4 Probability

In section A, we have already studied the basic concepts of probability and solved a large number of examples. We learnt about events, theorems of events, independent and dependent events, conditional probability, multiplication law of probability and the law of total probability etc. In this chapter, we will take these ideas further, and also learn about random variables and their distributions.

4.1 BAYE'S THEOREM

If E_1, E_2, E_3, …, E_n are mutually exclusive and exhaustive events associated with a random experiment and A is any event associated with the experiment, then

$$P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\Sigma P(E_i)\, P(A \mid E_i)}, \text{ where } i = 1, 2, 3, \ldots, n.$$

Proof. By the law of total probability, we have

$$\begin{aligned} P(A) &= P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2) + \ldots + P(E_n)\, P(A \mid E_n) \\ &= \Sigma P(E_i)\, P(A \mid E_i) \qquad \ldots(i) \end{aligned}$$

Also by multiplication law of probability, we have

$$P(A \cap E_i) = P(A)\, P(E_i \mid A) = P(E_i)\, P(A \mid E_i),\ i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{P(A)},\ i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\Sigma P(E_i)\, P(A \mid E_i)},\ i = 1, 2, 3, \ldots, n \qquad \text{(by using } (i))$$

The probability $P(E_i \mid A)$ means finding the probability of event E_i given that event A has occurred. Probability $P(E_i)$ was already known — so it was ***a priori probability.*** $P(E_i \mid A)$ is to be calculated after the knowledge that event A has happened — so it is called ***posteriori probability***.

For example, suppose that in a factory, 60% products are manufactured by machine M_1 and 40% by machine M_2. Machine M_1 produces 1% defective items and machine M_2 produces 2% defective items, and let

E_1 = event that product is manufactured by machine M_1,

E_2 = event that product is manufactured by machine M_2 and

A = event that product is defective.

Then from given information, we have

$$P(E_1) = 0{\cdot}60,\ P(E_2) = 0{\cdot}40,\ P(A \mid E_1) = 0{\cdot}01,\ P(A \mid E_2) = 0{\cdot}02.$$

From law of total probability, we can calculate that the probability of a product being defective,

$$P(A) = P(E_1) \,.\, P(A \mid E_1) + P(E_2) \,.\, P(A \mid E_2)$$
$$= (0{\cdot}60)\ (0{\cdot}01) + (0{\cdot}40)\ (0{\cdot}02) = 0{\cdot}006 + 0{\cdot}008 = 0{\cdot}014.$$

Thus, 1·4% products of the factory are defective.

If the event A happens *i.e.* if we pick up a product and find that it is defective, $P(E_1 \mid A)$ means finding the probability that it was manufactured by machine M_1. You may think that it is 60% as we are given that 60% products are manufactured by machine M_1. However, according to Bayes' theorem,

$$P(E_1 \mid A) = \frac{P(E_1) \,.\, P(A/E_1)}{P(E_1) \,.\, P(A/E_1) + P(E_2) \,.\, P(A/E_2)}$$

$$= \frac{(0{\cdot}60)\,(0{\cdot}01)}{(0{\cdot}60)\,(0{\cdot}01) + (0{\cdot}40)\,(0{\cdot}02)} = \frac{0{\cdot}006}{0{\cdot}006 + 0{\cdot}008} = \frac{3}{7} \ \textit{i.e. } 43\% \text{ approximately.}$$

Thus if we pick up a product, there is 60% chance that it came from machine M_1 and 40% chance that it came from machine M_2. However, when we examine the product and find it to be defective — we revise our probabilities (from $P(E_1)$ to $P(E_1 \mid A)$ etc.) and say that there is 43% chance that this product came from machine M_1 and 57% chance that it came from machine M_2. Thus, *a priori* probability of 60% is revised to *posteriori* probability 43% with the *additional information* that the product is defective. This is what real life is all about. When we receive additional information, we revise our first opinions.

ILLUSTRATIVE EXAMPLES

Example 1. *Bag I contains 2 white and 3 red balls and bag II contains 4 white and 5 red balls. A bag is taken at random and a ball is drawn from it. If the ball drawn is red, find the probability that it was drawn from bag I.*

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = bag I is taken,

E_2 = bag II is taken and

A = ball drawn is red.

Then $P(E_1) = \frac{1}{2} = P(E_2)$.

$P(A \mid E_1)$ = P(drawing a red ball from bag I) = $\frac{3}{5}$,

$P(A \mid E_2)$ = P(drawing a red ball from bag II) = $\frac{5}{9}$.

We want to find $P(E_1 \mid A)$.

By using Baye's theorem, we have

$$P(E_1 \mid A) = \frac{P(E_1)\, P(A \mid E_1)}{P(E_1) P(A \mid E_1) + P(E_2)\, P(A \mid E_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{3}{5}}{\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{5}{9}} = \frac{\frac{3}{5}}{\frac{3}{5} + \frac{5}{9}} = \frac{3}{5} \cdot \frac{45}{52} = \frac{27}{52}.$$

Example 2. *There are two groups of subjects, one of which consists of 5 science and 3 Engineering subjects; and the other consists of 3 Science and 5 Engineering subjects. An unbiased die is rolled. If number 1 or 6 turn up, a subject is selected at random from the first group, otherwise a subject is selected from the second group. If ultimately an Engineering subject is selected, find the probability that it is selected from second group.*

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = die turns up with number 1 or 6 *i.e.* selecting first group of subjects;

E_2 = die turns up with number 2, 3, 4 or 5 *i.e.* selecting second group of subjects and

A = Engineering subject is selected.

Then $P(E_1) = \frac{2}{6} = \frac{1}{3}$ and $P(E_2) = \frac{4}{6} = \frac{2}{3}$.

$P(A \mid E_1)$ = P(selecting Engineering subject from first group) $= \frac{3}{8}$,

$P(A \mid E_2)$ = P(selecting Engineering subject from second group) $= \frac{5}{8}$.

We want to find $P(E_2 \mid A)$.

By Baye's theorem, we have

$$P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{P(E_1)P(A \mid E_1) + P(E_2)\,P(A \mid E_2)}$$

$$= \frac{\frac{2}{3}\cdot\frac{5}{8}}{\frac{1}{3}\cdot\frac{3}{8} + \frac{2}{3}\cdot\frac{5}{8}} = \frac{10}{13}.$$

Example 3. *An insurance company insured 1500 scooter drivers, 2500 car drivers and 4500 truck drivers. The probability of a scooter, a car and a truck driver meeting with an accident is 0·01, 0·02 and 0·04 respectively. If one of the insured person meets with an accident, find the probability that he is a scooter driver.* *(I.S.C. 2007)*

Solution. Let E_1, E_2, E_3 and A be the events defined as follows :

E_1 = insured person is a scooter driver,

E_2 = insured person is a car driver,

E_3 = insured person is a truck driver and

A = person meets with an accident.

Total number of insured persons = 1500 + 2500 + 4500 = 8500.

$\therefore$ $P(E_1) = \frac{1500}{8500} = \frac{3}{17}$, $P(E_2) = \frac{25}{8500} = \frac{5}{17}$, $P(E_3) = \frac{4500}{8500} = \frac{9}{17}$ and

$P(A \mid E_1)$ = P(a scooter driver meeting with an accident) = 0·01,

$P(A \mid E_2) = 0{\cdot}02$, $P(A \mid E_3) = 0{\cdot}04$.

We want to find $P(E_1 \mid A)$.

By Baye's theorem, we have

$$P(E_1 \mid A) = \frac{P(E_1)\,P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)\,P(A \mid E_2) + P(E_3)\,P(A \mid E_3)}$$

$$= \frac{\frac{3}{17} \times 0{\cdot}01}{\frac{3}{7} \times 0{\cdot}01 + \frac{5}{17} \times 0{\cdot}02 + \frac{9}{17} \times 0{\cdot}04} = \frac{3 \times 0{\cdot}01}{3 \times 0{\cdot}01 + 5 \times 0{\cdot}02 + 9 \times 0{\cdot}04}$$

$$= \frac{0{\cdot}03}{0{\cdot}49} = \frac{3}{49}.$$

Example 4. *There are two groups of bags. Group I consists of 3 bags, each containing 5 green balls and 3 blue balls. Group II consists of 2 bags, each containing 2 green and 4 blue balls. A bag has been selected and a ball has been drawn at random from one of the bags and is found to be green. Find the probability that this green ball has been drawn from a bag of group II.*

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = selecting a bag from group I,

E_2 = selecting a bag from group II and

A = green ball has been drawn.

Since there are 5 bags and group I consists of 3 bags and group II consists of 2 bags, therefore,

$$P(E_1) = \frac{3}{5} \text{ and } P(E_2) = \frac{2}{5}.$$

If E_1 has occurred, then a bag from group I has been chosen. The bag chosen contains 5 green balls and 3 blue balls,

$$\therefore \quad P(A \mid E_1) = \frac{5}{8}.$$

Similarly $P(A \mid E_2) = \frac{2}{6} = \frac{1}{3}.$

We want to find $P(E_2 \mid A)$.

By Baye's theorem, we have

$$P(E_2 \mid A) = \frac{P(E_2)P(A \mid E_2)}{P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2)}$$

$$= \frac{\frac{2}{5}.\frac{1}{3}}{\frac{3}{5}.\frac{5}{8}+\frac{2}{5}.\frac{1}{3}} = \frac{\frac{2}{15}}{\frac{3}{8}+\frac{2}{15}} = \frac{2}{15} \times \frac{120}{61} = \frac{16}{61}.$$

Example 5. *A letter is known to have come either from TATANAGAR or KOLKATA. On the envelope, only the two consecutive letters TA are visible. What is the probability that the letter has come from (i) KOLKATA (ii) TATANAGAR.*

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = letter has come from KOLKATA,

E_2 = letter has come from TATANAGAR and

A = two consecutive visible letters are TA.

Letter can come either from KOLKATA or TATANAGAR, so

$$P(E_1) = \frac{1}{2} = P(E_2).$$

The word KOLKATA has 7 letters, so there are 6 groups of two consecutive letters—KO, OL, LK, KA, AT, TA. Only one of these is 'TA'.

$\therefore \quad P(A \mid E_1)$ = probability of event A when E_1 has occurred *i.e.* when letter has come from KOLKATA

$$= \frac{1}{6}.$$

The word TATANAGAR has 9 letters, so there are 8 groups of two consecutive letters—TA, AT, TA, AN, NA, AG, GA, AR. Two out of these are 'TA'.

$\therefore \quad P(A \mid E_2)$ = probability of event A when E_2 has occurred *i.e.* when the letter has come from TATANAGAR

$$= \frac{2}{8} = \frac{1}{4}.$$

(*i*) We want to find $P(E_1 \mid A)$.

By Baye's theorem, we have

$$P(E_1 \mid A) = \frac{P(E_1)\,P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

$$= \frac{\frac{1}{2}.\frac{1}{6}}{\frac{1}{2}.\frac{1}{6}+\frac{1}{2}.\frac{1}{4}} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{4}} = \frac{1}{6} \times \frac{12}{5} = \frac{2}{5}.$$

(*ii*) We want to find $P(E_2 \mid A)$.

By Baye's theorem, we have

$$P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

$$= \frac{\frac{1}{2}.\frac{1}{4}}{\frac{1}{2}.\frac{1}{6} + \frac{1}{2}.\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{1}{6} + \frac{1}{4}} = \frac{1}{4} \times \frac{12}{5} = \frac{3}{5}.$$

Example 6. *For A, B and C the chances of being selected as the manager of a firm are 4 : 1 : 2 respectively. The respective probabilities for them to introduce a radical change in marketing strategy are 0·3, 0·8 and 0·5 respectively. If the change does takes place, find the probability that it is due to the appointment of B.* *(I.S.C. 2013)*

Solution. Let E_1, E_2, E_3 and E be the events as defined below :

E_1 = A is selected as manager.

E_2 = B is selected as manager,

E_3 = C is selected as manager and

E = radical change occurs in marketing strategy.

$$P(E_1) = \frac{4}{4+1+2} = \frac{4}{7},\; P(E_2) = \frac{1}{4+1+2} = \frac{1}{7} \text{ and } P(E_3) = \frac{2}{4+1+2} = \frac{2}{7}.$$

Given $P(E \mid E_1) = 0{\cdot}3$, $P(E \mid E_2) = 0{\cdot}8$, $P(E \mid E_3) = 0{\cdot}5$.

We want to find the probability that the radical change in marketing strategy has occurred due to the appointment of B *i.e.* we want to find $P(E_2 \mid E)$.

By Baye's theorem, we have

$$P(E_2 \mid E) = \frac{P(E_2)\,P(E \mid E_2)}{P(E_1)P(E \mid E_1) + P(E_2)\,P(E \mid E_2) + P(E_3)\,P(E \mid E_3)}$$

$$= \frac{\frac{1}{7} \times 0{\cdot}8}{\frac{4}{7} \times 0{\cdot}3 + \frac{1}{7} \times 0{\cdot}8 + \frac{2}{7} \times 0{\cdot}5} = \frac{0{\cdot}8}{1{\cdot}2 + 0{\cdot}8 + 1}$$

$$= \frac{0{\cdot}8}{3} = \frac{8}{30} = \frac{4}{15}.$$

Example 7. *A bag contains 3 green and 7 white balls. Two balls are drawn one by one at random without replacement. If the second ball drawn is green, what is the probability that the first ball drawn is also green?*

Solution. Let E_1, E_2 and A be the events as defined below :

E_1 = first ball drawn is green,

E_2 = first ball drawn is white and

A = second ball drawn is green.

As the bag contains 3 green and 7 white balls,

$$P(E_1) = \frac{3}{10} \text{ and } P(E_2) = \frac{7}{10}.$$

The second ball is drawn from the bag without replacement.

When E_1 has occurred *i.e.* when a green ball has been drawn, then the bag contains 9 balls out of which 2 are green and 7 are white, so

$P(A \mid E_1)$ = probability of drawing second green ball when one green ball has already been drawn

$$= \frac{2}{9}.$$

When E_2 has occurred *i.e.* when a white ball has been drawn, then the bag contains 9 balls out of which 3 are green and 6 are white, so

$P(A \mid E_2)$ = probability of drawing second green ball when one white ball has already been drawn

$$= \frac{3}{9} = \frac{1}{3}.$$

We want to find $P(E_1 \mid A)$.

By Baye's theorem, we have

$$P(E_1 \mid A) = \frac{P(E_1)\, P(A \mid E_1)}{P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2)}$$

$$= \frac{\frac{3}{10} \cdot \frac{2}{9}}{\frac{3}{10} \cdot \frac{2}{9} + \frac{7}{10} \cdot \frac{1}{3}} = \frac{\frac{2}{3}}{\frac{2}{3} + \frac{7}{3}} = \frac{2}{3} \times \frac{3}{9} = \frac{2}{9}.$$

Example 8. *Bag I contains 3 white and 4 black balls and bag II contains 4 white and 5 black balls. One ball is transferred from bag I to bag II and then a ball is drawn at random from bag II. If the ball so drawn is white, find the probability that the transferred ball is black.*

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = black ball is transferred from bag I to bag II,

E_2 = white ball is transferred from bag I to bag II and

A = white ball has been drawn from bag II.

Then $P(E_1) = \frac{4}{7}$ and $P(E_2) = \frac{3}{7}$.

When E_1 has occurred *i.e.* when a black ball has been transferred from bag I to bag II, then the bag II has 4 white and 6 black balls.

$\therefore$ $P(E \mid E_1)$ = probability of drawing a white ball from bag II when E_1 has occurred

$$= \frac{4}{10} = \frac{2}{5}.$$

When E_2 has occurred *i.e.* when a white ball has been transferred from bag I to bag II, then the bag II has 5 white and 5 black balls.

$\therefore$ $P(A \mid E_2)$ = probability of drawing a white ball from bag II when E_2 has occurred

$$= \frac{5}{10} = \frac{1}{2}.$$

We want to find $P(E_1 \mid A)$.

By Baye's theorem, we have

$$P(E_1 \mid A) = \frac{P(E_1) P(A \mid E_1)}{P(E_1)\, P(A \mid E_1) + P(E_2) P(A \mid E_2)}$$

$$= \frac{\frac{4}{7} \cdot \frac{2}{5}}{\frac{4}{7} \cdot \frac{2}{5} + \frac{3}{7} \cdot \frac{1}{2}} = \frac{8}{35} \times \frac{70}{31} = \frac{16}{31}.$$

Example 9. *Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin 3 times and notes the number of heads. If she gets a 1, 2, 3, or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw a 1, 2, 3 or 4 with the die?*

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = girl gets 5 or 6 on throw of a die,

E_2 = girl gets 1, 2, 3 or 4 on throw of a die and

A = the girl gets exactly one head.

$$P(E_1) = \frac{2}{6} = \frac{1}{3} \text{ and } P(E_2) = \frac{4}{6} = \frac{2}{3}.$$

When E_1 has occurred, the girl tosses a coin three times and the sample space for this experiment is {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}. Thus sample space contains 8 equally likely outcomes, out of which exactly one head occurs 3 times.

$\therefore$ $P(A \mid E_1)$ = probability of getting exactly one head when E_1 has already occurred

$= \frac{3}{8}$.

When E_2 has occurred, the girl tosses a coin once and the sample space for this experiment = {H, T}.

$\therefore$ $P(A \mid E_2)$ = probability of getting exactly one head when E_2 has already occurred

$= \frac{1}{2}$.

We want to find $P(E_2 \mid A)$.

By Baye's theorem, we get

$$P(E_2 \mid A) = \frac{P(E_2)\, P(A \mid E_2)}{P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2)}$$

$$= \frac{\frac{2}{3} \cdot \frac{1}{2}}{\frac{1}{3} \cdot \frac{3}{8} + \frac{2}{3} \cdot \frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{8} + \frac{1}{3}} = \frac{1}{3} \times \frac{24}{11} = \frac{8}{11}.$$

Example 10. *A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.*

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = die shows six *i.e.* six has occurred,

E_2 = die does not show six *i.e.* six has not occurred and

A = the man reports that six has occurred.

We wish to calculate the probability that six has actually occurred given that the man reports that six occurs *i.e.* $P(E_1 \mid A)$.

Now, $P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{5}{6}$,

$P(A \mid E_1)$ = probability that the man reports that six occurs given that six has occurred

= probability that the man is telling the truth = $\frac{3}{4}$ and

$P(A \mid E_2)$ = probability that the man reports that six occurs given that six has not occurred

= probability that the man does not speak truth = $\frac{1}{4}$.

By Bayes' theorem,

$$P(E_1 \mid A) = \frac{P(E_1)\, P(A \mid E_1)}{P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2)}$$

$$= \frac{\frac{1}{6} \cdot \frac{3}{4}}{\frac{1}{6} \cdot \frac{3}{4} + \frac{5}{6} \cdot \frac{1}{4}} = \frac{3}{8}.$$

Example 11. *In a test, an examinee either guess or copies or knows the answer to a multiple choice question with four choices and only one correct option. The probability that he makes a guess is* $\frac{1}{3}$. *The probability that he copies the answer is* $\frac{1}{6}$. *The probability that the answer is correct, given that he copied it, is* $\frac{1}{8}$. *Find the probability that he knows the answer to the question, given that he correctly answered it.*

Solution. Let E_1, E_2, E_3 and A be the events defined as follows :

E_1 = the examinee guesses the answer,

E_2 = the examinee copies the answer,

E_3 = the examinee knows the answer and

A = the examinee has answered the question correctly.

$$P(E_1) = \frac{1}{3},\ P(E_2) = \frac{1}{6} \text{ (given).}$$

As E_1, E_2 and E_3 are mutually exclusive and exhaustive events, $P(E_1) + P(E_2) + P(E_3) = 1$

$$\Rightarrow\ P(E_3) = 1 - P(E_1) - P(E_2) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}.$$

When E_1 has occurred, then the examinee guesses. Since there are four choices and only one is correct, the probability that he answers correctly given that he has made a guess is $\frac{1}{4}$ *i.e.* $P(A \mid E_1) = \frac{1}{4}$.

$P(A \mid E_2)$ = probability that he answers correctly given that he has copied

$$= \frac{1}{8}.$$

When E_3 has occurred *i.e.* the examinee knows the answer, then the probability that he answers correctly given that he knows the answer is 1 (sure event) *i.e.* $P(A \mid E_3) = 1$.

We want to find $P(E_3 \mid A)$.

By Baye's theorem, we have

$$P(E_3 \mid A) = \frac{P(E_3)\,P(A \mid E_3)}{P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2) + P(E_3)\,P(A \mid E_3)}$$

$$= \frac{\frac{1}{2}.1}{\frac{1}{3}.\frac{1}{4} + \frac{1}{6}.\frac{1}{8} + \frac{1}{2}.1} = \frac{\frac{1}{2}}{\frac{1}{12} + \frac{1}{48} + \frac{1}{2}}$$

$$= \frac{1}{2} \times \frac{48}{4 + 1 + 24} = \frac{24}{29}.$$

Example 12. *A bag contains 4 balls. Two balls are drawn at random and are found to be white. What is the probability that all balls are white?*

Solution. Let E_1, E_2, E_3 and A be the events defined as follows :

E_1 = the bag contains 2 white balls and 2 non-white balls

E_2 = the bag contains 3 white balls and one non-white ball

E_3 = the bag contains all four white balls and

A = two white balls have been drawn from the bag.

$$\text{Then } P(E_1) = \frac{1}{3},\ \ P(E_2) = \frac{1}{3},\ \ P(E_3) = \frac{1}{3}$$

$P(A \mid E_1)$ = probability of drawing 2 white balls when E_1 has occurred *i.e.* the bag contains 2 white and 2 non-white balls

$$= \frac{{}^2C_2}{{}^4C_2} = \frac{1}{6}$$

$P(A \mid E_2) =$ probability of drawing 2 white balls when E_2 has occurred *i.e.* the bag contains 3 white balls and one non-white ball

$$= \frac{{}^3C_2}{{}^4C_2} = \frac{3.2}{1.2} \times \frac{1.2}{4.3} = \frac{1}{2} \text{ and}$$

$P(A \mid E_3) =$ probability of drawing 2 white balls when E_3 has occurred *i.e.* the bag contains all four white balls

$$= \frac{{}^4C_2}{{}^4C_2} = 1.$$

We want to find $P(E_3 \mid A)$

By Baye's theorem, we get

$$P(E_3 \mid A) = \frac{P(E_3)P(A \mid E_3)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + P(E_3)P(A \mid E_3)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1} = \frac{1}{\frac{1}{6} + \frac{1}{2} + 1}$$

$$= \frac{1}{\frac{10}{6}} = \frac{6}{10} = \frac{3}{5}.$$

Hence, the required probability $= \frac{3}{5}$.

Example 13. *Coloured balls are distributed in three bags as shown in the following table :*

Bag	*Colour*		
	Black	*White*	*Red*
I	*1*	*2*	*3*
II	*2*	*4*	*1*
III	*4*	*5*	*3*

A bag is selected at random and then two balls are drawn from the selected bag. They happen to be black and red. What is the probability that they have come from bag III ?

Solution. Let E_1, E_2, E_3 and A be the events defined as follows :

E_1 = bag I is selected,

E_2 = bag II is selected,

E_3 = bag III is selected and

A = one black and one red ball has been drawn from the selected bag.

As the bags are selected at random,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}.$$

Two balls are drawn randomly from the selected bag.

Total number of balls in bag I = 1 + 2 + 3 = 6.

$P(A \mid E_1) =$ probability of drawing one black and one red ball when bag I has been selected

$$= \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1 \times 3}{15} = \frac{1}{5}.$$

Total number of balls in bag II = 2 + 4 + 1 = 7.

$P(A \mid E_2) =$ probability of drawing one black and one red ball when bag II has been selected

$$= \frac{{}^2C_1 \times {}^1C_1}{{}^7C_2} = \frac{2 \times 1}{21} = \frac{2}{21}.$$

Total number of balls in bag III = 4 + 5 + 3 = 12.

$P(A\,|\,E_3)$ = probability of drawing one black and one red ball when bag III has been selected

$$= \frac{{}^{4}C_1 \times {}^{3}C_1}{{}^{12}C_2} = \frac{4\times 3}{66} = \frac{2}{11}.$$

We want to find $P(E_3\,|\,A)$.

By Baye's theorem, we have

$$P(E_3\,|\,A) = \frac{P(E_3)\,P(A\,|\,E_3)}{P(E_1)P(A\,|\,E_1)+P(E_2)P(A\,|\,E_2)+P(E_3)P(A\,|\,E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{2}{11}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{2}{21}+\frac{1}{3}\times\frac{2}{11}}$$

$$= \frac{\frac{2}{11}}{\frac{1}{5}+\frac{2}{21}+\frac{2}{11}} = \frac{\frac{2}{11}}{\frac{231+110+210}{1155}}$$

$$= \frac{2}{11}\times\frac{1155}{551} = \frac{2\times 105}{551} = \frac{210}{551}.$$

Example 14. *A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both clubs. Find the probability of the lost card being a club.*

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = lost card is of clubs,

E_2 = lost card is not of clubs and

A = two cards drawn are both of clubs.

Then $P(E_1) = \frac{13}{52} = \frac{1}{4}$ and $P(E_2) = \frac{39}{52} = \frac{3}{4}$.

When one card is lost, number of remaining cards in the pack = 51.

When E_1 has occurred *i.e.* a card of clubs is lost, then the probability of drawing 2 cards of clubs from the remaining pack = $\frac{{}^{12}C_2}{{}^{51}C_2}$,

so $$P(A\,|\,E_1) = \frac{{}^{12}C_2}{{}^{51}C_2} = \frac{12.11}{1.2} = \frac{1.2}{51.50} = \frac{22}{425}.$$

When E_2 has occurred *i.e.* when a card of clubs is not lost, then the probability of drawing 2 cards of clubs from the remaining pack = $\frac{{}^{13}C_2}{{}^{51}C_2}$,

so $$P(A\,|\,E_2) = \frac{{}^{13}C_2}{{}^{51}C_2} = \frac{13.12}{1.2} = \frac{1.2}{51.50} = \frac{26}{425}.$$

We want to find $P(E_1\,|\,A)$.

By Baye's theorem, we have

$$P(E_1\,|\,A) = \frac{P(E_1)\,P(A\,|\,E_1)}{P(E_1)\,P(A\,|\,E_1)+P(E_2)\,P(A\,|\,E_2)}$$

$$= \frac{\frac{1}{4}\cdot\frac{22}{425}}{\frac{1}{4}\cdot\frac{22}{425}+\frac{3}{4}\cdot\frac{26}{425}} = \frac{22}{22+78} = \frac{22}{100} = \frac{11}{50}.$$

EXERCISE 4.1

1. Bag A contains 2 white and 4 red balls and bag B contains 5 white and 3 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag B.

2. A bag contains 4 red and 4 black balls. Another bag contains 2 red and 6 black balls. One of the bags is selected at random and a ball is drawn from the bag. If the ball is red, what is the probability that it came from the first bag ?

3. A box contains 3 blue and 2 red balls while another box contains 2 blue and 5 red balls. A box is picked up randomly and a ball is drawn out.

(*i*) What are the chances of getting a blue ball ?

(*ii*) If the ball drawn is blue, what is the probability that it came from the first box ?

4. A class consists of 50 students out of which there are 10 girls. In the class 2 girls and 5 boys are rank holders in an examination. If a student is selected at random from the class and is found to be a rank holder, what is the probability that the selected student is a girl? (*I.S.C. 2008*)

5. Suppose 5 men out of 100 and 25 women out of 1000 are good orators. An orator is chosen at random. Find the probability that a male person is selected. Assume that there are equal number of men and women.

6. There are 4 boys and 2 girls in room number 1 and 5 boys and 3 girls in room number 2. A girl from one of the rooms laughed loudly. What is the probability that the girl who laughed loudly was from room number 2?

7. It is known that 60% students in a college reside in hostel while remaining 40% are day scholars. 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade. If one randomly chosen student has A grade, what is the probability that he lives in the hostel?

8. A company has two plants to manufacture scooters. Plant I manufactures 80% of the scooters and plant II manufactures 20%. At plant I, 85 out of 100 scooters are rated as of standard quality and at plant II, 65 out of 100 scooters are rated as of standard quality. A scooter is chosen at random and is found to be of standard quality. What is the probability that it has come from plant I? (*I.S.C. 2006*)

9. In a class, 5% of the boys and 10% of the girls have an IQ of more than 150. In this class, 60% of the students are boys. If a student is selected at random and found to have an IQ of more than 150, find the probability that the student is a boy.

10. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer. Assuming that the student who guesses at the answer will be correct with a probability $\frac{1}{4}$, what is the probability that he knew the answer given that he answered it correctly?

11. (*i*) Anshul speaks truth 4 times out of 5. A coin is tossed. Anshul reports that it is a head. What is the probability that it is actually a head?

(*ii*) A speaks truth 8 times out of 10 times. A die is rolled. He reports that it was 5. What is the probability that it was actually 5?

(*iii*) A man is known to speak truth 3 times out of 5 times. He throws a die and reports that it a number greater than 4. Find the probability that it is actually a number greater than 4.

12. There are 4 bags each containing 6 white balls and 3 black balls, and 3 bags each containing 2 white and 4 black balls. If a bag is picked up and a black ball is drawn, what is the chance that it came from the first group ?

13. Urn I has 2 white and 3 black balls, urn II has 4 white and 1 black ball and urn III has 3 white and 4 black balls. An urn is selected at random and a ball is drawn at random.

(*i*) What is the probability of drawing a white ball ?

(*ii*) If the ball drawn is white, what is the probability that urn I was selected ?

14. In the above question, a die having 3 red, 2 yellow and 1 green face is thrown to select the urn. If a red face turns up, we pick urn I, if a yellow face turns up we pick urn II, otherwise we pick urn III. Then we draw a ball out of the urn.

(*i*) What is the probability of drawing a white ball ?

(*ii*) If the ball drawn is white, what is the probability that the die had turned up with a red face ?

15. Each of three identical jewellery boxes has two drawers. In each drawer of the first box there is a gold watch. In each drawer of the second box there is a silver watch. In one drawer of the third box there is a gold watch while in the other there is a silver watch. If we select a box at random, open one of the drawers and find it to contain a silver watch, what is the probability that the other drawer has the gold watch ?

16. A firm produces steel pipes in three plants A, B and C with daily production of 500, 1000 and 2000 units respectively. It is known that fractions of defective output produced by the three plants are respectively 0·005, 0·008 and 0·010. A pipe is selected at random from a day's total production and found to be defective. What is the probability that it came from first plant ? (*I.S.C. 2005, 2000*)

17. A factory has three machines A, B and C producing 1500, 2500 and 3000 bulbs per day, respectively. Machine A produces 1·5% defective bulbs, machine B produces 2% defective bulbs and machine C produces 2·5% defective bulbs. At the end of the day, a bulb is drawn at random and it is found to be defective. What is the probability that this defective bulb has been produced by machine B? (*I.S.C. 2010*)

18. An insurance company insured 4000 doctors, 8000 teachers and 12000 engineers. The probability of a doctor, a teacher and an engineer dying before the age of 58 years are 0·01, 0·03 and 0·05 respectively. If one of the insured person dies before the age of 58 years, find the probability that he is a doctor. (*I.S.C. 2009*)

19. There are three coins. One is a two headed coin, another is a biased coin that comes up tails 25% of the times and third is an unbiased coin. One of the three coins is chosen at random and tossed. If it shows heads, what is the probability that it was the two headed coin?

20. It is known that 40% of the students in a certain college are girls and 50% of the students are above the median height. If 2/3 of the boys are above the median height, what is the probability that a randomly selected student who is below median height is girl ?

Hint. Let number of students = 100, girls 40 and boys 60. $\frac{2}{3}$ of the boys are above median height $\Rightarrow$ $\frac{1}{3}$ of boys are below median height $\Rightarrow$ 20 boys are below median height $\Rightarrow$ 30 girls are below median height.

28. Three bags contain balls as shown in the table below :

Bag	*Number of white balls*	*Number of blue balls*	*Number of red balls*
I	1	2	3
II	2	1	1
III	4	3	2

A bag is chosen at random and two balls are drawn from it. They happen to be red and white. What is the probability that they came from bag III ?

29. Bag A contains 2 white, 1 black and 3 red balls; Bag B contains 3 white, 2 black and 4 red balls; Bag C contains 4 white, 3 black and 2 red balls. One bag is chosen at random and 2 balls are drawn at random from that Bag. If the randomly drawn balls are happened to be red and black, what is the probability that both balls come from Bag B? (*I.S.C. 2011*)

30. There are three urns. Urn I contains 1 white, 2 black and 3 red balls. Urn II contains 2 white, 1 black and 1 red ball. Urn III contains 4 white, 5 black and 3 red balls. One urn is chosen at random and two balls drawn without replacement. They happen to be white and red. What is the probability that they are from urn I, II or III ?

31. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart.

4.2 RANDOM VARIABLES/FUNCTIONS

A random variable is often described as a variable whose values are determined by chance i.e. by the outcomes of a random experiment.

If to each point of sample space we assign a real number, we then have a *function* defined on the sample space. It is called a *random function* or *stochastic function*. Loosely, it is called a *random variable* or *stochastic variable*. Thus, *a random variable is a function whose domain is the sample space of a random experiment and whose range is a subset of real numbers.*

For example, let S be the sample space of a simultaneous throw of two coins. Then S = {HH, HT, TH, TT}.

Let X denote the number of heads. Then X is a random variable. It can have values 0, 1 and 2 only. Thus, domain of X is S and range is {0, 1, 2}. Note that X(HT) = 1, X(TT) = 0 etc.

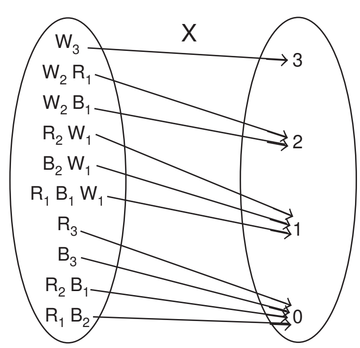
Let Y denote the number of tails. Then Y is also a random variable, having domain S and range {0, 1, 2}.

Note that though domain and range of X and Y is same, $X \neq Y$ because X(HH) = 2 while Y(HH) = 0. Also note that many other random variables can be defined over this sample space; for example, number of heads minus number of tails, square of the number of heads etc.

Some other examples of random variables are :

(*i*) Three balls are drawn simultaneously from a bag containing 5 red, 3 black and 4 white balls. Let X be a variable defined as the number of white balls drawn. Then X is a random variable with domain = sample space S = $\{W_3, R_3, B_3, W_2 R_1, W_2 B_1, R_2W_1, R_2 B_1, B_2 R_1, B_2W_1, B_1R_1W_1\}$ where W_1, W_2, W_3 denote one, two, three white balls respectively etc. and range = {0, 1, 2, 3}.

Note that $X(W_3) = 3$, $X(W_2 B_1) = 2$, $X(R_2 B_1) = 0$, $X(R_3) = 0$ etc. The random function X is shown in the adjoining diagram.



19. Five dice are thrown simultaneously. If the occurrence of an even number in a single die is considered success, find the probability of atmost three successes.

20. A pair of dice is thrown 6 times. Getting a total of 7 on the two dice is considered a success. Find the probability of getting

(*i*) atleast 5 successes (*ii*) exactly 5 successes.

21. An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be atleast 4 successes.

22. Suppose that a radio tube inserted into a certain type of set has a probability of 0·2 of functioning more than 500 hours. If we test 4 tubes, what is the probability that exactly k of them function for more than 500 hours, where $k = 0, 1, 2, 3$ and 4 ?

23. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random (*i*) 1 (*ii*) 0 (*iii*) less than 2 bolts will be defective.

24. A die is thrown 6 times. What is the probability that there will be (*i*) no ace (*ii*) not more than one ace (*iii*) more than 4 aces ? (Note that *ace* means a number 1, or one dot, on the upper face of the die).

25. The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he fire so that the probability of hitting the target atleast once is more than 0·99?

Hint. Let there be n trials, then $1 - \left(\frac{1}{4}\right)^n > \frac{99}{100} \Rightarrow \frac{1}{100} > \left(\frac{1}{4}\right)^n \Rightarrow 100 < 4^n \Rightarrow 4^n > 100$, which is satisfied if n is atleast 4.

26. An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

(*i*) all will bear 'X' mark

(*ii*) not more than 2 will bear 'Y' mark

(*iii*) atleast one ball will bear 'Y' mark

(*iv*) the number of balls with 'X' mark and 'Y' mark will be equal.

Hint. (*ii*) Required probability $= P(y \leq 2) = P(x \geq 4)$.

27. A certain brand of razor blades is sold in packets of 5. The following is the frequency distribution of packets according to the number of faulty blades in them :

Number of faulty blades	0	1	2	3	4 or more
Number of packets	80	17	2	1	0

Find the number of faulty blades per packet. Assuming that the distribution is binomial, estimate the probability that a blade taken at random from a packet will be faulty.

Hint. Mean $= np = 0\left(\frac{80}{100}\right) + 1\left(\frac{17}{100}\right) + 2\left(\frac{2}{100}\right) + 3\left(\frac{1}{100}\right) + 0 = \frac{6}{25}$.

ANSWERS

EXERCISE 4.1

1. $\frac{9}{25}$. **2.** $\frac{2}{3}$. **3.** (*i*) $\frac{31}{70}$ (*ii*) $\frac{21}{31}$. **4.** $\frac{2}{7}$.

5. $\frac{2}{3}$. **6.** $\frac{9}{17}$. **7.** $\frac{9}{13}$. **8.** $\frac{68}{81}$. **9.** $\frac{3}{7}$.

10. $\frac{12}{13}$. **11.** (*i*) $\frac{4}{5}$ (*ii*) $\frac{4}{9}$ (*iii*) $\frac{3}{7}$.

12. $\frac{2}{5}$. **13.** (*i*) $\frac{19}{35}$ (*ii*) $\frac{14}{57}$.

14. (*i*) $\frac{113}{210}$ (*ii*) $\frac{42}{113}$. **15.** $\frac{1}{3}$. **16.** $\frac{5}{61}$. **17.** $\frac{20}{59}$.

18. $\frac{1}{22}$. **19.** $\frac{4}{9}$. **20.** $\frac{3}{5}$. **21.** $\frac{90}{1089}$. **22.** $\frac{1}{2}$.

23. $\frac{1}{15}, \frac{2}{5}, \frac{8}{15}, 0$. **24.** $\frac{156}{947}$. **25.** $\frac{9}{58}$. **26.** $\frac{18}{67}$. **27.** $\frac{81}{85}$.

28. $\frac{5}{17}$. **29.** $\frac{170}{939}$. **30.** $\frac{33}{118}, \frac{55}{118}, \frac{30}{118}$. **31.** $\frac{11}{50}$.

EXERCISE 4.2

1. $-5, -3, -1, 1, 3, 5$. **2.** (*i*) $\frac{1}{10}$ (*ii*) $\frac{3}{10}, \frac{19}{100}, \frac{3}{10}$.

3. (*i*) $\frac{1}{6}$ (*ii*) $\frac{1}{2}, 1, \frac{1}{2}$.

4. (*i*) $\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$ (*ii*) $\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$.

6.

Z	0	1	2
P(Z)	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

7. 0, 1, 2; yes; $\begin{pmatrix} 0 & 1 & 2 \\ \frac{10}{21} & \frac{10}{21} & \frac{1}{21} \end{pmatrix}$.

8. $\begin{pmatrix} 0 & 1 & 2 \\ \frac{3}{28} & \frac{15}{28} & \frac{5}{14} \end{pmatrix}$. **9.** $\begin{pmatrix} 0 & 1 & 2 \\ \frac{188}{221} & \frac{32}{221} & \frac{1}{221} \end{pmatrix}$. **10.** $\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{4}{35} & \frac{18}{35} & \frac{12}{35} & \frac{1}{35} \end{pmatrix}$.

11. $\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{21} & \frac{5}{14} & \frac{10}{21} & \frac{5}{42} \end{pmatrix}$.

12.

X	0	1	2	3
P(X)	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$

13. $\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{30}{91} & \frac{45}{91} & \frac{15}{91} & \frac{1}{91} \end{pmatrix}$. **14.** $\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{28}{143} & \frac{70}{143} & \frac{40}{143} & \frac{5}{143} \end{pmatrix}$.

15. (*i*)

X	0	1	2	3
P(X)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(*ii*) $\frac{2}{3}$ (*iii*) $\frac{1}{3}$.

16.

X	0	1	2	3	4
P(X)	$\frac{194580}{270725}$	$\frac{69184}{270725}$	$\frac{6768}{270725}$	$\frac{192}{270725}$	$\frac{1}{270725}$

EXERCISE 4.3

1. (*i*) 2, 1, 1 (*ii*) 1·6, 2·24, 1·497 (approx.) (*iii*) 0, 1·2, 1·095 (approx.)

2. $\frac{7}{6}; \frac{17}{36}$. **3.** (*i*) 0·1 (*ii*) 0·8; 2·16. **4.** 0·7; 0·21.

5. $\frac{7}{2}; \frac{35}{12}$. **6.** $\begin{pmatrix} 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ \frac{2}{15} & \frac{1}{15} & \frac{2}{15} & \frac{3}{15} & \frac{1}{15} & \frac{2}{15} & \frac{3}{15} & \frac{1}{15} \end{pmatrix}; \frac{263}{15}, \frac{1076}{225}, \frac{2\sqrt{269}}{15}$.