3

Planes

INTRODUCTION

Intuitively we understand what a plane is and we know many of its properties. A particular plane can be specified in several ways, for example,

(*i*) one and only one plane can be drawn through three non-collinear points, therefore, three non-collinear points specify a particular plane.

(*ii*) one and only one plane can be drawn to contain two concurrent lines, therefore, two concurrent lines specify a particular plane.

(*iii*) one and only one plane can be drawn through two parallel lines, therefore, two parallel lines specify a particular plane.

(*iv*) one and only one plane can be drawn perpendicular to a given direction and at a given distance from the origin, therefore, a normal to a plane and the distance of the plane from the origin specify a particular plane.

(*v*) one and only one plane can be drawn through a given point and perpendicular to a given direction, therefore, a point on the plane and a normal to the plane specify a particular plane.

(*vi*) One and only one plane can be drawn through a given point and parallel to two given lines. Therefore, a point on the plane and two lines parallel to the plane specify a particular plane.

There are many other ways of specifying a particular plane but (*iv*) and (*v*) mentioned above are most useful as these lead to a very simple form of the **vector equation** of a plane.

Further, it is worthwhile to note that if a line is perpendicular to a plane, then it is perpendicular to every line in the plane.

Formally, a plane is a surface such that if any two (distinct) points are taken on it then the line containing these points lies completely in it i.e. every point of the line lies in it.

3.1 EQUATION OF A PLANE IN DIFFERENT FORMS

In this section, we shall find vector and cartesian equations of a plane in different forms :

3.1.1 Plane perpendicular to a given direction and at a given distance from the origin

Vector equation

Let a plane, say p_1, be at a distance p from the origin O and let $\hat{n}$ be a unit vector perpendicular to the plane ($\hat{n}$ being directed away from O) as shown in fig. 3.1.

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Let N be the foot of perpendicular from O to the plane p_1, then $\overrightarrow{ON} = p\,\hat{n}$.

Let P with position vector $\vec{r}$ be any point, then P lies in the plane iff

$$\overrightarrow{NP} \perp \hat{n}$$

(because $\hat{n}$ is perpendicular to the plane p_1, it is perpendicular to every line which lies in the plane)

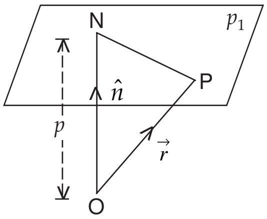

Fig. 3.1.

i.e. iff $\overrightarrow{NP} \bullet \hat{n} = 0$

i.e. iff $(\vec{r} - p\,\hat{n}) \bullet \hat{n} = 0$ *i.e.* iff $\vec{r} \bullet \hat{n} - p\,\hat{n} \bullet \hat{n} = 0$ but $\hat{n} \bullet \hat{n} = 1$

i.e. iff $\vec{r} \bullet \hat{n} = p$ ...(1)

which is the required equation of the plane p_1.

Cartesian equation

Let $< l, m, n >$ be the direction cosines of $\overrightarrow{ON}$ and P be (x, y, z), then

$$\hat{n} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k} \text{ and } \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}.$$

On substituting these values in (1), we get

$$(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}) \bullet (l\,\hat{i} + m\,\hat{j} + n\,\hat{k}) = p$$

i.e. $lx + my + nz = p$ ...(2)

This is known as **normal form.**

Corollary 1. Standard form of the vector equation

If the vector $\vec{n}$ is perpendicular to the plane p_1, then $\hat{n} = \dfrac{\vec{n}}{|\vec{n}|}$.

Therefore, the vector equation (1) of the plane p_1 can be written as $\vec{r} \bullet \dfrac{\vec{n}}{|\vec{n}|} = p$

or $\vec{r} \bullet \vec{n} = d$, where $d = |\vec{n}|\, p$.

The equation $\vec{r} \bullet \vec{n} = d$ is called the **standard form** of the vector equation of a plane.

Corollary 2. General form of the cartesian equation

If $<$ A, B, C $>$ are the direction numbers of the vector $\vec{n}$, then

$\vec{n} = \mathrm{A}\,\hat{i} + \mathrm{B}\,\hat{j} + \mathrm{C}\,\hat{k}$ and the equation $\vec{r} \bullet \vec{n} = d$ gives

$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = d \Rightarrow \mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$ where $\mathrm{D} = -d$.

Conversely, the equation $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$ can be written as $\vec{r} \bullet (\mathrm{A}\,\hat{i} + \mathrm{B}\,\hat{j} + \mathrm{C}\,\hat{k}) = d$ where $\mathrm{D} = -d$ *i.e.* $\vec{r} \bullet \vec{n} = d$, which represents a plane with $<$ A, B, C $>$ as direction numbers of a normal to the plane. It follows that every first degree equation in x, y, z *i.e.* $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$, A, B, C not all zero, represents a plane and $<$ A, B, C $>$ are direction numbers of a normal to this plane.

The equation $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$ is called the **general form** of the cartesian equation.

Direction numbers of any normal to a plane are called ***attitude numbers*** *of the plane.*

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3.1.2 Plane passing through a given point and perpendicular to a given direction

Vector form

Let a plane, say p_1, pass through the point A with position vector $\vec{a}$ and be perpendicular to a direction along the vector $\vec{n}$.

Let P with position vector $\vec{r}$ be any point, then P lies in the plane iff $\overrightarrow{AP} \perp \vec{n}$

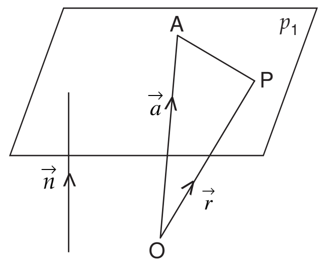

Fig. 3.2.

i.e. iff $\overrightarrow{AP} \cdot \vec{n} = 0$

i.e. iff $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ ...(1)

Cartesian form

Let the given point A be (x_1, y_1, z_1) and P be (x, y, z) and $< A, B, C >$ be the direction numbers of $\vec{n}$, then

$$\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}, \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \quad \vec{r} - \vec{a} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$$

and $\quad \vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$.

Substituting these values in (1), we get

$$((x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}) \cdot (A\hat{i} + B\hat{j} + C\hat{k}) = 0$$

i.e. $\quad A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$ ...(2)

This is known as **one point form.**

3.1.3 Position of a line with respect to a plane

To find condition(s) for a line to

(i) lie in a plane *(ii) be parallel to a plane* *(iii) meet a plane in a unique point.*

Vector form

Let the given line be $\vec{r} = \vec{a} + \lambda \vec{b}$ ...(1)

and the given plane be $\vec{r} \cdot \vec{n} = d$ ...(2)

Any point on the line (1) is $\vec{a} + \lambda \vec{b}$. This point will lie in the plane (2)

iff $(\vec{a} + \lambda \vec{b}) \cdot \vec{n} = d$

i.e. iff $(\vec{a} \cdot \vec{n} - d) + \lambda \vec{b} \cdot \vec{n} = 0$ ...(3)

(*i*) The line (1) lies in the plane (2) iff every point on the line (1) lies in the plane (2)

i.e. iff (3) is satisfied by every real value of λ *i.e.* iff (3) is an identity

i.e. iff $\vec{a} \cdot \vec{n} - d = 0$ and $\vec{b} \cdot \vec{n} = 0$

i.e. iff $\vec{a} \cdot \vec{n} = d$ and $\vec{b} \cdot \vec{n} = 0$.

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(*ii*) The line (1) is parallel to the plane (2) iff no point of the line (1) lies in the plane (2)

i.e. iff (3) is not satisfied by any real value of λ

i.e. iff $\vec{b} \cdot \vec{n} = 0$ and $\vec{a} \cdot \vec{n} - d \neq 0$

i.e. iff $\vec{b} \cdot \vec{n} = 0$ and $\vec{a} \cdot \vec{n} \neq d$.

(*iii*) The line (1) meets the plane (2) in a unique point iff (3) has a unique value of λ *i.e.* iff $\vec{b} \cdot \vec{n} \neq 0$.

Cartesian form

Let the given line be $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ ...(1)

and the given plane be $Ax + By + Cz + D = 0$ ...(2)

Any point on the line (1) is $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$.

This point lies in the plane $Ax + By + Cz + D = 0$

if $A(x_1 + a\lambda) + B(y_1 + b\lambda) + C(z_1 + c\lambda) + D = 0$

i.e. if $(Ax_1 + By_1 + Cz_1 + D) + (Aa + Bb + Cc)\lambda = 0$ ...(3)

(*i*) The line (1) lies in the plane (2) if every point on line (1) lies in the plane (2)

i.e. if (3) is satisfied by every real value of λ

i.e. if (3) is an identity

i.e. if $\boldsymbol{Ax_1 + By_1 + Cz_1 + D = 0}$ ***and*** $\boldsymbol{Aa + Bb + Cc = 0}$.

(*ii*) The line (1) is parallel to the plane (2) if no point of line (1) lies in the plane (2)

i.e. if (3) is not satisfied by any real value of λ

i.e. if $\boldsymbol{Aa + Bb + Cc = 0}$ ***and*** $\boldsymbol{Ax_1 + By_1 + Cz_1 + D \neq 0}$.

(*iii*) The line (1) meets the plane (2) in a unique point if (3) has a unique value of λ *i.e.* if $\boldsymbol{Aa + Bb + Cc \neq 0}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the vector equation of a plane which is at a distance of 7 units from the origin and which is normal to the vector* $3\hat{i} + 5\hat{j} - 6\hat{k}$.

Solution. Here $p = 7$ and $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$,

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{3^2 + 5^2 + (-6)^2}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

The vector equation of the plane is $\vec{r} \cdot \hat{n} = p$

i.e. $\vec{r} \cdot \dfrac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}} = 7$ or $\vec{r} \cdot (3\hat{i} + 5\hat{j} - 6\hat{k}) = 7\sqrt{70}$.

Example 2. *Find the direction cosines of the perpendicular from the origin to the plane* $\vec{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$.

Solution. The equation of the plane is

$\vec{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) = -1$ or $\vec{r} \cdot (-6\hat{i} + 3\hat{j} + 2\hat{k}) = 1$

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It is of the form $\vec{r} \bullet \vec{n} = d$, where $\vec{n}$ is the normal vector from origin to the plane.

Here $\vec{n} = -6\hat{i} + 3\hat{j} - 2\hat{k}$, therefore, direction numbers of the normal from origin to the plane are $< -6, 3, 2, >$.

Dividing by $\sqrt{(-6)^2 + 3^2 + 2^2} = 7$, the direction cosines of the normal from origin to the given plane are $< -\frac{6}{7}, \frac{3}{7}, \frac{2}{7} >$.

Example 3. *Find the length of perpendicular from origin to the plane* $\vec{r} \bullet (3\hat{i} - 4\hat{j} - 12\hat{k}) + 39 = 0$. *Also write the unit vector normal to this plane directed from origin to the plane.*

Solution. The equation of the plane is $\vec{r} \bullet (3\hat{i} - 4\hat{j} - 12\hat{k}) = -39$

or $\vec{r} \bullet (-3\hat{i} + 4\hat{j} + 12\hat{k}) = 39$

or $\vec{r} \bullet \vec{n} = 39$, where $\vec{n} = -3\hat{i} + 4\hat{j} + 12\hat{k}$

or $\vec{r} \bullet \frac{\vec{n}}{|\vec{n}|} = \frac{39}{|\vec{n}|}$, where $|\vec{n}| = \sqrt{(-3)^2 + 4^2 + 12^2} = 13$

or $\vec{r} \bullet \hat{n} = p$, where $p = \frac{39}{|\vec{n}|} = \frac{39}{13} = 3$.

Hence, the length of perpendicular from origin to the plane = 3 units and the unit vector normal to the given plane directed from origin to the plane

$$= \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{13}(-3\hat{i} + 4\hat{j} + 12\hat{k}).$$

Example 4. *Find the cartesian equation of the plane whose vector equation is*

$$\vec{r} \bullet (5\hat{i} + 7\hat{j} - 11\hat{k}) = 14.$$

Solution. The vector equation of the plane is

$$\vec{r} \bullet (5\hat{i} + 7\hat{j} - 11\hat{k}) = 14 \qquad \ldots(i)$$

Since $\vec{r}$ is the position vector of any point P (x, y, z) on the given plane, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Substituting this value of $\vec{r}$ in (i), we get

$$(x\hat{i} + y\hat{j} + z\hat{k}) \bullet (5\hat{i} + 7\hat{j} - 11\hat{k}) = 14$$

or $5x + 7y - 11z = 14$, which is the cartesian equation of the plane.

Example 5. *Find the vector equation of the plane whose cartesian equation is* $3x - 7y + 9z + 12 = 0$.

Solution. The cartesian equation of the plane is

$$3x - 7y + 9z + 12 = 0 \qquad \ldots(i)$$

If $\vec{r}$ is the position vector of any point P (x, y, z) on plane (i), then the equation (i) can be written as

$$(x\hat{i} + y\hat{j} + z\hat{k}) \bullet (3\hat{i} - 7\hat{j} + 9\hat{k}) + 12 = 0$$

or $\vec{r} \bullet (3\hat{i} - 7\hat{j} + 9\hat{k}) + 12 = 0$, which is the vector equation of the given plane.

Example 6. *In each of the following problems, determine the direction cosines of the normal to the plane and its distance from the origin :*

(i) $2x + 3y - z - 5 = 0$ *(ii)* $5y + 8 = 0$.

Solution. (i) The equation of the plane is $2x + 3y - z - 5 = 0$

It can be written as $2x + 3y - z = 5 \qquad \ldots(1)$

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If $\vec{r}$ is the position vector of any point P (x, y, z) on the plane, then (1) can be written as

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 5 \text{ or } \vec{r} \cdot \vec{n} = 5, \text{ where } \vec{n} = 2\hat{i} + 3\hat{j} - \hat{k}$$

or $\vec{r} \cdot \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{5}{|\vec{n}|}$, where $|\vec{n}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$

or $\vec{r} \cdot \hat{n} = p$, where $p = \dfrac{5}{|\vec{n}|} = \dfrac{5}{\sqrt{14}}$ and

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{14}} = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} - \frac{1}{\sqrt{14}}\hat{k}.$$

Hence, the direction cosines of the normal to the plane are $< \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}, -\dfrac{1}{\sqrt{14}} >$ and the perpendicular distance from the origin to the plane $= \dfrac{5}{\sqrt{14}}$ units.

(*ii*) The equation of the plane is $5y + 8 = 0$.

It can be written as $-5y = 8$ or $0x - 5y + 0z = 8$ ...(1)

If $\vec{r}$ is the position vector of any point P (x, y, z) on the plane, then (1) can be written as

$$\vec{r} \cdot (0\hat{i} - 5\hat{j} + 0\hat{k}) = 8$$

or $\vec{r} \cdot \vec{n} = 8$, where $= \vec{n} = 0\hat{i} - 5\hat{j} + 0\hat{k}$

or $\vec{r} \cdot \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{8}{|\vec{n}|}$, where $|\vec{n}| = \sqrt{0^2 + (-5)^2 + 0^2} = 5$

or $\vec{r} \cdot \hat{n} = p$, where $p = \dfrac{8}{|\vec{n}|} = \dfrac{8}{5}$ and

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{0\hat{i} - 5\hat{j} + 0\hat{k}}{5} = 0\hat{i} - \hat{j} + 0\hat{k}.$$

Hence, the direction cosines of the normal to the plane are $< 0, -1, 0 >$ and the perpendicular distance from the origin to the plane $= \dfrac{8}{5}$ units.

Example 7. *Find the vector as well as cartesian equation of the plane which is at a distance of 6 units from the origin and the direction numbers of a normal to this plane are < 2, 2, – 1 >.*

Solution. Direction numbers of a normal to the plane are $< 2, 2, -1 >$.

Dividing by $\sqrt{2^2 + 2^2 + (-1)^2} = 3$, direction cosines of a normal to the plane are $< \dfrac{2}{3}, \dfrac{2}{3}, -\dfrac{1}{3} >$.

$\therefore$ Unit vector normal to the plane $= \dfrac{2}{3}\hat{i} + \dfrac{2}{3}\hat{j} - \dfrac{1}{3}\hat{k}$.

The distance of the plane from the origin = 6 units (given), the vector equation of the plane is $\vec{r} \cdot \hat{n} = p$

i.e. $\vec{r} \cdot \left(\dfrac{2}{3}\hat{i} + \dfrac{2}{3}\hat{j} - \dfrac{1}{3}\hat{k}\right) = 6$ or $\vec{r} \cdot (2\hat{i} + 2\hat{j} - \hat{k}) = 18$.

Its cartesian equation is $2x + 2y - z = 18$.

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Example 8. *Find the equation of the plane such that the length of perpendicular from the origin to the plane is 5 units and this perpendicular makes angles of 60° and 45° with x-axis and y-axis respectively.*

Solution. Let $< l, m, n >$ be the direction cosines of the perpendicular drawn from origin to the plane, then

$$l = \cos 60° = \frac{1}{2} \text{ and } m = \cos 45° = \frac{1}{\sqrt{2}}.$$

We know that $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + n^2 = 1 \Rightarrow n^2 = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4} \Rightarrow n = \pm\frac{1}{2}.$$

Also length of perpendicular from origin to the plane $= p = 5$ (given).

As n has two values, there are two planes satisfying the given conditions.

The equations of the required planes are

$$\frac{1}{2}x + \frac{1}{\sqrt{2}}y \pm \frac{1}{2}z = 5 \qquad | \; lx + my + nz = p$$

i.e. $x + \sqrt{2}y + z = 10$, $x + \sqrt{2}y - z = 10$.

Example 9. *Find the vector and the cartesian equations of the plane passing through the point (5, 2, – 4) and perpendicular to the line with direction ratios < 2, 3, – 1 >.*

Solution. The plane passes through the point with position vector $\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$ and a normal vector to the plane is $\vec{n} = 2\hat{i} + 3\hat{j} - \hat{k}$.

The equation of the plane is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

i.e. $(\vec{r} - (5\hat{i} + 2\hat{j} - 4\hat{k})) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0$

or $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = (5\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k})$

or $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 5.2 + 2.3 + (-4)(-1)$

or $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 20.$

Its cartesian equation is $2x + 3y - z = 20$.

Example 10. *The position vectors of two points A and B are $3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} - 2\hat{j} - 4\hat{k}$ respectively. Find the vector equation of the plane passing through B and perpendicular to the vector $\overrightarrow{AB}$.*

Solution. $\overrightarrow{AB}$ = P.V. of B – P.V. of A $= (\hat{i} - 2\hat{j} - 4\hat{k}) - (3\hat{i} + \hat{j} + 2\hat{k})$

$= -2\hat{i} - 3\hat{j} - 6\hat{k}$.

As the plane passes through B and is perpendicular to the vector $\overrightarrow{AB}$, its equation is

$(\vec{r} - (\hat{i} - 2\hat{j} - 4\hat{k})) \cdot (-2\hat{i} - 3\hat{j} - 6\hat{k}) = 0 \qquad | \; (\vec{r} - \vec{a}) \cdot \vec{n} = 0$

or $(\vec{r} - (\hat{i} - 2\hat{j} - 4\hat{k})) \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) = 0$ (Dividing by – 1)

or $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) = (\hat{i} - 2\hat{j} - 4\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 6\hat{k})$

or $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) = 1 . 2 + (-2) . 3 + (-4) . 6$

or $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) + 28 = 0$.

Example 11. *If the line drawn from the point (– 10, 5, 4) meets a plane at right angles at the point (4, – 1, 2), find the equation of the plane.*

Solution. Let the given points (4, – 1, 2), (– 10, 5, 4) be A, B respectively.

Direction ratios of the line AB are

$< -10 - 4, 5 + 1, 4 - 2 >$ *i.e.* $< -14, 6, 2 >$ *i.e.* $< 7, -3, -1 >$.

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The required plane passes through A and is perpendicular to the line AB *i.e.* having the line AB with direction ratios $< 7, -3, -1 >$ as a normal, therefore, the equation of the plane is

$$7(x - 4) + (-3)(y - (-1)) + (-1)(z - 2) = 0 \quad | A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

or $7(x - 4) - 3(y + 1) - (z - 2) = 0$

or $7x - 3y - z - 29 = 0.$

Example 12. *Find the vector equation of the straight line passing through (1, 2, 3) and perpendicular to the plane* $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0.$

Solution. The equation of the plane is $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$ *i.e.* $\vec{r} \cdot (-\hat{i} - 2\hat{j} + 5\hat{k}) = 9$, which is of the form $\vec{r} \cdot \vec{n} = d$.

The vector $\vec{n} = -\hat{i} - 2\hat{j} + 5\hat{k}$ is along a normal to the given plane. So, the required line is along the direction of the vector $-\hat{i} - 2\hat{j} + 5\hat{k}$.

The required line passes through the point with position vector $\hat{i} + 2\hat{j} + 3\hat{k}$, therefore, the vector equation of the line is

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(-\hat{i} - 2\hat{j} + 5\hat{k}), \lambda \text{ is a parameter.}$$

Example 13. *Show that the line* $\vec{r} = 4\hat{i} - 7\hat{k} + \lambda(4\hat{i} - 2\hat{j} + 3\hat{k})$ *is parallel to the plane* $\vec{r} \cdot (5\hat{i} + 4\hat{j} - 4\hat{k}) = 7.$

Solution. Compare the line $\vec{r} = 4\hat{i} - 7\hat{k} + \lambda(4\hat{i} - 2\hat{j} + 3\hat{k})$ with $\vec{r} = \vec{a} + \lambda\vec{b}$,

here $\vec{a} = 4\hat{i} - 7\hat{k}$ and $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$.

Compare the plane $\vec{r} \cdot (5\hat{i} + 4\hat{j} - 4\hat{k}) = 7$ with $\vec{r} \cdot \vec{n} = d$;

here $\vec{n} = 5\hat{i} + 4\hat{j} - 4\hat{k}$ and $d = 7$.

The given line is parallel to the given plane iff $\vec{b} \cdot \vec{n} = 0$ and $\vec{a} \cdot \vec{n} \neq d$

i.e. iff $(4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (5\hat{i} + 4\hat{j} - 4\hat{k}) = 0$ and $(4\hat{i} - 7\hat{k}) \cdot (5\hat{i} + 4\hat{j} - 4\hat{k}) \neq 7$

i.e. iff $4.5 + (-2).4 + 3.(-4) = 0$ and $4.5 + 0.4 + (-7).(-4) \neq 7$

i.e. iff $0 = 0$ and $48 \neq 7$, which are both true.

Hence, the given line is parallel to the given plane.

Example 14. *Show that the line* $\vec{r} = 2\hat{i} + 3\hat{j} + \lambda(7\hat{i} - 5\hat{k})$ *lies in the plane*

$$\vec{r} \cdot (5\hat{i} - 3\hat{j} + 7\hat{k}) = 1.$$

Solution. Compare the line $\vec{r} = 2\hat{i} + 3\hat{j} + \lambda(7\hat{i} - 5\hat{k})$ with $\vec{r} = \vec{a} + \lambda\vec{b}$, here $\vec{a} = 2\hat{i} + 3\hat{j}$ and $\vec{b} = 7\hat{i} - 5\hat{k}$.

Compare the plane $\vec{r} \cdot (5\hat{i} - 3\hat{j} + 7\hat{k}) = 1$ with $\vec{r} \cdot \vec{n} = d$;

here $\vec{n} = 5\hat{i} - 3\hat{j} + 7\hat{k}$ and $d = 1$.

The given line lies in the given plane iff $\vec{b} \cdot \vec{n} = 0$ and $\vec{a} \cdot \vec{n} = d$

i.e. iff $(7\hat{i} - 5\hat{k}) \cdot (5\hat{i} - 3\hat{j} + 7\hat{k}) = 0$ and $(2\hat{i} + 3\hat{j}) \cdot (5\hat{i} - 3\hat{j} + 7\hat{k}) = 1$

i.e. iff $7.5 + 0.(-3) + (-5).7 = 0$ and $2.5 + 3.(-3) + 0.7 = 1$

i.e. iff $0 = 0$ and $1 = 1$, which are both true.

Hence, the given line lies in the given plane.

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Example 15. *Find the coordinates of the point where the line*

$$\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4} \text{ meets the plane } x + y + 4z = 6.$$

Solution. The given line is $\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4}$ ...(*i*)

Any point on the line (*i*) is

$P(-1 + 2\lambda, -2 + 3\lambda, -3 + 4\lambda)$,

where λ is any arbitrary real number.

The point P will lie on the plane

$x + y + 4z = 6$ ...(*ii*)

if $(-1 + 2\lambda) + (-2 + 3\lambda) + 4(-3 + 4\lambda) = 6$

$\Rightarrow$ $21\lambda = 21 \Rightarrow \lambda = 1$.

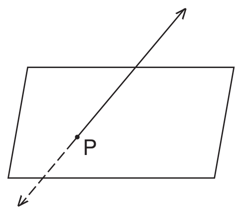

Fig. 3.3.

Substituting this value of λ, the point P is

$(-1 + 2 \times 1, -2 + 3 \times 1, -3 + 4 \times 1)$ *i.e.* $(1, 1, 1)$.

Example 16. *Find the coordinates of the point where the line joining the points (1, –2, 3) and (2, –1, 5) cuts the plane $x - 2y + 3z = 19$. Hence, find the distance of this point from the point (5, 4, 1).* *(I.S.C. 2008)*

Solution. The equation of the line passing through the points (1, –2, 3) and (2, –1, 5) is

$$\frac{x-1}{2-1} = \frac{y+2}{-1-(-2)} = \frac{z-3}{5-3} \quad i.e. \quad \frac{x-1}{1} = \frac{y+2}{1} = \frac{z-3}{2} \qquad ...(i)$$

Any point on line (*i*) is $P(1 + t, -2 + t, 3 + 2t)$.

The point P will be on the plane $x - 2y + 3z = 19$

if $(1 + t) - 2(-2 + t) + 3(3 + 2t) = 19$

$\Rightarrow$ $5t = 5 \Rightarrow t = 1$.

Substituting this value of t, the point P is (2, –1, 5).

Hence, the line joining the given points cuts the given plane at the point (2, –1, 5).

Required distance = distance between points (2, –1, 5) and (5, 4, 1)

$$= \sqrt{(5-2)^2 + (4-(-1))^2 + (1-5)^2}$$

$$= \sqrt{3^2 + 5^2 + 4^2} = \sqrt{50} = 5\sqrt{2} \text{ units.}$$

Example 17. *Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured along a line parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.*

Solution. Let A be (1, –2, 3). The equations of the line through A and parallel to given line are

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} \qquad ...(i)$$

Any point on the line (*i*) is

$P(1 + 2\lambda, -2 + 3\lambda, 3 - 6\lambda)$

where λ is an arbitrary real number.

The point P will lie on the plane

$x - y + z = 5$ ...(*ii*)

if $(1 + 2\lambda) - (-2 + 3\lambda) + (3 - 6\lambda) = 5 \Rightarrow \lambda = \frac{1}{7}$.

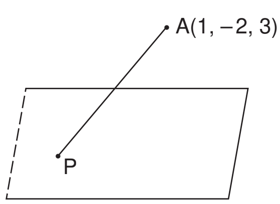

Fig. 3.4.

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Substituting this value of λ, we obtain the point of intersection $P\left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$.

Therefore, the required distance = AP = $\sqrt{\left(\frac{9}{7}-1\right)^2+\left(-\frac{11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$ = 1 unit.

Example 18. *Find the coordinates of the foot of perpendicular drawn from the origin to the plane $2x - 3y + 4z - 6 = 0$.*

Solution. The given plane is

$$2x - 3x + 4z - 6 = 0 \qquad \ldots(i)$$

Direction ratios of a normal to the plane (i) are < 2, − 3, 4 >.

Therefore, the equation of the line through origin O (0, 0, 0) and perpendicular to the plane (i) are

$$\frac{x-0}{2} = \frac{y-0}{-3} = \frac{z-0}{4} \qquad \ldots(ii)$$

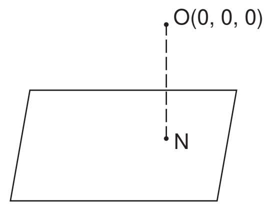

Fig. 3.5.

Any point on the line (ii) is

N $(2\lambda, -3\lambda, 4\lambda)$, where λ is an arbitrary real number.

If it lies on the line (i), then

$$2 \,.\, 2\lambda - 3 \,.\, (-3\lambda) + 4 \,.\, 4\lambda - 6 = 0 \;\Rightarrow 29\lambda = 6 \Rightarrow \lambda = \frac{6}{29}.$$

$\therefore$ The foot of perpendicular from origin to the given plane is

$$\left(2 \,.\, \frac{6}{29}, -3 \,.\, \frac{6}{29}, 4 \,.\, \frac{6}{29}\right) \text{ i.e. } \left(\frac{12}{29}, -\frac{18}{29}, \frac{24}{29}\right).$$

Example 19. *Find the length and the foot of perpendicular from the point (1, 1, 2) to the plane $2x - 2y + 4z + 5 = 0$.*

Solution. The given plane is $2x - 2y + 4z + 5 = 0$ ...(i)

$\therefore$ direction numbers of a normal to (i) are

< 2, −2, 4 > *i.e.* < 1, −1, 2 >.

Hence, the equations of the line through the given point P (1, 1, 2) and perpendicular to the given plane (i) are

$$\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-2}{2} \qquad \ldots(ii)$$

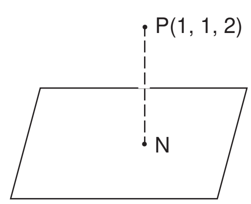

Fig. 3.6.

Any point on the line (ii) is

N $(1+\lambda, 1-\lambda, 2+2\lambda)$

If it lies on the plane (i), then

$$2(1+\lambda) - 2(1-\lambda) + 4(2+2\lambda) + 5 = 0$$

$$\Rightarrow 12\lambda = -13 \;\Rightarrow\; \lambda = -\frac{13}{12}.$$

$\therefore$ The foot of perpendicular from P to the plane is

$$N\left(1-\frac{13}{12}, 1+\frac{13}{12}, 2-\frac{26}{12}\right) \text{ i.e. } \left(-\frac{1}{12}, \frac{25}{12}, -\frac{1}{6}\right).$$

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$\therefore$ The length of perpendicular from P to the given plane (*i*)

$$= \text{NP} = \sqrt{\left(-\frac{1}{12}-1\right)^2 + \left(\frac{25}{12}-1\right)^2 + \left(-\frac{1}{6}-2\right)^2}$$

$$= \sqrt{\frac{(13)^2+(13)^2+(26)^2}{(12)^2}} = \sqrt{\frac{(13)^2\,(6)}{(12)^2}} = \frac{13}{12}\sqrt{6} \text{ units.}$$

Example 20. *Find the image of the point (1, 2, 3) in the plane $x + 2y + 4z = 38$.*

Solution. First, we note that the point P (1, 2, 3) does not lie in the plane

$$x + 2y + 4z = 38 \qquad \ldots(i)$$

Let M be the foot of perpendicular from P to the plane (*i*).

Direction ratios of a normal to the plane (*i*) are < 1, 2, 4 >, therefore, the equations of the line MP are

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{4} \qquad \ldots(ii)$$

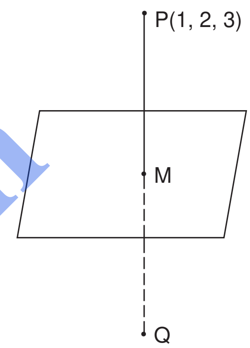

Fig. 3.7.

Any point on the line (*ii*) is Q $(1+\lambda, 2+2\lambda, 3+4\lambda)$, where λ is an arbitrary real number. We want to find a suitable value of λ so that Q becomes the image of the point in the plane (*i*) *i.e.* M becomes mid-point of the segment PQ

i.e. coordinates of M are $\left(\frac{1+1+\lambda}{2}, \frac{2+2+2\lambda}{2}, \frac{3+3+4\lambda}{2}\right)$

i.e. M $\left(\frac{2+\lambda}{2}, 2+\lambda, 3+2\lambda\right)$.

Since M lies on the plane (*i*), we get

$$\frac{2+\lambda}{2} + 2(2+\lambda) + 4(3+2\lambda) = 38$$

$\Rightarrow$ $2+\lambda+8+4\lambda+24+16\lambda = 76 \Rightarrow 21\lambda = 42 \Rightarrow \lambda = 2$.

Substituting this value of λ, the point Q is (3, 6, 11).

EXERCISE 3.1

1. Find the direction cosines of the normal to the plane $2x + 3y - z - 7 = 0$.
2. Find a unit normal vector to the plane $2x - 2y - z - 5 = 0$.
3. Find the cartesian equation of the following planes :

   (*i*) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$ (*ii*) $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

   (*iii*) $\vec{r} \cdot ((s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}) = 15$.
4. Find the vector equations of the following planes :

   (*i*) $3x + 7y - 6z = 5$ (*ii*) $9x - 2y - 5z + 3 = 0$.
5. If O be the origin and the coordinates of P be (1, 2, –3), then find the equation of the plane through P and perpendicular to OP.
6. Find the equation of the plane passing through the point (2, –3, 1) and perpendicular to the line whose direction ratios are < 3, –1, 5 >.
7. Find the equation of the plane passing through the point (1, 2, 3) and perpendicular to the line $\frac{x}{-2} = \frac{y}{4} = \frac{z}{3}$. *(I.S.C. 2002)*

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8. Show that the line $\frac{x-2}{1} = \frac{y+2}{-1} = \frac{z-3}{4}$ is parallel to the plane $x + 5y + z = 7$.

9. If the line $\frac{x+1}{3} = \frac{y-2}{4} = \frac{z+6}{5}$ is parallel to the plane $2x - 3y + kz = 0$, then find the value of k.

10. If (3, 6, 11) is the image of the point (1, 2, 3) in the plane $x + 2y + 4z + k = 0$, then find the value of k.

11. If $(3, k, 6)$ is the image of the point (1, 3, 4) in the plane $x - y + z = 5$, then find the value of k.

12. Write the image of the point (–2, 3, 5) in the plane XOY.

13. Find the vector equation of a plane which is at a distance of 5 units from the origin and which is normal to the vector $3\hat{i} + 2\hat{j} - 6\hat{k}$.

14. If the vector equation of a plane is $\vec{r} \cdot (2\hat{i} + 2\hat{j} - \hat{k}) = 21$, find the length of perpendicular from the origin to the plane.

15. Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{29}}$ units from the origin and its normal vector from the origin is $2\hat{i} - 3\hat{j} + 4\hat{k}$. Also find its cartesian equation.

16. In each of the following problems, determine the direction cosines of the normal to the plane and its distance from the origin :

(i) $2x - 3y + 4z - 6 = 0$ (ii) $x + y + z = 1$

(iii) $z = 2$ (iv) $3x + 5 = 0$.

17. Find the vector equation of a plane which is at a distance of 5 units from the origin and has < 2, –1, 2 > as the direction numbers of a normal to the plane.

18. Find the vector equation of a plane which is at a distance of $2\sqrt{3}$ units from the origin and a normal to the plane makes equal acute angles with coordinate axes.

Hint. Since a normal makes equal acute angle with coordinate axes, $l = m = n$, where l, m, n are direction cosines of the normal.

But $l^2 + m^2 + n^2 = 1 \Rightarrow 3l^2 = 1 \Rightarrow l = \frac{1}{\sqrt{3}}$

$\Rightarrow$ unit vector normal to the plane from origin $= \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$.

Use $\vec{r} \cdot \hat{n} = p$.

19. Find the vector and the cartesian equations of the plane

(i) that passes through the point (1, 0, – 2) and normal vector to the plane is $\hat{i} + \hat{j} - \hat{k}$.

(ii) that passes through the point (1, 4, 6) and normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

20. Find the vector and cartesian equation of the plane passing through the point (1, 2, 3) and perpendicular to the line with direction ratios < 2, 3, –4 >.

21. (i) Find the vector and the cartesian equation of the plane through the point with position vector $2\hat{i} - \hat{j} + \hat{k}$ and perpendicular to the vector $4\hat{i} + 2\hat{j} - 3\hat{k}$.

(ii) If the foot of perpendicular drawn from the origin to a plane is (12, – 4, –3), find the equation of the plane in vector as well as cartesian form.

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ANSWERS

EXERCISE 3.1

1. $<\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}>$

2. $\pm \frac{1}{3}(2\hat{i} - 2\hat{j} - \hat{k})$.

3. (*i*) $x + y - z = 2$ (*ii*) $2x + 3y - 4z = 1$
(*iii*) $(s - 2t)x + (3 - t)y + (2s + t)z = 15$.

4. (*i*) $\vec{r} \cdot (3\hat{i} + 7\hat{j} - 6\hat{k}) = 5$ (*ii*) $\vec{r} \cdot (9\hat{i} - 2\hat{j} - 5\hat{k}) + 3 = 0$.

5. $x + 2y - 3z - 14 = 0$.

6. $3x - y + 5z - 14 = 0$.

7. $2x - 4y - 3z + 15 = 0$.

9. $\frac{6}{5}$.

10. -38.

11. 1.

12. $(-2, 3, -5)$.

13. $\vec{r} \cdot (3\hat{i} + 2\hat{j} - 6\hat{k}) = 35$.

14. 7 units.

15. $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 6$; $2x - 3y + 4z = 6$.

16. (*i*) $< \frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{6}{\sqrt{29}} >$; $\frac{6}{\sqrt{29}}$ units (*ii*) $< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$; $\frac{1}{\sqrt{3}}$ units
(*iii*) $< 0, 0, 1 >$; 2 units (*iv*) $< -1, 0, 0 >$; $\frac{5}{3}$ units.

17. $\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 15$.

18. $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 6$.

19. (*i*) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 3$; $x + y - z = 3$
(*ii*) $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) + 1 = 0$; $x - 2y + z + 1 = 0$.

20. $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) + 4 = 0$; $2x + 3y - 4z + 4 = 0$.

21. (*i*) $\vec{r} \cdot (4\hat{i} + 2\hat{j} - 3\hat{k}) = 3$; $4x + 2y - 3z = 3$
(*ii*) $\vec{r} \cdot (12\hat{i} - 4\hat{j} - 3\hat{k}) = 169$; $12x - 4y - 3z = 169$.

22. (*i*) $x + 5y - 6z + 19 = 0$ (*ii*) $4x - 7y + 3z - 28 = 0$.

23. $\vec{r} \cdot (5\hat{i} + 2\hat{j} + 2\hat{k}) = 1$; $\frac{1}{\sqrt{33}}$ units.

24. 5 units ; $\frac{1}{7}(-2\hat{i} + 3\hat{j} - 6\hat{k})$.

25. $-\frac{2}{7}x + \frac{3}{7}y - \frac{6}{7}z = 2$; 2 units.

26. $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$.

27. $\vec{r} = \hat{i} - \hat{j} + 2\hat{k} + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$.

28. (*ii*) 5.

30. $(1, 1, 1)$.

31. $\frac{x-4}{1} = \frac{y-6}{1} = \frac{z-2}{2}$.

33. 13 units.

34. $(-5, 6, 1)$; No.

35. $\vec{r} = 2\hat{i} - 3\hat{j} - 5\hat{k} + \lambda(6\hat{i} - 3\hat{j} + 5\hat{k})$; $\left(\frac{76}{35}, -\frac{108}{35}, -\frac{34}{7}\right)$.

36. 6 units.

37. (*i*) $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (*ii*) $\left(0, \frac{18}{25}, \frac{24}{25}\right)$.

38. (*i*) $\sqrt{11}$ units; $(5, 2, 6)$; $\frac{x-2}{3} = \frac{y-3}{-1} = \frac{z-7}{-1}$ (*ii*) $\frac{13}{12}\sqrt{6}$ units, $\left(-\frac{1}{12}, \frac{25}{12}, -\frac{1}{6}\right)$.

39. (*i*) $(3, 1, 6)$.

EXERCISE 3.2

1. $7y + 4z - 5 = 0$

2. $12x + 15y - 14z - 68 = 0$.

3. $x - y - 7z + 23 = 0$.

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