, where the curve y = $9x^2$ and y-axis meet i.e. A(0,4)

•Point B, where the curve y = $9x^2$ and y=4 meet

•Point C, where the curve y = $9x^2$ and y=1 meet

•Point D, where the y-axis and y=1 meet i.e. D(0,1).

Consider the curve, y = $9x^2$

Now,

$$x = \frac{1}{3}\sqrt{y}$$

Area of the required region = Area of ABCD.

$$\text{Area of ABCD} = \int_1^4 \text{x dy} = \frac{1}{3}\int_1^4 \sqrt{y}\,\text{dy}$$

$$= \frac{1}{3}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = \frac{1}{3}\times\frac{2}{3}\left[y^{\frac{3}{2}}\right]_1^4$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \frac{2}{9}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{2}{9}(8-1) = \frac{14}{9}$$

The Area of the required region = $\frac{14}{9}$ sq. units

17. Question

Find the area of the region enclosed between the circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$

Answer

Given the boundaries of the area to be found are,

• First circle, $x^2 + y^2 = 1$ ---(1)

• Second circle, $(x-1)^2 + y^2 = 1$ ---- (2)

From the equation, of the first circle, $x^2 + y^2 = 1$

• the vertex at (0,0) i.e. the origin

• the radius is 1 unit.

From the equation, of the second circle, $(x-1)^2 + y^2 = 1$

• the vertex at (1,0) i.e. the origin

• the radius is 1 unit.

Now to find the point of intersection of (1) and (2), substitute $y^2 = 1-x^2$ in (2)

$(x-1)^2 + (1-x^2) = 1$

$x^2 + 1 - 2x + 1 - x^2 = 1$

$$x = -\frac{1}{2}$$

Substituting x in (1), we get $y = \pm\frac{\sqrt{3}}{2}$

So the two points, A and B where the circles (1) and (2) meet are $A = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $B = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

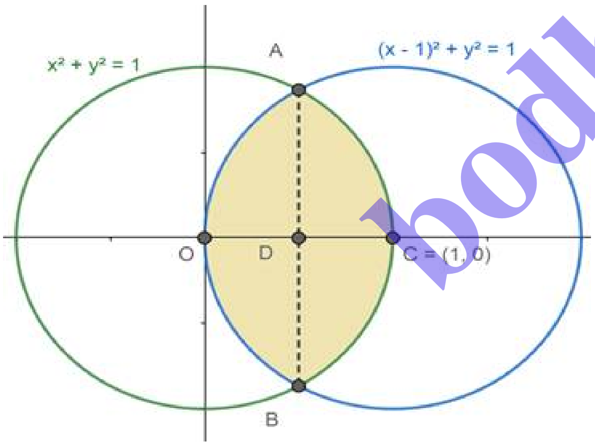
The line connecting AB, will be intersecting the x-axis at $D = \left(\frac{1}{2}, 0\right)$

As x and y have even powers for both the circles, they will be symmetrical about the x-axis and y-axis.

Here the circle, $x^2 + y^2 = 1$, can be re-written as

$y^2 = 1 - x^2$

$y = \sqrt{(1 - x^2)}$ ----- (3)



Now, the area to be found will be the area is

Area of the required region = Area of OABC.

Area of OABC = Area of AOC + Area of BOC

[area of AOC = area of BOC as the circles are symmetrical about the y-axis]

Area of OABC = 2 × Area of AOC

Area of OABC = 2 (Area of OAD + Area of ADC)

[area of OAD = area of ADC as the circles are symmetrical about the x-axis]

Area of OABC = 2 (2 × Area of ADC)

Area of OABC = 4 × Area of ADC

Area of ADC is under the first circle, thus $y = \sqrt{(1-x^2)}$ is the equation.

$$\text{Area of OABC} = 4\int_{\frac{1}{2}}^{1} y\,dx = 4\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,dx$$

$$= 4\int_{\frac{1}{2}}^{1} \sqrt{(1)^2 - x^2}\,dx = 3\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{\frac{1}{2}}^{1}$$

[Using the formula, $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$]

$$= 4\left[\frac{1\sqrt{1-1^2}}{2} + \frac{1}{2}\sin^{-1}(1)\right] - 4\left[\frac{\frac{1}{2}\sqrt{1-\left(\frac{1}{2}\right)^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{2}}{1}\right)\right]$$

$$= 4\left(0+\frac{\pi}{4}\right) - 4\left(\frac{\frac{\sqrt{3}}{4}}{2} + \frac{1}{2}\left(\frac{\pi}{6}\right)\right) = \pi - \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

[$\sin^{-1}(1) = 90°$ and $\sin^{-1}\frac{1}{2} = 30^0$]

The Area of the required region $= \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq. units

18. Question

Sketch the region common to the circle $x^2+y^2=16$ and the parabola $x^2=6y$. Also, find the area of the region, using integration.

Answer

Given the boundaries of the area to be found are,

• the circle, $x^2 + y^2 = 16$ ---(1)

• the parabola, $x^2 = 6y$ ---- (2)

From the equation, of the first circle, $x^2 + y^2 = 16$

• the vertex at (0,0) i.e. the origin

• the radius is 4 unit.

From the equation, parabola, $x^2 = 6y$

• the vertex at (0,0) i.e. the origin

• Symmetric about the y-axis, as it has the even power of x.

Now to find the point of intersection of (1) and (2), substitute $x^2 = 6y$ in (1)

$6y + y^2 = 16$

$y^2 + 6y - 16 = 0$

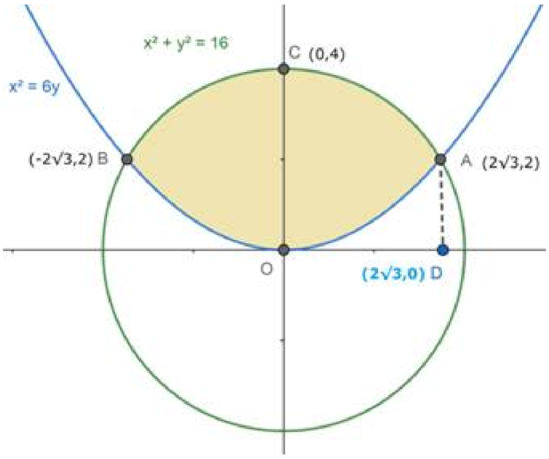
$$y = \frac{-6 \pm \sqrt{6^2 - 4(1)(-16)}}{2(1)} = \frac{-6 \pm \sqrt{36+64}}{2} = \frac{-6 \pm \sqrt{100}}{2} = \frac{-6 \pm 10}{2}$$

y = 2 (or) y = -8

as x cannot be imaginary, y = 2

Substituting x in (2), we get x = ± 2√3

So the two points, A and B where (1) and (2) meet are A = (2√3,2) and B = (-2√3,2)



As x and y have even powers for both the circle and parabola, they will be symmetrical about the x-axis and y-axis.

Consider the circle, $x^2 + y^2 = 16$, can be re-written as

$$y^2 = 16 - x^2$$

$$y = \sqrt{(16 - x^2)}$$ ----- (3)

Consider the parabola, $x^2 = 6y$, can be re-written as

$$y = \frac{x^2}{6}$$ ----- (4)

Let us drop a perpendicular from A on to x-axis. The base of the perpendicular is D = (2√ 3, 0)

Now, the area to be found will be the area is

Area of the required region = Area of OACBO.

Area of OABCO= Area of OCAO+ Area of OCBO

[area of OCBO= area of OCAOas the circle is symmetrical about the y-axis]

Area of OACBO= 2 × Area of OCAO---- (5)

Area of OCAO= Area of OCAD - Area of OADO

Area of OCAOis

$$\text{Area of OCAO} = \int_0^{2\sqrt{3}} \sqrt{16 - x^2}\,dx - \frac{1}{6}\int_0^{2\sqrt{3}} x^2$$

$$= \left[\frac{x\sqrt{16 - x^2}}{2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^{2\sqrt{3}} - \frac{1}{6}\left[\frac{x^3}{3}\right]_0^{2\sqrt{3}}$$

[Using the formula, $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ and $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \left\{\left[\frac{2\sqrt{3}\sqrt{16-\ (2\sqrt{3})^2}}{2} + 8\ \sin^{-1}\left(\frac{2\sqrt{3}}{4}\right)\right] - \left[\frac{0\sqrt{16-\ 0^2}}{2} + 8\ \sin^{-1}\left(\frac{0}{4}\right)\right]\right\}$$

$$-\ \frac{1}{6}\left[\frac{(2\sqrt{3})^3}{3} - \frac{(0)^3}{3}\right]$$

$$= \left\{\left[\frac{2\sqrt{3}\sqrt{16-\ 12}}{2} + 8\ \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] - [0 + 8\ \sin^{-1}(0)]\right\} - \frac{1}{6}\left[\frac{48\sqrt{3}}{3} - 0\right]$$

[$\sin^{-1}(1) = 90°$ and $\sin^{-1}\frac{\sqrt{3}}{2} = 60^0$]

$$= \left\{\left[2\sqrt{3} + 8\left(\frac{\pi}{3}\right)\right] - 0\right\} - \left[\frac{8\sqrt{3}}{3}\right] = \frac{8\pi}{3} + \frac{6\sqrt{3}-8\sqrt{3}}{3} = \frac{8\pi}{3} + \frac{2\sqrt{3}}{3}$$

The Area of OCAO= $\left(\frac{8\pi+2\sqrt{3}}{3}\right)$ sq. units

Now substituting the area of OCAOin equation (5)

Area of OACBO= 2 × Area of OCAO

$$= 2\left(\frac{8\pi+2\sqrt{3}}{3}\right) = \frac{16\pi+4\sqrt{3}}{3}$$

Area of the required region is $\frac{16\pi+4\sqrt{3}}{3}$ sq. units.

19. Question

Sketch the region common to the circle$x^2+y^2=25$and the parabola $y^2=8x$. Also, find the area of the region, using integration.

Answer

Given the boundaries of the area to be found are,

• the circle, $x^2 + y^2 = 25$ ---(1)

• the parabola, $y^2 = 8x$ ---- (2)

From the equation, of the first circle, $x^2 + y^2 = 25$

• the vertex at (0,0) i.e. the origin

• the radius is 5 units.

From the equation, of the parabola , $y^2 = 8x$

• the vertex at (0,0) i.e. the origin

• Symmetric about the x-axis, as it has the even power of y.

Now to find the point of intersection of (1) and (2), substitute $y^2 = 8x$ in (1)

$x^2 + 8x = 25$

$x^2 + 8x - 25= 0$

$$x = \frac{-8 \pm \sqrt{8^2 - 4(1)(-25)}}{2(1)} = \frac{-8 \pm \sqrt{64+100}}{2} = \frac{-8 \pm 2\sqrt{41}}{2}$$

as y cannot be imaginary, we reject the negative value of x

so $x = -4 + \sqrt{41}$

So the two points, A and B are the points where (1) and (2) meet.

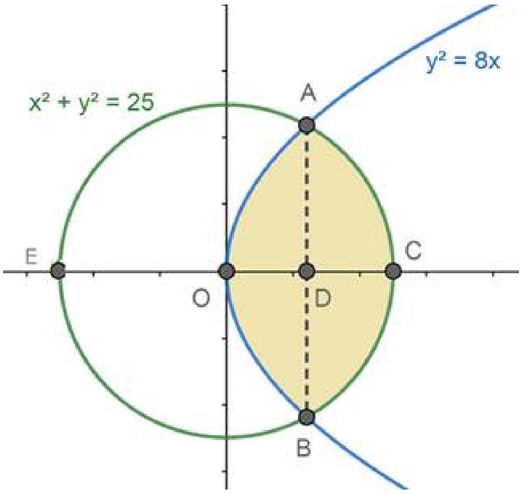
The line AB meets the x-axis at D = [(√41 -4),0]

Substitute y = 0 in (1),

$x^2 + 0 = 25$

$x = \pm 5$

So the circle intersects the x-axis at C(5,0)and E(-5,0)



As x and y have even powers for the circle, they will be symmetrical about the x-axis and y-axis.

Consider the circle, $x^2 + y^2 = 25$, can be re-written as

$$y^2 = 25 - x^2$$

$$y = \sqrt{(25 - x^2)} \text{ ----- (3)}$$

Consider the parabola, $y^2 = 8x$, can be re-written as

$$y = \sqrt{8x} \text{ ----- (4)}$$

Now, the area to be found will be the area is

Area of the required region = Area of OACBO.

Area of OABCO= Area of OCAO+ Area of OCBO

[area of OCBO= area of OCAOas the circle is symmetrical about the y-axis]

Area of OACBO= 2 × Area of OCAO---- (5)

Area of OCAO= Area of OADO+ Area of DACD

Area of OCAOis

$$\text{Area of OCAO} = \int_{0}^{\sqrt{41}-4} \sqrt{8x}\,dx - \int_{\sqrt{41}-4}^{5} \sqrt{25 - x^2}\,dx$$

$$= 2\sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\sqrt{41}-4} - \left[\frac{x\sqrt{25 - x^2}}{2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_{\sqrt{41}-4}^{5}$$

[Using the formula, $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ and $\int x^n dx = \frac{x^{n+1}}{n+1}$]

Let √41 – 4 = a

$$= \frac{4\sqrt{2}}{3}\left[a^{\frac{3}{2}} - 0\right] - \left\{\left[\frac{5\sqrt{25-5^2}}{2} + \frac{25}{2}\sin^{-1}\left(\frac{5}{5}\right)\right] - \left[\frac{a\sqrt{25-a^2}}{2} + \frac{25}{2}\sin^{-1}\left(\frac{a}{5}\right)\right]\right\}$$

$$= \frac{4\sqrt{2}}{3}\left[a^{\frac{3}{2}} - 0\right] - \left\{\left[0 + \frac{25}{2}\sin^{-1}(1)\right] - \left[\frac{a\sqrt{25-a^2}}{2} + \frac{25}{2}\sin^{-1}\left(\frac{a}{5}\right)\right]\right\}$$

[$\sin^{-1}(1) = 90°$ and $\sin^{-1}(0) = 0°$]

$$= \frac{4\sqrt{2}}{3}\left[a^{\frac{3}{2}} - 0\right] - \left\{\left[0 + \frac{25}{2}\left(\frac{\pi}{2}\right)\right] - \left[\frac{a\sqrt{25-a^2}}{2} + \frac{25}{2}\sin^{-1}\left(\frac{a}{5}\right)\right]\right\}$$

$$= \frac{4\sqrt{2}}{3}\left[a^{\frac{3}{2}}\right] - \left[\frac{25\pi}{4}\right] + \left[\frac{a\sqrt{25-a^2}}{2} + \frac{25}{2}\sin^{-1}\left(\frac{a}{5}\right)\right]$$

The Area of OCAO= $\frac{4\sqrt{2}}{3}\left[a^{\frac{3}{2}}\right] - \left[\frac{25\pi}{4}\right] + \left[\frac{a\sqrt{25-a^2}}{2} + \frac{25}{2}\sin^{-1}\left(\frac{a}{5}\right)\right]$ sq. units, where a = √41 - 4

Now substituting the area of OCAOin equation (5)

Area of OACBO= 2 × Area of OCAO

$$= 2\left\{\frac{4\sqrt{2}}{3}\left[a^{\frac{3}{2}}\right] - \left[\frac{25\pi}{4}\right] + \left[\frac{a\sqrt{25-a^2}}{2} + \frac{25}{2}\sin^{-1}\left(\frac{a}{5}\right)\right]\right\}$$

$$= \frac{8\sqrt{2}}{3}\left[a^{\frac{3}{2}}\right] - \left[\frac{25\pi}{2}\right] + \left[a\sqrt{25-a^2} + 25\sin^{-1}\left(\frac{a}{5}\right)\right]$$

Area of the required region is $\frac{8\sqrt{2}}{3}\left[a^{\frac{3}{2}}\right] - \left[\frac{25\pi}{2}\right] + \left[a\sqrt{25-a^2} + 25\sin^{-1}\left(\frac{a}{5}\right)\right]$ sq. units, where a = √41 - 4

20. Question

Draw a rough sketch of the region $\left\{(x, y) : y^2 \le 3x, 3x^2 + 3y^2 \le 16\right\}$ and find the area enclosed by the region, using the method of integration.

Answer

Given the boundaries of the area to be found are,

$R = \{(x,y): y^2 \le 3x,\ 3x^2 + 3y^2 \le 16\}$

This can be written as

$R_1 = \{(x,y): y^2 \le 3x\}$

$R_2 = \{(x,y): 3x^2 + 3y^2 \le 16\}$

Then $Area\ required\ is\ =\ R_1 \cap R_2$

From R_1, we can say that , $y^2 = 3x$ is a parabola

$y^2 = 3x$ ---- (1)

• With vertex at (0,0) i.e. the origin

• Symmetric about the x-axis, as it has the even power of y

From R_1, we can say that , $3x^2 + 3y^2 = 16$ is a circle

$3x^2 + 3y^2 = 16$ ----- (2)

• the vertex at (0,0) i.e. the origin

• the radius of $\frac{4}{\sqrt{3}}$ units

Now to find the point of intersection of (1) and (2), substitute $y^2 = 3x$ in (2)

$3x^2 + 3(3x) = 16$

$3x^2 + 9x - 16 = 0$

$$x = \frac{-9 \pm \sqrt{9^2 - 4(3)(-16)}}{2(3)} = \frac{-9 \pm \sqrt{81 + 192}}{6} = \frac{-9 \pm \sqrt{273}}{6}$$

as y cannot be imaginary, we reject the negative value of x

so $x = \frac{-9+\sqrt{273}}{6}$

So the two points, A and B are the points where (1) and (2) meet.

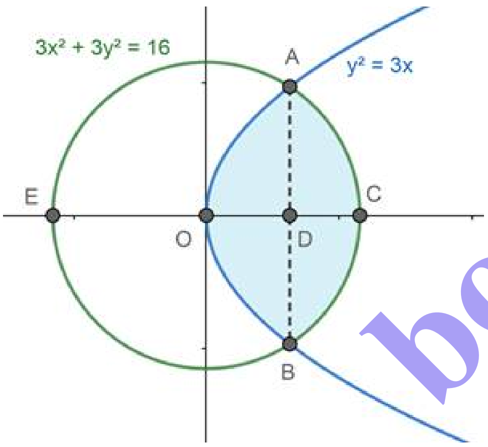
The line AB meets the x-axis at $D = \left(\frac{-9+\sqrt{273}}{6}, 0\right)$

Substitute y = 0 in (2),

$3x^2 + 0 = 16$

$$x = \pm\frac{4}{\sqrt{3}}$$

So the circle intersects the x-axis at $C = \left(\frac{4}{\sqrt{3}}, 0\right)$ and $E = \left(-\frac{4}{\sqrt{3}}, 0\right)$



As x and y have even powers for the circle, they will be symmetrical about the x-axis and y-axis.

Consider the circle, $3x^2 + 3y^2 = 16$, can be re-written as

$$x^2 + y^2 = \frac{16}{3}$$

$y = \sqrt{\left(\frac{16}{3} - x^2\right)}$ ----- (3)

Consider the parabola, $y^2 = 3x$, can be re-written as

$y = \sqrt{3x}$ ----- (4)

Now, the area to be found will be the area is

Area of the required region = Area of OACBO.

Area of OABCO= Area of OCAO+ Area of OCBO

[area of OCBO= area of OCAOas the circle is symmetrical about the y-axis]

Area of OACBO= 2 × Area of OCAO---- (5)

Area of OCAO= Area of OADO+ Area of DACD

Area of OCAOis

$$\text{Area of OCAO} = \int_0^{\frac{-9+\sqrt{273}}{6}} \sqrt{3}\mathrm{x}\,\mathrm{dx} + \int_{\frac{-9+\sqrt{273}}{6}}^{\frac{4}{\sqrt{3}}} \sqrt{\frac{16}{3} - \mathrm{x}^2}\,\mathrm{dx}$$

$$= \sqrt{3}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{-9+\sqrt{273}}{6}} + \left[\frac{x\sqrt{\frac{16}{3} - x^2}}{2} + \frac{\frac{16}{3}}{2}\sin^{-1}\left(\frac{x}{\frac{4}{\sqrt{3}}}\right)\right]_{\frac{-9+\sqrt{273}}{6}}^{\frac{4}{\sqrt{3}}}$$

[Using the formula, $\int \sqrt{a^2 - x^2}\,\mathrm{dx} = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ and $\int x^n dx = \frac{x^{n+1}}{n+1}$]

Let $\frac{-9+\sqrt{273}}{6} = a$

$$= \frac{2\sqrt{3}}{3}\left(a^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$$

$$+ \left\{\left[\frac{\frac{4}{\sqrt{3}}\sqrt{\frac{16}{3} - (\frac{4}{\sqrt{3}})^2}}{2} + \frac{\frac{16}{3}}{2}\sin^{-1}\left(\frac{\frac{4}{\sqrt{3}}}{\frac{4}{\sqrt{3}}}\right)\right]\right.$$

$$\left. - \left[\frac{a\sqrt{\frac{16}{3} - (a)^2}}{2} + \frac{\frac{16}{3}}{2}\sin^{-1}\left(\frac{a}{\frac{4}{\sqrt{3}}}\right)\right]\right\}$$

$$= \frac{2\sqrt{3}}{3}\left(a^{\frac{3}{2}}\right) + \left\{\left[0 + \frac{16}{6}\sin^{-1}(1)\right] - \left[\frac{a\sqrt{\frac{16}{3} - (a)^2}}{2} + \frac{16}{6}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)\right]\right\}$$

[$\sin^{-1}(1) = 90°$ and $\sin^{-1}(0) = 0°$]

$$= \frac{2\sqrt{3}}{3}\left(a^{\frac{3}{2}}\right) + \left\{\left[\frac{16}{6}\left(\frac{\pi}{2}\right)\right] - \left[\frac{a\sqrt{\frac{16}{3} - (a)^2}}{2} + \frac{16}{6}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)\right]\right\}$$

$$= \frac{2\sqrt{3}}{3}\left(a^{\frac{3}{2}}\right) + \left[\frac{4\pi}{3}\right] - \left[\frac{a\sqrt{\frac{16}{3} - (a)^2}}{2} - \frac{8}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)\right]$$

The Area of OCAO= $\frac{2\sqrt{3}}{3}\left(a^{\frac{3}{2}}\right) + \left[\frac{4\pi}{3}\right] - \left[\frac{a\sqrt{\frac{16}{3} - (a)^2}}{2} - \frac{8}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)\right]$ sq. units, where $a = \frac{-9+\sqrt{273}}{6}$

Now substituting the area of OCAOin equation (5)

Area of OACBO= 2 × Area of OCAO

$$= 2\left\{\frac{2\sqrt{3}}{3}\left(a^{\frac{3}{2}}\right) + \left[\frac{4\pi}{3}\right] - \left[\frac{a\sqrt{\frac{16}{3} - (a)^2}}{2} + \frac{8}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)\right]\right\}$$

$$= \frac{4\sqrt{3}}{3}\left(a^{\frac{3}{2}}\right) - \left[\frac{8\pi}{3}\right] - a\sqrt{\frac{16}{3} - (a)^2} - \frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)$$

Area of the required region is $\frac{4\sqrt{3}}{3}\left(a^{\frac{3}{2}}\right) - \left[\frac{8\pi}{3}\right] - a\sqrt{\frac{16}{3} - (a)^2} - \frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)$ sq. units, where $a = \frac{-9+\sqrt{273}}{6}$

21. Question

Draw a rough sketch and find the area of the region bounded by the parabolas $y^2=4x$ and $x^2=4y$, using the method of integration.

Answer

Given the boundaries of the area to be found are,

• the first parabola, $y^2 = 4x$ ---(1)

• the second parabola, $x^2 = 4y$ ---- (2)

From the equation, of the first parabola, $y^2 = 4x$

• the vertex at (0,0) i.e. the origin

• Symmetric about the x-axis, as it has the even power of y.

From the equation, of the second parabola, $x^2 = 4y$

• the vertex at (0,0) i.e. the origin

• Symmetric about the y-axis, as it has the even power of x.

Now to find the point of intersection of (1) and (2), substitute $y = \frac{x^2}{4}$ in (1)

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

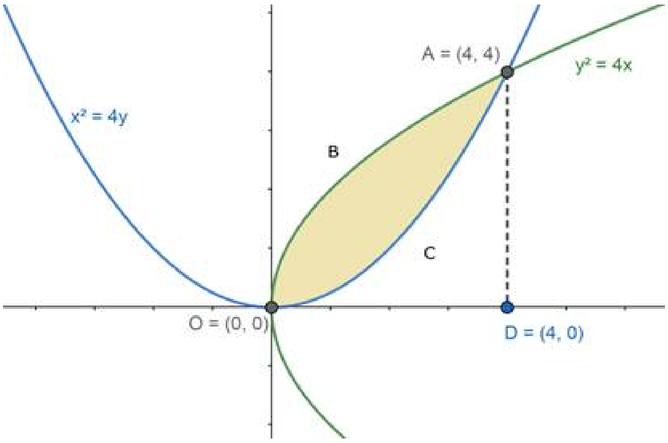
$x^4 = 64x$

$x(x^3 - 64) = 0$

x = 0 (or) x = 4

Substituting x in (2), we get y = 0 (or) y = 4

So the two points, A and B where (1) and (2) meet are A = (4,4) and O= (0,0)



Consider the first parabola, $y^2 = 4x$, can be re-written as

$y = 2\sqrt{x}$ ----- (3)

Consider the parabola, $x^2 = 4y$, can be re-written as

$y = \frac{x^2}{4}$ ----- (4)

Let us drop a perpendicular from A on to x-axis. The base of the perpendicular is D = (4, 0)

Now, the area to be found will be the area is

Area of the required region = Area of OBACO.

Area of OBACO= Area of OBADO- Area of OCADO

Area of OBACOis

$$\text{Area of OBACO} = \int_0^4 2\sqrt{x}\,dx - \frac{1}{4}\int_0^4 x^2$$

$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

[Using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \frac{4}{3}[4^{\frac{3}{2}} - 0^{\frac{3}{2}}] - \frac{1}{12}[4^3 - 0^3]$$

$$= \frac{4}{3}(8) - \frac{1}{12}(64)$$

$$= \frac{32 - 16}{3} = \frac{16}{3}$$

The Required Area of OBACO= $\left(\frac{16}{3}\right)$ sq. units

22. Question

Find by integration the area bounded by the curve $y^2=4ax$ and the lines $y=2a$ and $x=0$.

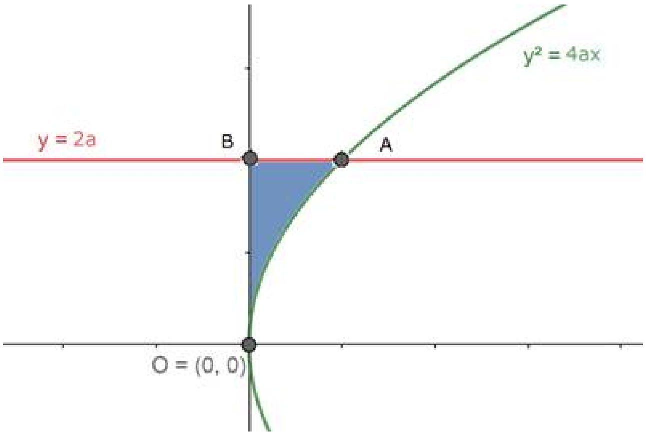
Answer

Given the boundaries of the area to be found are,

• The curve $y^2 = 4ax$

• $y = 2a$ (a line parallel to x-axis)

• x = 0 (y-axis)



As per the given boundaries,

• The curve y^2 =4ax, has only the positive numbers as y has even power, so it is about the x-axis equally distributed on both sides.

• y= 2a is parallel to x-axis with 2a units from the x-axis.

The boundaries of the region to be found are,

•Point A, where the curve y^2 = 4ax and y=2a meet i.e. A(2a,2a)

•Point B, where the curve y^2 = 4ax and y-axis meet i.e. B(0,2a)

•Point O, is the origin

Consider the curve y^2 = 4ax,

$$x = \frac{y^2}{4a}$$

Area of the required region = Area of OBA.

$$\text{Area of OBA} = \int_0^{2a} \text{x dy} = \int_0^{2a} \frac{y^2}{4a} \text{dy}$$

$$= \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{2a}$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \frac{1}{12a}[(2a)^3 - 0^3]$$

$$= \frac{8a^3}{12a} = \frac{2a^2}{3}$$

The Area of the required region $= \frac{2a^2}{3}$ sq. units

23. Question

Find the area between the curve y=$\frac{x}{\pi} + 2\sin^2 x$, the axis and the ordinates x=0 and x=π.

Answer

Given

• Curve is $y = \frac{x}{\pi} + 2\sin^2 x$

• x = 0 and

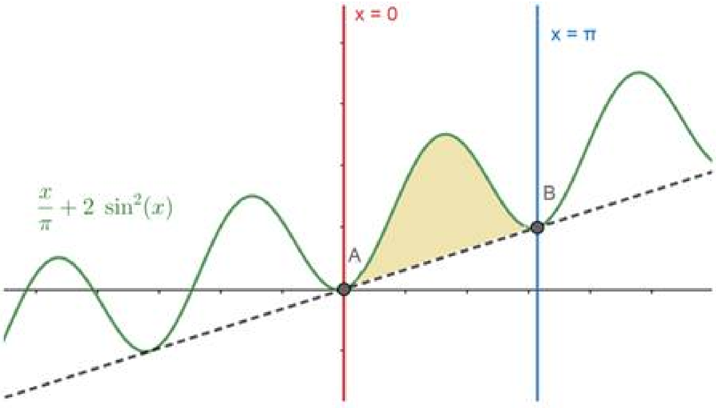
• x = π

The given curve $y = \frac{x}{\pi} + 2\sin^2 x$ is similar toy = $\sin^2$ x.

Now consider the y values for some random x values between 0 and π for the function y = $\sin^2$x.

x	y
0	0
$\frac{\pi}{6}$	$\frac{1}{4}$
$\frac{\pi}{4}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{3}{4}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{4}$
π	0

From the table we can clearly draw the graph for $y = \frac{x}{\pi} + 2\sin^2 x$



The required area under the curve is given by:

$$Area = \int_0^{\pi} \left[\frac{x}{\pi} + 2\,sin^2 x\right] dx$$

[using the property cos 2x = 1- 2sin^2 x]

$$= \frac{1}{\pi}\int_0^{\pi} [x]\,dx \;+ 2\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} + 2\left\{\frac{1}{2}[x]_0^{\pi} - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi}\right\}$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int \cos x\,dx = \sin x$]

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{0^2}{2}\right] + 2\left\{\frac{1}{2}[\pi - 0] - \frac{1}{4}[\sin 2(\pi) - \sin 2(0)]\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] + \left\{[\pi - 0] - \frac{1}{2}[\sin 2(\pi)] + 0]\right\}$$

$$= \left[\frac{\pi}{2}\right] + \left\{[\pi] - \frac{1}{2}[\sin 2(\pi)]\right\}$$

$$= \left[\frac{3\pi}{2}\right] - \frac{1}{2}[0]$$

[as sin π = 0, then sin 2π = 0]

$$= \frac{3\pi}{2}$$

Hence the required area of the curve $y = \frac{x}{\pi} + 2\,sin^2 x$ from x = 0 to x= π is = $\frac{3\pi}{2}$ sq. units.

24. Question

Find the area of bounded by the curve y=cos x, the x-axis and the ordinates x=0 and x=2π.

Answer

Given

• Curve is y = cos x

• X- axis

• x = 0 and

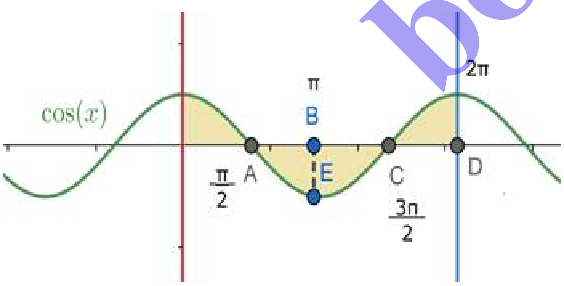
• x = 2π

The given curve is y = cos x.

Now consider the y values for some random x values between 0 and 2π for the function y = cos x.

X	Y
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0
�	-1
$\frac{3\pi}{2}$	0
2π	1

From the table we can clearly draw the graph for y = cos x



From the given curve, we can say that,

For $0 < x < \frac{\pi}{2}$, y = cos x

For $\frac{\pi}{2} < x < \frac{3\pi}{2}$, y = -cos x

For $\frac{3\pi}{2} < x < 2\pi$, y = cos x

The required area under the curve is given by:

Area required = Area under of OA + Area of ABC + Area under AC

$$Area\, required = \int_0^{\frac{\pi}{2}} \cos x\ dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-\cos x)\ dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\ dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x\ dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\ dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\ dx$$

$$= (\sin x)_0^{\frac{\pi}{2}} - (\sin x)_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + (\sin x)_{\frac{3\pi}{2}}^{2\pi}$$

[using the formula, $\int \cos x\, dx = \sin x$]

$$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right] + \left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$$

$$= [1 - 0] - [-1 - 1] + [0 - (-1)] = 1 + 2 + 1 = 4$$

[as $\sin\frac{\pi}{2} = 1$, sin 2π = 0, $\sin\frac{3\pi}{2} = -1$, sin 0 = 0]

Hence the required area of the curve y = cos x from x = 0 to x=2π is 4 sq. units.

25. Question

Compare the areas under the curves $y=\cos^2 x$ and $y=\sin^2 x$ between x=0 and x=π.

Answer

Given

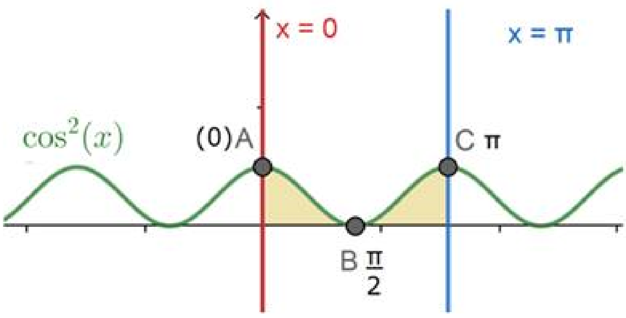
- First curve $y = \cos^2 x$
- Second curve $y = \sin^2 x$
- x= 0
- x= π

Consider the curve $y = \cos^2 x$

Now consider the y values for some random x values between 0 and π for the function $y = \cos^2 x$.

X	Y
0	1
$\frac{\pi}{6}$	$\frac{3}{4}$
$\frac{\pi}{4}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{4}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$\frac{3}{4}$
π	1

From the table we can clearly draw the graph for y = $\cos^2$ x



The required area under the curve is given by:

$$Area\ required = \int_0^{\pi} \cos^2 x\ \ dx$$

$$= \int_0^{\pi} \cos^2 x\ \ dx\ = \int_0^{\pi} \frac{1+\cos 2x}{2} dx$$

[using the property cos 2x = 2 $\cos^2$ x - 1]

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$

[using the formula, $\int \cos x\, dx = \sin x$]

$= \frac{1}{2}\left\{[\pi - 0] + \frac{1}{2}[\sin 2(\pi) - \sin 2(0)]\right\}$

$= \left\{\frac{\pi}{2} + \frac{1}{4}[0 - 0]\right\} = \frac{\pi}{2}$

[as sin 2π = 0, sin 0 = 0]

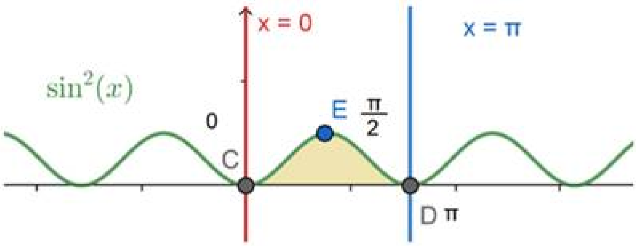
Hence the required area of the curve y = $\cos^2$ x from x = 0 to x=π is = $\frac{\pi}{2}$ sq. units. ------ (1)

Consider the curve y = $\sin^2$ x

Now consider the y values for some random x values between 0 and π for the function y = $\sin^2$ x.

X	Y
0	0
$\frac{\pi}{6}$	$\frac{1}{4}$
$\frac{\pi}{4}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{3}{4}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{4}$
π	0

From the table we can clearly draw the graph for y = $\sin^2$ x



The required area under the curve is given by:

$$Area\ required = \int_0^{\pi} \sin^2 x \ \ dx$$

$$= \int_0^{\pi} \sin^2 x \ \ dx \ = \int_0^{\pi} \frac{1-\cos 2x}{2} dx$$

[using the property cos 2x = 1- 2 $\sin^2$ x]

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

[using the formula, $\int \sin x \, dx = \cos x$]

$$= \frac{1}{2}\left\{[\pi - 0] - \frac{1}{2}[\sin 2(\pi) - \sin 2(0)]\right\}$$

$$= \left\{\frac{\pi}{2} - \frac{1}{4}[0-0]\right\}$$

[as sin 2π = 0, sin 0 = 0]

$$= \frac{\pi}{2}$$

Hence the required area of the curve y = $\sin^2$ x from x = 0 to x=π is = $\frac{\pi}{2}$ sq. units. ----- (2)

From (1) and (2), we can clearly state that, the areas under

y = $\cos^2$ x and y = $\sin^2$ x are similar which is = $\frac{\pi}{2}$ sq. units.

26. Question

Using integration, find the area of the triangle, the equations of whose sides are y=2x+1, y=3x+1 and x=4.

Answer

Given,

• ABC is a triangle

• Equation of side AB of y = 2x + 1

• Equation of side BC of y = 3x+1

• Equation of side CA of x=4

By solving AB & BC we get the point B,

AB : y = 2x + 1 , BC: y = 3x + 1

2x + 1 = 3x + 1

x= 0

by substituting x = 0 in AB we get y = 1

The point B = (0,1)

By solving BC & CA we get the point C,

AC : x = 4 , BC: y = 3x + 1

y = 12 + 1

y = 13

The point C = (4,13)

By solving AB & AC we get the point A,

AB : y = 2x+1 , AC : x = 4

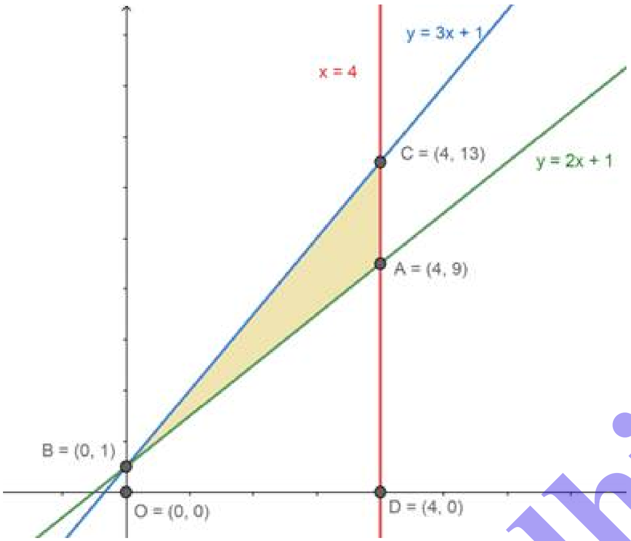
y = 8 +1

y = 9

The point A = (4,9)

These points are used for obtaining the upper and lower bounds of the integral.

From the given information, the area under the triangle (colored) can be given by the below figure.



From above figure we can clearly say that, the area between ABC is the area to be found.

The required area is

Area of ABC = Area under OBCD – Area under OBAD

Now, the combined area under the rABC is given by

Area under rABC

=Area under AB + Area under BC - Area under AC

Area of the rABC=

$$Area\ of\ ABC = \int_0^4 (3x+1)dx - \int_0^4 (2x+1)dx$$

$$= \left[\frac{3x^2}{2} + x\right]_0^4 - \left[\frac{2x^2}{2} + x\right]_0^4$$

$$= \left\{\left[\frac{3(4)^2}{2} - \frac{3(0)^2}{2}\right] + [4-0]\right\} - \{[\,4^2 - 0\,] + [4-0]\}$$

$$= \left\{\left[\frac{3(16)}{2}\right] + 4\right\} - \{[\,16\,] + [4]\}$$

= 24+4 – 20 = 8

Therefore, area of the rABC is 8 sq.units.

27. Question

Find area of region $\{(x, y) : x^2 \le y \le x\}$

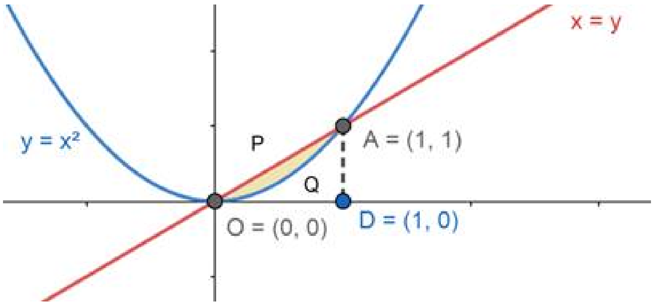
Answer

Given,

• R = {(x,y): $x^2 \le y \le x$}

From the set we have the curve, $y = x^2$ ------ (1)

Also the line equation y = x ------ (2)



As per the given boundaries,

• The curve $y = x^2$, has only the positive numbers as x has even power, so it is about the y-axis equally distributed on both sides.

• y = x is a line passing through the origin.

The boundaries of the region to be found are,

•Point A, where the curve $y = x^2$ and y=x meet, i.e. A (1,1)

•Point O, which is the origin

Drop a perpendicular D on the x-axis from A, where D = (1,0)

Now,

Area of the required region = Area of OPAQO.

Area of OPAQ• = Area of OPAD• – Area of OQADO

$$\text{Area of OPAQO} = \int_0^1 y\,dx - \int_0^1 y\,dx$$

$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \frac{1}{2}(1^2 - 0^2) - \frac{1}{3}(1^3 - 0^3) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

The Area of the required region = $\frac{1}{6}$ sq. units

28. Question

Find the area of the region bounded by the curve y^2=2y-x and the y-axis.

Answer

Given the boundaries of the area O befound are,

• Curve is $y^2 = 2y - x$

• Y-axis.

Consider the curve, $y^2 = 2y - x$

$y^2 - 2y = -x$

by adding 1 on both sides

$y^2 - 2y + 1 = -(x-1)$

$(y-1)^2 = -(x-1)$

From the above equation, we can say that, the given equation is that of a parabola with vertex at A(1,1)

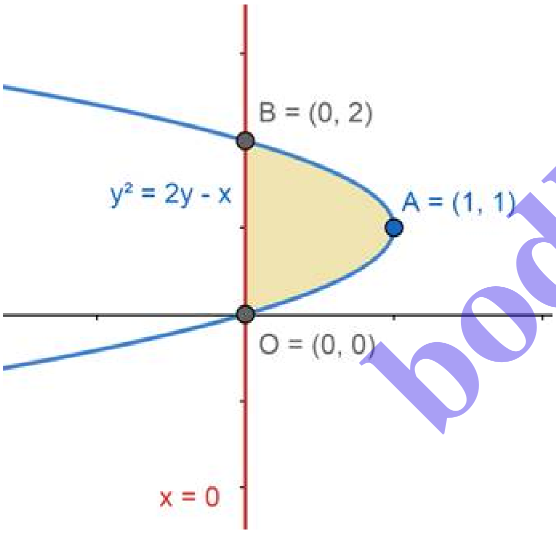
Consider the line x = 0 which is the y-axis, now substituting x = 0 in the curve equation we get

$y^2 - 2y = 0$

$y(y-2)=0$

y = 0 (or)y = 2

So , the parabola meets the y-axis at 2 points, B (0,2) and •(0,0)



As per the given boundaries,

• The parabola y^2 = 2y-x, with vertex at A(1,1).

• X= 0 which is the y-axis.

The boundaries of the region to be found are,

•Point A, where the curve y^2 = 2y-x has the extreme end the vertex i.e. A (1,1)

•Point O, which is the origin

•Point B, where the curve y^2 = 2y-x and the y – axis meet i.e. B (0,2)

Consider the curve,

$y^2 = 2y - x$

$x = 2y - y^2$

Area of the required region = Area of OBAO.

$$\text{Area of OBAO} = \int_0^1 x\,dy$$

$$= \int_0^1 (2y - y^2)\,dy$$

$$= \left[\frac{2y^2}{2}\right]_0^1 - \left[\frac{y^3}{3}\right]_0^1$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= [1^2 - 0^2] - \frac{1}{3}[1^3 - 0^3] = 1 - \frac{1}{3} = \frac{2}{3}$$

The Area of the required region $= \frac{2}{3}$ sq. units

29. Question

Draw a rough sketch of the curves y=sin x and y=cos x, as x varies from 0 to $\frac{\pi}{2}$, and find the area of the region enclosed between them and the x-axis.

Answer

Given

• First curve y = cos x

• Second curve y = sinx

• x= 0

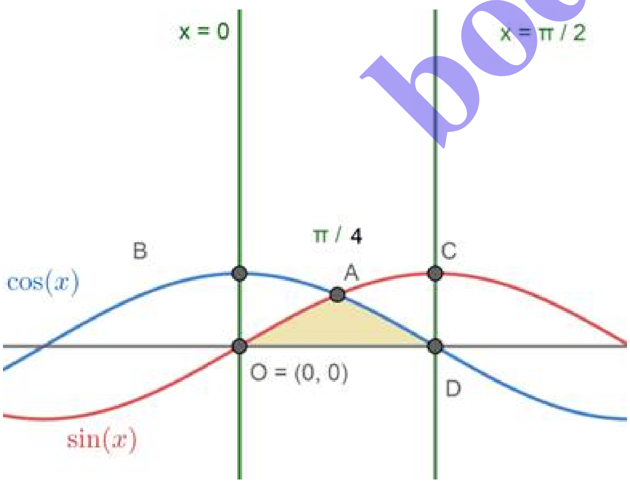
• $x = \frac{\pi}{2}$

Consider the curves y = cosx and y = sin x

Now consider the y values for some random x values between 0 and $\frac{\pi}{2}$ for the functions y = cos x and y = sin x.

y = cos x		Y = sin x	
X	Y	X	Y
0	1	0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	1

From the above table we can clearly draw the below graphs.



The required area under the curve is given by:

Area of OAD = Area under the curve OA + Area under the curve AD

$$Area\, required = \int_0^{\frac{\pi}{4}} y_{OA}\, dx \;+\; \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y_{AD}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sin x \ dx \ + \ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \ dx$$

$$= [-\cos x]_0^{\frac{\pi}{4}} + [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

[using the formula, $\int \cos x \, dx = \sin x$ and $\int \sin x \, dx = -\cos x$]

$$= -\left[\cos\left(\frac{\pi}{4}\right) - \cos 0\right] + [\sin\frac{\pi}{2} - \sin\frac{\pi}{4}]$$

$$= -\left(\frac{1}{\sqrt{2}} - 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \frac{2}{\sqrt{2}} = 2 - \sqrt{2}$$

Thus the area under the curves y = cos x and y = sin x is 2 - √2 sq. units.

30. Question

Find the area of the region bounded by the parabola $y^2=2x+1$ and the lines x-y=1.

Answer

Given the boundaries of the area O befound are,

• Curve is $y^2 = 2x +1$

• Line x-y = 1

Consider the curve

$y^2 = 2x +1$

$$(y-0)^2 = 2\left(x + \frac{1}{2}\right)$$

This clearly shows, the curve is a parabola with vertex $A\left(-\frac{1}{2}, 0\right)$

Consider the curve, $y^2 = 2x +1$ and substitute the line x = y +1 in the curve

$y^2 = 2(y+1) +1$

$y^2 = 2y +2 +1$

$y^2 = 2y +3$

$y^2 -2y -3 = 0$

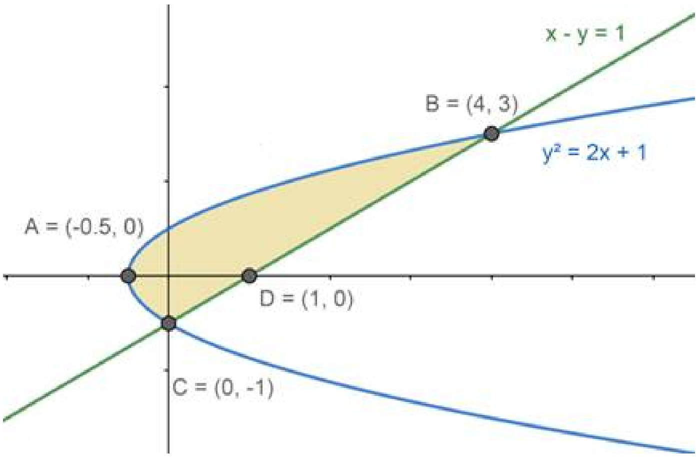
$$y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2}$$

y = 3 (or) y = -1

substituting y in x-y = 1

x = 4 (or) x = 0

So , the parabola meets the line x-y =1 at 2 points, B (4,3) and C (0,-1)



As per the given boundaries,

• The parabola $y^2 = 2x + 1$, with vertex at A(-0.5,0) and symmetric about the x-axis as y has even powers.

• Line x-y = 1

The boundaries of the region to be found are,

•Point A, where the curve $y^2 = 2x + 1$ has the extreme end the vertex i.e. A (-0.5,0)

•Point B, where the curve $y^2 = 2x + 1$ and the line x-y = 1 meet i.e. B (4,3)

•Point C, where the curve $y^2 = 2x + 1$ and the line x-y = 1 meet i.e. B (0,-1) on the negative y

•Point D, where the line x-y = 1 meets the x-axis i.e. D(1,0)

Consider the curve,

$y^2 = 2x + 1$

$2x = y^2 - 1$

$$x = \frac{y^2 - 1}{2}$$

Consider the line x – y = 1

x = y +1

Area of the required region = Area of ABDC

Area of ABDC = Area above CDB – Area above CAB

$$\text{Area of ABDC} = \int_{-1}^{3} x_{CDB}\, dy - \int_{-1}^{3} x_{ABDC}\, dy$$

$$= \int_{-1}^{3} (y+1)\, dy - \frac{1}{2}\int_{-1}^{3} (y^2 - 1)\, dy$$

$$= \left[\frac{y^2}{2} + y\right]_{-1}^{3} - \frac{1}{2}\left[\frac{y^3}{3} - y\right]_{-1}^{3}$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \left\{\frac{1}{2}[3^2 - (-1)^2] + (3 - (-1))\right\} - \frac{1}{2}\left\{\frac{1}{3}[3^3 - (-1)^3] - [3 - (-1)]\right\}$$

$$= \frac{1}{2}[9 - 1] + 4 - \frac{1}{2}\left\{\frac{1}{3}[27 + 1] - [4]\right\} = \frac{1}{2}[8] - \frac{1}{2}\left\{\frac{28}{3} - [4]\right\}$$

$$= 4 + 4 - \frac{14}{3} + 2 = \frac{30 - 14}{3} = \frac{16}{3}$$

The Area of the required region $= \frac{16}{3}$ sq. units

31. Question

Find the area of the region bounded by the curve y=2x-x^2 and the straight line y=-x.

Answer

Given the boundaries of the area O befound are,

• Curve is y = 2x – x^2

• Line y = -x

Consider the curve

y = 2x – x^2

x^2 -2x = - y

adding 1 on both sides

x^2 – 2x +1 = -(y-1)

$(x-1)^2$ = -(y-1)

This clearly shows, the curve is a parabola with vertex B (1,1)

Consider the curve, y = 2x – x^2 and substitute the line -x = y in the curve

-x = 2x – x^2

x^2 – 2x – x = 0

x^2 – 3x = 0

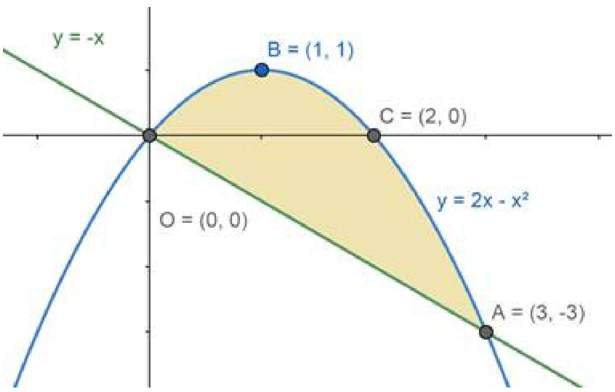
x(x-3) = 0

x = 3 (or) x = 0

substituting x in y = -x

y = -3 (or) y = 0

So , the parabola meets the line y = -x at 2 points, A (3,-3) and •(0,0)



As per the given boundaries,

• The parabola y = 2x - x^2, with vertex at B(1,1).

• Line y = -x

The boundaries of the region to be found are,

•Point A, where the curve y = 2x - x^2 and the line y = -x meet i.e. A (3,-3)

•Point B, where the curve y = 2x - x^2 has the extreme end the vertex i.e. B (1,1)

•Point C, where the curve y = 2x - x^2 and the line y= -x meet i.e. C (2,0)

•Point O, the origin

Area of the required region = Area of OACBO

Area of OACBO= Area under OBCA – Area under line OA

$$\text{Area of OBCA} = \int_0^3 y_{OBCA}\,dx - \int_0^3 y_{OA}\,dx$$

$$= \int_0^3 2x - x^2\,dx - \int_0^3 (-x)\,dx$$

$$= \int_0^3 (2x - x^2 + x\,)dx = \int_0^3 3x - x^2\,dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \left\{\frac{3}{2}[3^2 - (0)^2]\right\} - \frac{1}{3}\{[3^3 - (0)^3]\}$$

$$= \frac{3}{2}(9) - \frac{1}{3}(27) = \frac{81-54}{6} = \frac{27}{6} = \frac{9}{2}$$

The Area of the required region = $\frac{9}{2}$ sq. units

32. Question

Find the area of the region bounded by the curve $(y-1)^2=4(x+1)$ and the line y=x-1.

Answer

Given the boundaries of the area O befound are,

• Curve is $(y-1)^2$ = 4 (x+1)

• Line y = x-1

Consider the curve

$(y-1)^2$ = 4 (x+1)

Substitute y = x-1

$(x-1-1)^2$ = 4(x+1)

$(x-2)^2$ = 4x+ 4

x^2 – 4x + 4 = 4x + 4

x^2 – 8x = 0

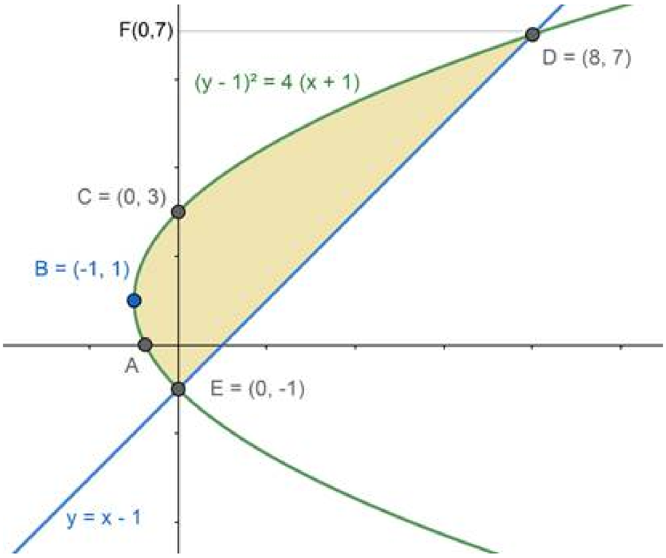
x(x-8) =0

x = 8 (or) x = 0

substituting x in y = x-1

y = 7 (or) y = -1

So , the parabola meets the line y = x-1 at 2 points, D (8,7) and E (0,-1)



As per the given boundaries,

• The parabola $(y-1)^2 = 4(x+1)$, with vertex at B(-1,1).

• Line y = x-1

The boundaries of the region to be found are,

•Point B, where the curve $(y-1)^2 = 4(x+1)$ has the extreme end the vertex i.e. B (-1,1)

•Point D, where the curve $(y-1)^2 = 4(x+1)$ and the line y= x + 1 meet i.e. D (8,7)

•Point E, where the curve $(y-1)^2 = 4(x+1)$ and the line y = x-1 meet i.e. E (0,-1)

•Point O, the origin

Consider the parabola,

$(y-1)^2 = 4(x+1)$

$$x = \frac{(y-1)^2 - 4}{4}$$

Area of the required region = Area of EABCDE

Area of EABCDE = Area above line ED - Area above EABCD

$$\text{Area of EABCDE} = \int_{-1}^{7} \mathrm{x_{ED}}\,\mathrm{dy} - \int_{-1}^{7} \mathrm{x_{EABCD}}\,\mathrm{dy}$$

$$= \int_{-1}^{7} (\mathrm{y}+1)\,\mathrm{dy} - \int_{-1}^{7} \left(\frac{(y-1)^2 - 4}{4}\right) \mathrm{dy}$$

$$= \int_{-1}^{7} (\mathrm{y}+1)\,\mathrm{dy} - \int_{-1}^{7} \left(\frac{y^2 - 2y + 1 - 4}{4}\right) \mathrm{dy}$$

$$= \int_{-1}^{7} (\mathrm{y}+1)\,\mathrm{dy} - \frac{1}{4}\int_{-1}^{7} (y^2 - 2y - 3)\,\mathrm{dy}$$

$$= \left[\frac{y^2}{2} + y\right]_{-1}^{7} - \frac{1}{4}\left\{\left[\frac{y^3}{3}\right] - 2\left[\frac{y^2}{2}\right] - 3y\right\}_{-1}^{7}$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \left\{\frac{1}{2}[7^2 - (-1)^2] + [7 - (-1)]\right\}$$
$$- \frac{1}{4}\left\{\frac{1}{3}[7^3 - (-1)^3] - [7^2 - (-1)^2] - 3[7 - (-1)]\right\}$$

$$= \left\{\frac{1}{2}[48] + [8]\right\} - \frac{1}{4}\left\{\frac{1}{3}[344] - [48] - 3[8]\right\}$$

$$= \{24 + 8\} - \left\{\frac{1}{3}[86] - [12] - 3[2]\right\} = 32 - \frac{86}{3} + 18 = 50 - \frac{86}{3} = \frac{150 - 86}{3}$$
$$= \frac{64}{3}$$

The Area of the required region $= \frac{64}{3}$ sq. units

33. Question

Find the area of the region bounded by the curve y=$\sqrt{x}$ and the line y=x.

Answer

Given the boundaries of the area O befound are,

• Curve is y = √x

• Line y = x

Consider the curve

$y^2 = x$

Substitute y = x

$(x)^2 = x$

$x^2 - x = 0$

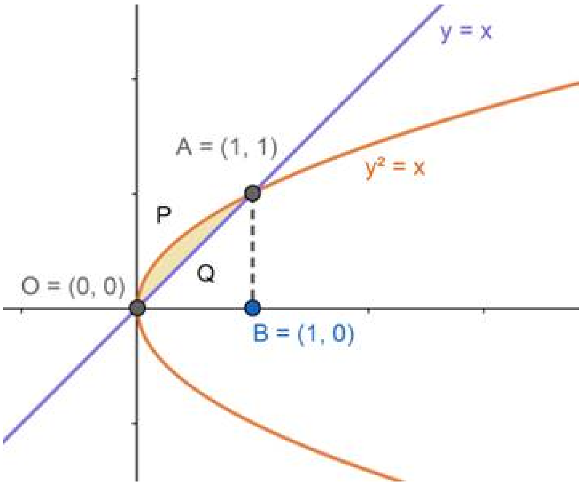
x(x-1) =0

x = 1 (or) x = 0

substituting x in y = x

y = 1 (or) y = 0

So , the parabola meets the line y = √x at 2 points, A (1,1) and •(0,0)



As per the given boundaries,

• The parabola $(y)^2 = x$, with vertex at O(0,0).

• Line y = x

The boundaries of the region to be found are,

•Point A, where the curve $(y)^2 = x$ and the line y= x meet i.e. A (1,1)

•Point O, the origin

Now, drop a perpendicular B on the x-axis from A, the point will be B(1,0)

Area of the required region = Area of OPAQO

Area of OPAQ•= Area under OPAB - Area under OQAB

$$\text{Area of OPAQO} = \int_0^1 y_{OPAB}\,dx - \int_0^1 y_{OQAB}\,dx$$

$$= \int_0^1 \sqrt{x}\,dx - \int_0^1 x\,dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1 = \frac{2}{3}\left[1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] - \frac{1}{2}[1^2 - 0]$$

[Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$$

The Area of the required region = $\frac{1}{6}$ sq. units

34. Question

Find the area of the region included between the parabola $y^2 = 3x$ and the circle $x^2+y^2-6x=0$, lying in the first quadrant.

Answer

Given the boundaries of the area to be found are,

• the circle, $x^2 + y^2 - 6x = 0$ ---(1)

• the parabola, $y^2 = 3x$ ---- (2)

• Area under 1st quadrant.

From the equation, of the first circle, $x^2 + y^2 - 6x = 0$

$x^2 - 6x + 9 + y^2 - 9 = 0$

$(x-3)^2 + y^2 = 9$

• the vertex at (3,0)

• the radius is 3 units.

From the equation, of the parabola , $y^2 = 3x$

• the vertex at (0,0) i.e. the origin

• Symmetric about the x-axis, as it has the even power of y.

Now to find the point of intersection of (1) and (2), substitute $y^2 = 3x$ in (1)

$x^2 + 3x - 6x = 0$

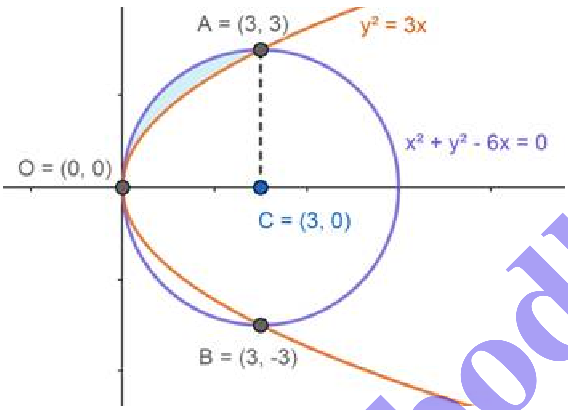
$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 3$ (or) $x = 0$

Substituting x in (2), we get $y = \pm 3$ or $y = 0$

So the three points, A, B and •where (1) and (2) meet are A = (3,3) , B = (3,-3) and O=(0,0)



Consider the circle, $x^2 + y^2 - 6x$, can be re-written as

$$y^2 = 6x - x^2$$

$y = \sqrt{9-(x-3)^2}$ ----- (3)

Consider the parabola, $y^2 = 3x$, can be re-written as

$y = \sqrt{3x}$ ----- (4)

Let us drop a perpendicular from A on to x-axis. The base of the perpendicular is C = (3, 0)

Now, the area to be found will be the area is

Area of the required region = Area between the circle and the parabola at OA.

Area of OA= Area under circle OAC - Area under parabola OAC

Area of OA is

$$\text{Area of OA} = \int_0^3 y_{\text{circle}}\,dx - \int_0^3 y_{\text{parabola}}\,dx$$

$$= \int_0^3 \sqrt{9-(x-3)^2}\,dx - \int_0^3 \sqrt{3x}\,dx$$

[Using the formula, $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ and $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \left[\frac{(x-3)\sqrt{9-(x-3)^2}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{x-3}{3}\right)\right]_0^3 - \sqrt{3}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$

$$= \left\{\left[\frac{(3-3)\sqrt{9-(3-3)^2}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{3-3}{3}\right)\right] - \left[\frac{(0-3)\sqrt{9-(0-3)^2}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{0-3}{3}\right)\right]\right\} - \frac{2\sqrt{3}}{3}\left[3^{\frac{3}{2}} - 0^{\frac{3}{2}}\right]$$

$$= \left\{\left[\frac{0\sqrt{9-(3-3)^2}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{0}{3}\right)\right] - \left[\frac{(-3)\sqrt{9-9}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{-3}{3}\right)\right]\right\} - \frac{2\sqrt{3}}{3}\left[3\sqrt{3}\right]$$

$$= \left\{\left[\frac{9}{2}(0)\right] - \left[0 + \frac{9}{2}\sin^{-1}(-1)\right]\right\} - 6 = -\frac{9}{2}\left(-\frac{\pi}{2}\right) - 6 = \frac{9\pi}{4} - 6$$

$$= \frac{3}{4}(3\pi - 8)$$

[$\sin^{-1}(-1) = -90°$]

Area of the required region is $\frac{3}{4}(3\pi - 8)$ sq. units.

35. Question

Find the area bounded by the curve y=cos x between x=0 to x=2π.

Answer

Given

• Curve is y = cos x

• x = 0 and

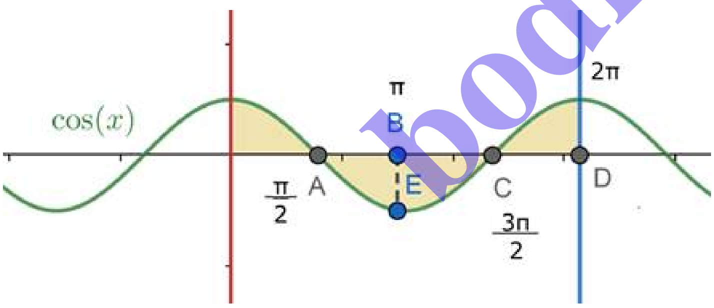
• x = 2π

The given curve is y = cos x.

Now consider the y values for some random x values between 0 and 2π for the function y = cos x.

X	Y
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0
�	-1
$\frac{3\pi}{2}$	0
2π	1

From the table we can clearly draw the graph for y = cos x



From the given curve, we can say that,

For $0 < x < \frac{\pi}{2}$, y = cos x

For $\frac{\pi}{2} < x < \frac{3\pi}{2}$, y = -cos x

For $\frac{3\pi}{2} < x < 2\pi$, y = cos x

The required area under the curve is given by:

Area required = Area under of OA + Area of ABC + Area under AC

$$Area\,required = \int_0^{\frac{\pi}{2}} \cos x \; dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-\cos x) \; dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x \; dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \; dx \; - \; \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \; dx \; + \; \int_{\frac{3\pi}{2}}^{2\pi} \cos x \; dx$$

$$= (\sin x)_0^{\frac{\pi}{2}} - (\sin x)_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + (\sin x)_{\frac{3\pi}{2}}^{2\pi}$$

[using the formula, $\int \cos x \, dx = \sin x$]

$$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right] + \left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$$

$$= [1 - 0] - [-1 - 1] + [0 - (-1)] = 1 + 2 + 1 = 4$$

[as $\sin\frac{\pi}{2} = 1$, sin 2π = 0, $\sin\frac{3\pi}{2} = -1$, sin 0 = 0]

Hence the required area of the curve y = cos x from x = 0 to x=2π is 4 sq. units.

36. Question

Using integration, find the area of the region in the first quadrant, enclosed by the x-axis, the line y=x and the circle $x^2+y^2=32$

Answer

Given the boundaries of the area to be found are,

• the circle, $x^2 + y^2 = 32$ ---(1)

• the line, y = x---- (2)

• Area should be in first quadrant.

From the equation, of the first circle, $x^2 + y^2 = 32$

• the vertex at (0,0) i.e. the origin

• the radius is 4√2 unit.

Now to find the point of intersection of (1) and (2), substitute y = x in (1)

$x^2 + x^2 = 32$

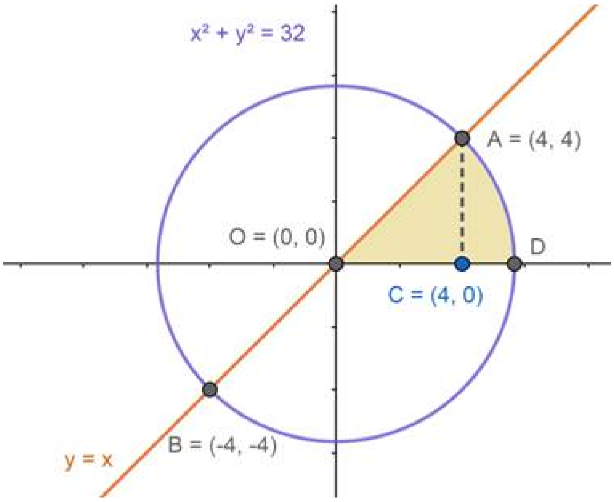
$2x^2 = 32$

$x^2 = 16$

x = ± 4

Substituting x in (2), we get y = ± 4

So the two points, A and B where (1) and (2) meet are A = (4,4) and B = (-4,-4)



As x and y have even powers for both the circles, they will be symmetrical about the x-axis and y-axis.

Consider the circle, $x^2 + y^2 = 32$, can be re-written as

$y^2 = 32 - x^2$

$y = \sqrt{(32 - x^2)}$ ----- (3)

Let us drop a perpendicular from A on to x-axis. The base of the perpendicular is C = (4, 0)

Now, the area to be found will be the area is

Area of the required region = Area of OADO.

Area of OADO= Area of OAC•+ Area of CADC

Area of OADOis

$$\text{Area of OADO} = \int_{4\sqrt{2}}^{4} \sqrt{32 - x^2}\,dx + \int_{0}^{4} x\,dx$$

$$= \left[\frac{x\sqrt{32 - x^2}}{2} + \frac{32}{2}\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_{4\sqrt{2}}^{4} + \left[\frac{x^2}{2}\right]_{0}^{4}$$

[Using the formula, $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ and $\int x^n dx = \frac{x^{n+1}}{n+1}$]

$$= \left\{\left[\frac{(4)\sqrt{32 - (4)^2}}{2} + \frac{32}{2}\sin^{-1}\left(\frac{4}{4\sqrt{2}}\right)\right] - \frac{(4\sqrt{2})\sqrt{32 - (4\sqrt{2})^2}}{2} - \frac{32}{2}\sin^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right)\right\} + \left\{\left[\frac{(4)^2}{2}\right] - \left[\frac{0^2}{2}\right]\right\}$$

$$= \left\{\left[\frac{(4)\sqrt{32 - 16}}{2} + \frac{32}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] - \frac{4\sqrt{2}\sqrt{32 - 32}}{2} - \frac{32}{2}\sin^{-1}(1)\right\} + \left\{\left[\frac{16}{2}\right]\right\}$$

[$\sin^{-1}(1) = 90°$ and $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0$]

$$= \left[8 + 16\left(\frac{\pi}{4}\right)\right] - \left[\frac{32}{2}(0)\right] - \{16\} = 8 + 4\pi - 8 = 4\pi$$

$= 4\pi$

Area of the required region is 4π sq. units.

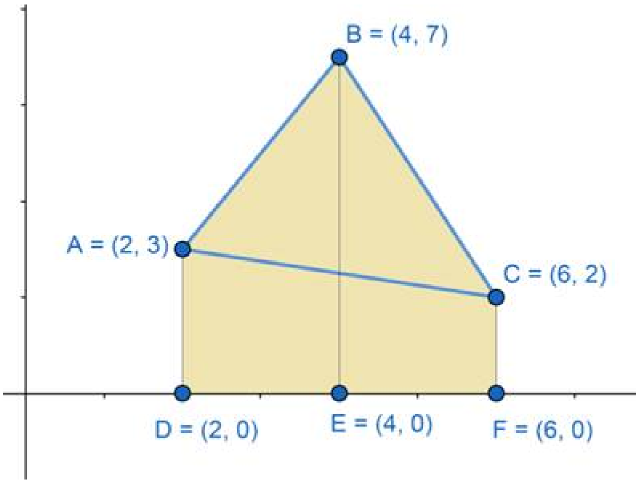
37. Question

Using integration, find the area of the triangle whose vertices are A(2,3), B(4,7) and C(6,2).

Answer

Given,

• A (2,3), B (4,7) and C (6,2) are the 3 vertices of a triangle.



From above figure we can clearly say that, the area between ABC and DEF is the area to be found.

For finding this area, we can consider the lines AB, BC and CA which are the sides of the given triangle. By calculating the area under these lines we can find the area of the complete region.

Consider the line AB,

If (x_1,y_1) and (x_2, y_2) are two points, the equation of a line passing through these points can be given by

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

Using this formula, equation of the line A(2,3) B =(4,7)

$$\frac{y-(3)}{7-3}=\frac{x-(2)}{4-(2)}$$

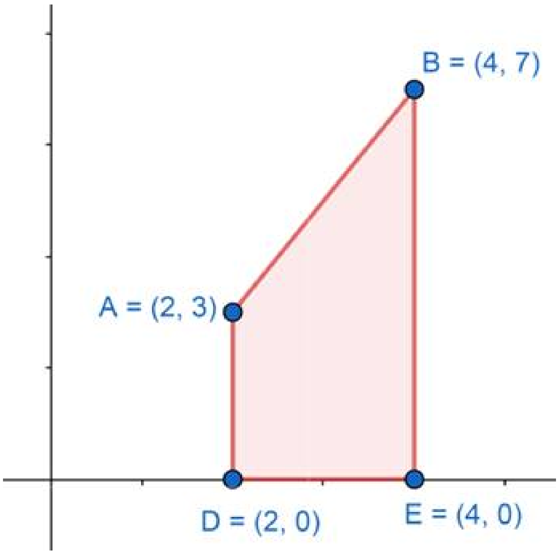
$$\frac{y-(3)}{4}=\frac{x-2}{2}$$

$$y=\frac{4}{2}(x-2)+3$$

y = 2x – 4 + 3

y = 2x-1

Consider the area under AB:



From the above figure, the area under the line AB will be given by,

$$Area\ of\ ABED = \int_2^4 y\, dx = \int_2^4 (2x-1)\, dx$$

$$= \int_2^4 (2x-1)\, dx = 2\left[\frac{x^2}{2}\right]_2^4 - [x]_2^4 = [x^2 - x]_2^4$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\, dx = cx$]

= [4^2 - 4] – [2^2 - 2] = 12-2= 10

Area of ABDE = 10 sq. units. ---- (1)

Consider the line BC,

Using this 2-point formula for line, equation of the line B(4,7) and C (6,2)

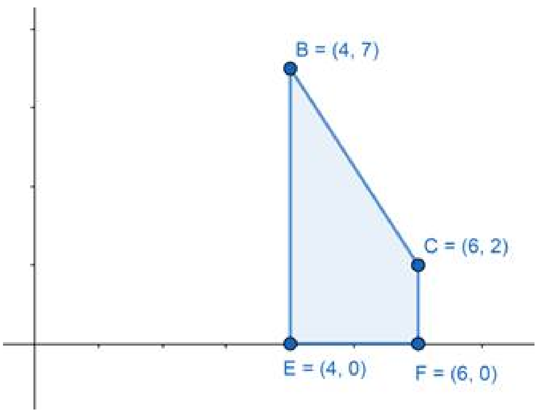
$$\frac{y-(7)}{2-7} = \frac{x-(4)}{6-(4)}$$

$$\frac{y-(7)}{-5} = \frac{x-4}{2}$$

$$y = \frac{5}{2}(4-x) + 7 = \frac{20-5x+14}{2} = \frac{34-5x}{2}$$

$$y = \frac{34-5x}{2}$$

Consider the area under BC:



From the above figure, the area under the line BC will be given by,

$$Area\ of\ BCDF = \int_4^6 y\,dx = \int_4^6 \left(\frac{34-5x}{2}\right) dx$$

$$= \int_4^6 \frac{1}{2}(34-5x)\,dx = \frac{1}{2}\left[34x - \frac{5x^2}{2}\right]_4^6$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

$$= \frac{1}{2}\left\{\left[34(6) - \frac{5(6)^2}{2}\right] - \left[34(4) - \frac{5(4)^2}{2}\right]\right\}$$

$$= \frac{1}{2}\left\{\left[204 - \frac{180}{2}\right] - \left[136 - \frac{80}{2}\right]\right\} = \frac{1}{2}[204-90] - \frac{1}{2}[136-40]$$

$$= \frac{114-96}{2} = 9$$

Area of BCFE = 9 sq. units. ---- (2)

Consider the line CA,

Using this 2-point formula for line, equation of the line C(6,2)and A(2,3)

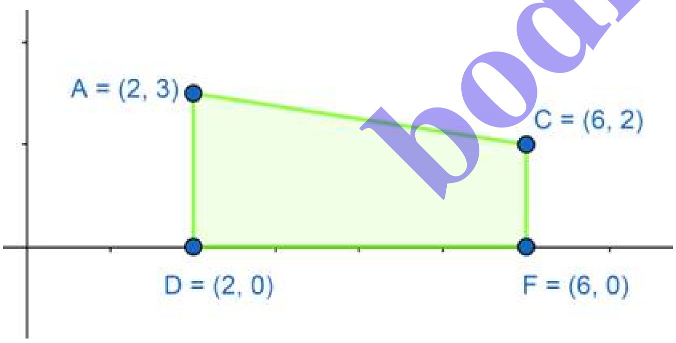
$$\frac{y-(2)}{3-2} = \frac{x-(6)}{2-(6)}$$

$$\frac{y-(2)}{1} = \frac{x-6}{-4}$$

$$y = \frac{1}{4}(6-x) + 2 = \frac{6-x+8}{4} = \frac{14-x}{4}$$

$$y = \frac{14-x}{4}$$

Consider the area under CA:



From the above figure, the area under the line CA will be given by,

$$Area\ of\ ACFE = \int_2^6 y\,dx = \int_2^6 \left(\frac{14-x}{4}\right) dx$$

$$= \int_2^6 \frac{1}{4}(14-x)\,dx = \frac{1}{4}\left[14x - \frac{x^2}{2}\right]_2^6$$

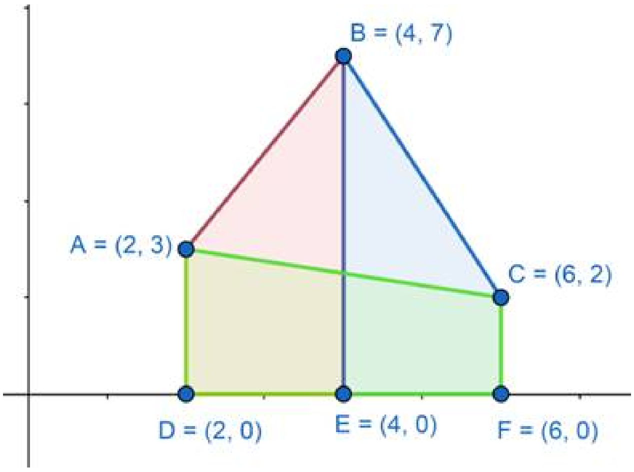
[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

$$= \frac{1}{4}\left\{\left[14(6) - \frac{6^2}{2}\right] - \left[14(2) - \frac{2^2}{2}\right]\right\}$$

$$= \frac{1}{4}\left\{\left[84 - \frac{36}{2}\right] - [28 - 2]\right\} = \frac{1}{4}[66 - 26] = \frac{40}{4} = 10$$

Area of ACFD = 10 sq.units ---- (3)

If we combined, the areas under AB, BC and AC in the below graph, we can clearly say that the area under AC (3) is overlapping the previous twoareas under AB & BC.



Now, the combined area under the rABC is given by

Area under rABC

=Area under AB + Area under BC - Area under AC

From (1), (2) and (3), we get

Area under rABC = 10+9-10 = 9

Therefore, area under rABC = 9 sq.units.

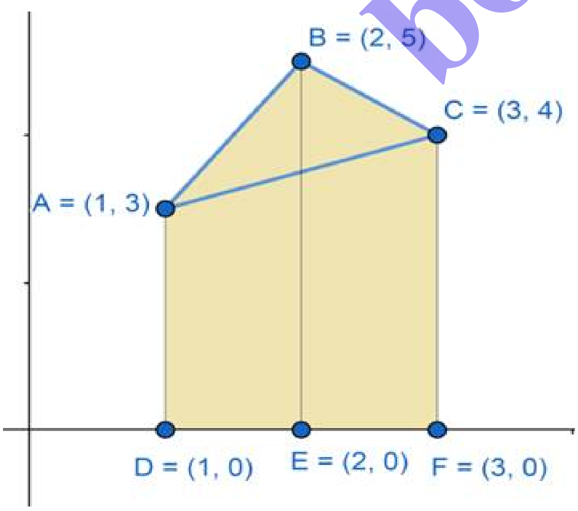
38. Question

Using integration, find the area of the triangle whose vertices are A(1,3), B(2,5) and C(3,4).

Answer

Given,

• A (1,3), B (2,5) and C (3,4) are the 3 vertices of a triangle.



From above figure we can clearly say that, the area between ABC and DEF is the area to be found.

For finding this area, we can consider the lines AB, BC and CA which are the sides of the given triangle. By calculating the area under these lines we can find the area of the complete region.

Consider the line AB,

If (x_1, y_1) and (x_2, y_2) are two points, the equation of a line passing through these points can be given by

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Using this formula, equation of the line A(1,3) B =(2,5)

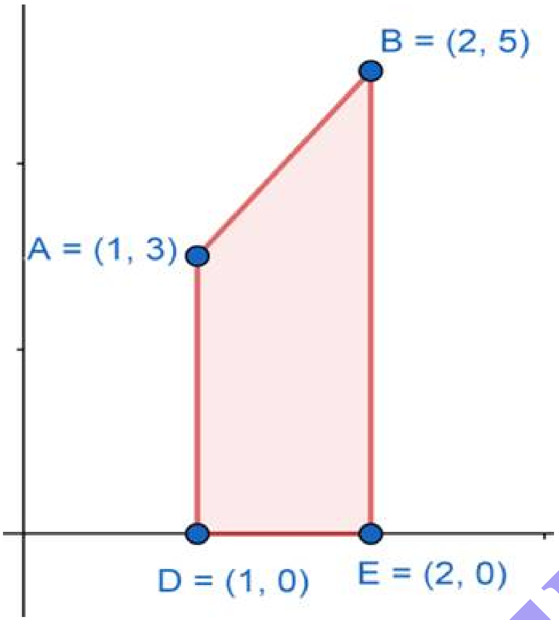
$$\frac{y-(3)}{5-3} = \frac{x-(1)}{2-(1)}$$

$$\frac{y-(3)}{2} = \frac{x-1}{1}$$

y = 2x – 2 + 3

y = 2x+1

Consider the area under AB:



From the above figure, the area under the line AB will be given by,

$$Area\ of\ ABED = \int_1^2 y\,dx = \int_1^2 (2x+1)\,dx$$

$$= \int_1^2 (2x+1)\,dx = 2\left[\frac{x^2}{2}\right]_1^2 + [x]_1^2 = [x^2 + x]_1^2$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

= $[2^2 + 2] - [1^2 + 1]$ = 6-2= 4

Area of ABDE = 4 sq. units. ---- (1)

Consider the line BC,

Using this 2-point formula for line, equation of the line B(2,5) and C (3,4)

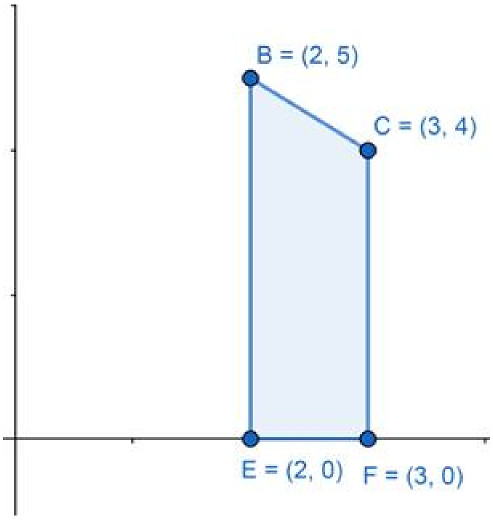
$$\frac{y-(5)}{4-5} = \frac{x-(2)}{3-(2)}$$

$$\frac{y-(5)}{-1} = \frac{x-2}{1}$$

y – 5 = 2 – x

y = 7-x

Consider the area under BC:



From the above figure, the area under the line BC will be given by,

$$Area\ of\ BCDF = \int_2^3 y\,dx = \int_2^3 (7-x)\,dx$$

$$= \int_2^3 (7-x)\,dx = \left[7x - \frac{x^2}{2}\right]_2^3$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

$$= \left\{\left[7(3) - \frac{(3)^2}{2}\right] - \left[7(2) - \frac{(2)^2}{2}\right]\right\}$$

$$= \left\{\left[21 - \frac{9}{2}\right] - [14-2]\right\} = \frac{42-9}{2} - 12 = \frac{33-24}{2} = \frac{9}{2}$$

Area of BCFE $= \frac{9}{2}$ sq. units. ---- (2)

Consider the line CA,

Using this 2-point formula for line, equation of the line C(3,4)and A(1,3)

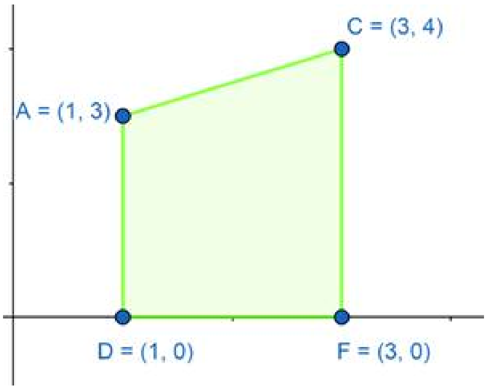
$$\frac{y-(4)}{3-4} = \frac{x-(3)}{1-(3)}$$

$$\frac{y-(4)}{-1} = \frac{x-3}{-2}$$

$$y = \frac{1}{2}(x-3) + 4 = \frac{x-3+8}{2} = \frac{x+5}{2}$$

$$y = \frac{x+5}{2}$$

Consider the area under CA:



From the above figure, the area under the line CA will be given by,

$$Area\ of\ ACFE = \int_1^3 y\,dx = \int_1^3 \left(\frac{x+5}{2}\right) dx$$

$$= \int_1^3 \frac{1}{2}(x+5)\,dx = \frac{1}{2}\left[\frac{x^2}{2} + 5x\right]_1^3$$

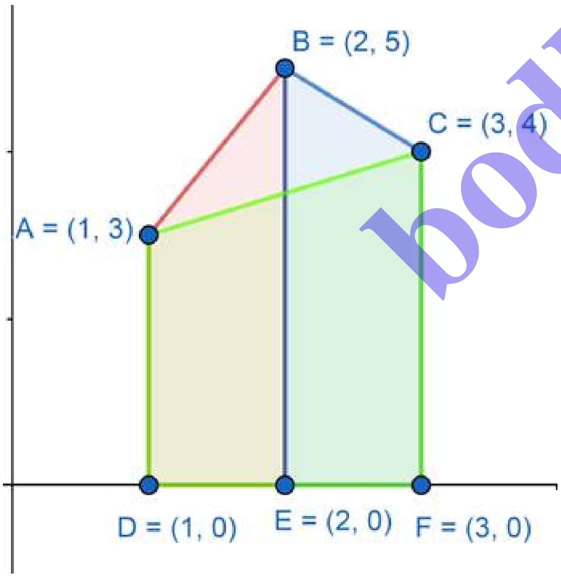
[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

$$= \frac{1}{2}\left\{\left[\frac{3^2}{2} + 5(3)\right] - \left[\frac{1^2}{2} + 5(1)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{9}{2} + 15\right] - \left[\frac{1}{2} + 5\right]\right\} = \frac{1}{2}\left[\frac{8}{2} + 10\right] = \frac{1}{2}(14) = 7$$

Area of ACFD = 7 sq.units ---- (3)

If we combined, the areas under AB, BC and AC in the below graph, we can clearly say that the area under AC (3) is overlapping the previous twoareas under AB & BC.



Now, the combined area under the rABC is given by

Area under rABC

=Area under AB + Area under BC - Area under AC

From (1), (2) and (3), we get

Area under rABC $= 4 + \frac{9}{2} - 7 = \frac{9}{2} - 3 = \frac{3}{2}$

Therefore, area under rABC $= \frac{3}{2}$ sq.units.

39. Question

Using integration, find the area of the triangular region bounded by the lines y=2x+1, y=3x+1 and x=4.

Answer

Given,

• ABC is a triangle

• Equation of side AB of y = 2x + 1

• Equation of side BC of y = 3x+1

• Equation of side CA of x=4

By solving AB & BC we get the point B,

AB : y = 2x+1 , BC: y = 3x+1

2x+1 = 3x+1

x = 0

by substituting x = 0 in AB we get y = 1

The point B = (0,1)

By solving BC & CA we get the point C,

AC : x = 4 , BC: y = 3x+1

y = 12+1 = 13

y = 13

The point C = (4,13)

By solving AB & AC we get the point A,

AB : y = 2x+1 , AC : x=4

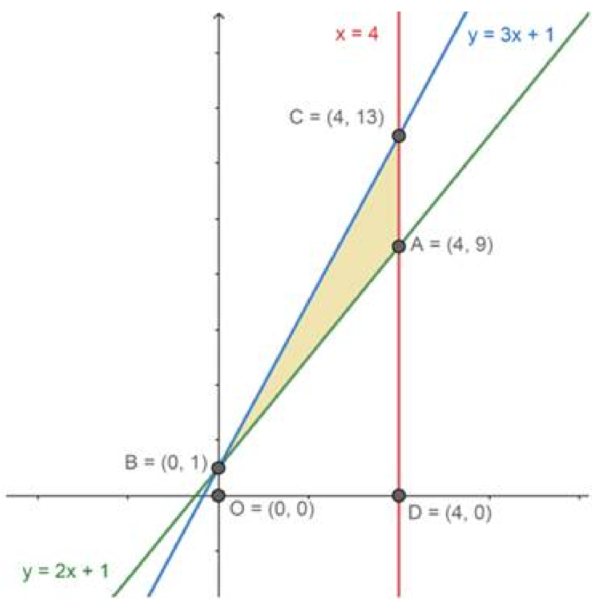
y = 8+1= 9

y = 9

The point A = (4,9)

These points are used for obtaining the upper and lower bounds of the integral.

From the given information, the area under the triangle (colored) can be given by the below figure.

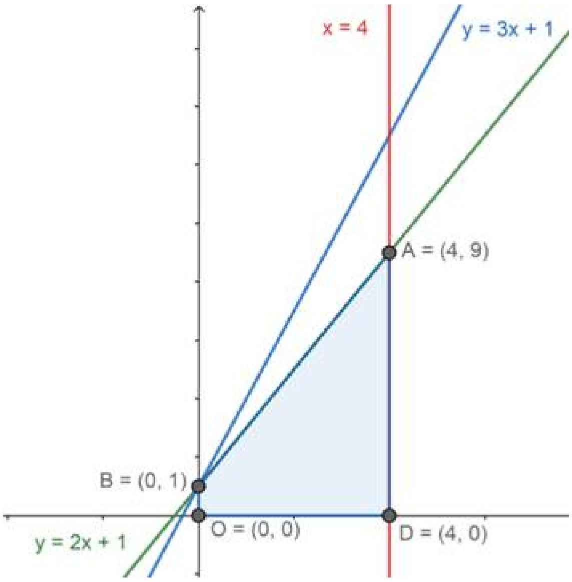


From above figure we can clearly say that, the area between ABC is the area to be found.

For finding this area, the line equations of the sides of the given triangle are considered. By calculating the area under these lines we can find the area of the complete region.

Consider the line AB, y = 4x+5

The area under line AB:



From the above figure, the area under the line AB will be given by,

$$Area\ of\ AB = \int_0^4 y_{AB}\,dx = \int_0^4 (2x+1)\,dx$$

$$= \int_0^4 (2x+1)\,dx = \left[\frac{2x^2}{2} + x\right]_0^4$$

[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

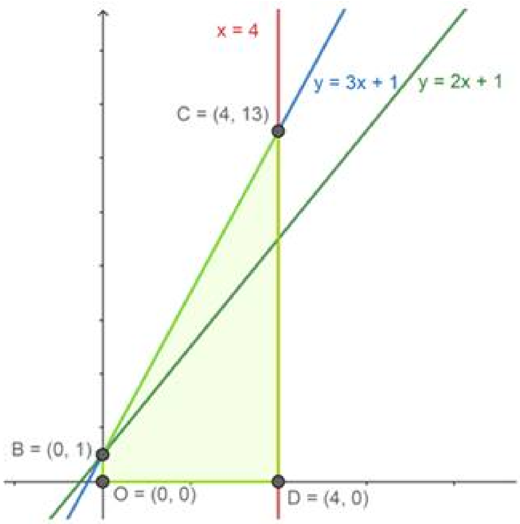
= {[(4^2) + (4)] – [$(0)^2$ + 0)]}

= 20

Area under AB = 20 sq. units. ---- (1)

Consider the line BC, y = 3x+1

Consider the area under BC:



From the above figure, the area under the line BC will be given by,

$$Area\ of\ BC = \int_0^4 y_{BC}\,dx = \int_0^4 (3x+1)\,dx$$

$$= \int_0^4 (3x+1)\,dx = \left[\frac{3x^2}{2} + x\right]_0^4$$

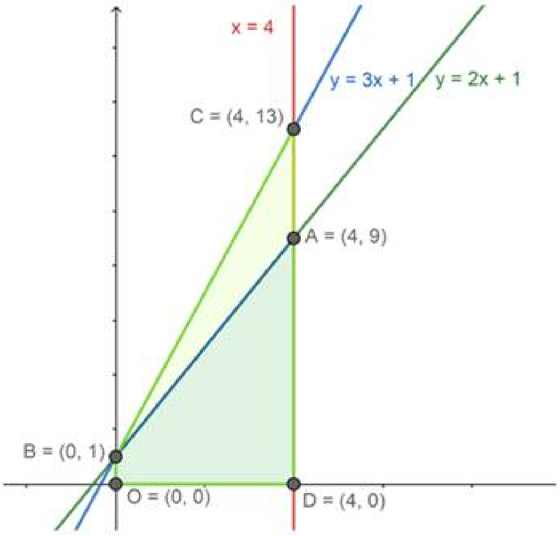
[using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int c\,dx = cx$]

$$= \left\{\left[\frac{3(4)^2}{2} + 4\right] - \left[\frac{(0)^2}{2} + (0)\right]\right\}$$

= 24+4-0= 28

Area under BC = 28 sq. units. ---- (2)

If we area under AB is removed from BC from the graph, we can obtain the area required.



Now, the combined area under the rABC is given by

Area under rABC =Area under BC - Area under AB

From (1), (2), we get

Area under rABC = 28 – 20 = 8

Therefore, area under rABC = 8 sq.units.