3 Linear Programming

INTRODUCTION

You are familiar with linear equations and linear inequations in one and two variables. They can be solved algebraically or graphically (by drawing a line diagram in case of one variable, and on two dimensional graph paper in case of two variables x and y). First, we will recapitulate how to solve a system of linear inequations. Then we will utilise this knowledge extensively in solving problems on linear programming.

3.1 GRAPHICAL SOLUTION OF LINEAR INEQUATIONS

A statement of any one of the following types :

(*i*) $x < a$ (*ii*) $x \leq a$ (*iii*) $x > a$ (*iv*) $x \geq a$

(*v*) $y < b$ (*vi*) $y \leq b$ (*vii*) $y > b$ (*viii*) $y \geq b$

where a, b are real numbers, is called a ***linear inequation (or inequality)*** in one variable x or y.

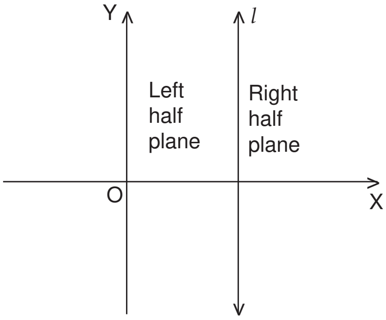
A statement of any one of the following types :

(*i*) $ax + by + c < 0$ (*ii*) $ax + by + c \leq 0$

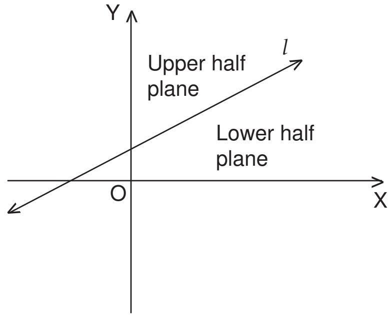
(*iii*) $ax + by + c > 0$ (*iv*) $ax + by + c \geq 0$

where a, b, c are real numbers and atleast one of a and b is non-zero, is called a *linear inequation (or inequality)* in two variables x and y.

We know that a straight line l divides the cartesian plane into two parts. Each part is called a *half plane*. A vertical line divides the plane into left half and right half planes and a non-vertical line divides the plane into lower half and upper half planes [see fig. (*i*) and (*ii*) given below].



(*i*)



(*ii*)

An ordered pair (α, β) of real numbers may or may not satisfy a given inequation (in one or two variables).

The set of all ordered pairs of real numbers which satisfy a given inequation is called the ***solution set*** *of the given inequation.*

Since there is one-one correspondence between the ordered pairs of real numbers and the points of a co-ordinate plane, therefore, we can represent the solution set of a given inequation (in one or two variables) by the points of a co-ordinate plane. *The set of all points whose co-ordinates satisfy a given inequation is called the* ***solution region*** or ***feasible region*** or ***graph of the inequation*** *and every point in this region is called the* ***feasible solution****.*

To find the graphical solution of an inequation in one variable

(*i*) Draw the straight line $x = a$ (or $y = b$) as the case may be.

(*ii*) The straight line $x = a$ divides the co-ordinate plane into two halves.

(*iii*) One half is the graph of $x < a$ and the other half is the graph of $x > a$. Shade the solution region of the given inequation

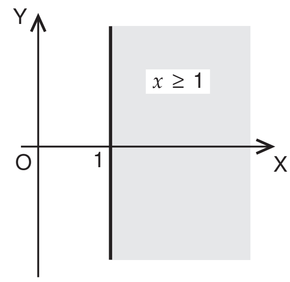
(*iv*) If an inequation is of the form $x \le a$ or $x \ge a$, then the points on the line $x = a$ are also included in the solution region and draw a dark line in the solution region.

(*v*) If an inequation is of the form $x < a$ or $x > a$, then the points on the line $x = a$ are not included in the solution region and draw a line (or broken line) in the solution region.

For example :

Let us consider the inequation $x \ge 1$ in one variable.

Draw the straight line $x = 1$, which is a vertical line. It divides the plane into two halves. The points which lie either to the right of the line $x = 1$ or on the line $x = 1$ satisfy the given inequation $x \ge 1$. Hence, the solution set of the given inequation is $\{(x, y); x, y \in \mathbf{R}, x \ge 1\}$. The solution region of the inequation $x \ge 1$ is shown shaded in adjoining figure.



To find the graphical solution of an inequation in two variables

(*i*) Draw the straight line $ax + by + c = 0$.

(*ii*) The straight line $ax + by + c = 0$ divides the co-ordinate plane into two halves.

(*iii*) One half is the graph of $ax + by + c < 0$ and the other half is the graph of $ax + by + c > 0$.

(*iv*) In order to identify the half plane represented by an inequation, take any point (α, β) not lying on the line $ax + by + c = 0$ and check whether it satisfies the inequation or not. If it satisfies, then the inequation represents the half plane containing the point and shade this region; otherwise the inequation represents the half plane which does not contain the point within it. For convenience, we take the point (0, 0). However, if (0, 0) lies on the line, then take any other point of the plane not lying on the line.

(*v*) If an inequation is of the form $ax + by + c \le 0$ or $ax + by + c \ge 0$, then the points on the line $ax + by + c = 0$ are also included in the solution region and draw a dark line in the solution region.

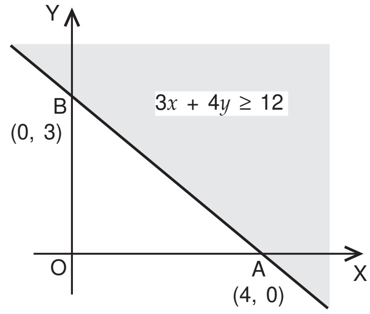
(*vi*) If an inequation is of the form $ax + by + c < 0$ or $ax + by + c > 0$, then the points on the line are not to be included in the solution region and draw a line (or broken line) in the solution region.

For example :

Let us consider the inequation $3x + 4y \ge 12$ in two variables.

First, draw the straight line

$$3x + 4y = 12 \qquad \ldots(i)$$



To draw the line (*i*), on putting $y = 0$, we get $x = 4$ and on putting $x = 0$, we get $y = 3$. Therefore, the straight line (*i*) passes through the points A (4, 0) and B (0, 3).

The straight line divides the co-ordinate plane into two halves. Further, we note that the point O (0, 0) lies below the line AB and it *does not satisfy* the given inequation $3x + 4y \geq 12$. $(\because 3.0 + 4.0 = 0 < 12)$.

Hence, the graph of the given inequation $3x + 4y \geq 12$ is that part of co-ordinate plane which lies above the line AB (including the points on the line AB). The required solution region of the given inequation is shown shaded in the above figure.

To solve a system of linear inequations

To solve a system of linear inequations (in one or two variables), proceed as follows :

(*i*) Draw the graphs (or solution regions) of all the given linear inequations.

(*ii*) Find the common part of the co-ordinate plane which satisfies all the given linear inequations.

(*iii*) This common part of the co-ordinate plane is the required solution of the given inequations. The corner points should be labelled clearly.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the inequations $3x + 2y > 5$ and $x + y \geq 1$ simultaneously.*

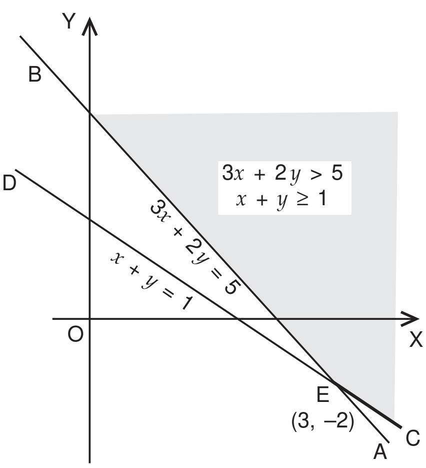
Solution. The given inequations are

$$3x + 2y > 5 \qquad \ldots(i)$$

and $$x + y \geq 1 \qquad \ldots(ii)$$

To draw the graph of $3x + 2y > 5$:

Draw the straight line $3x + 2y = 5$ which passes through the points $\left(\frac{5}{3}, 0\right)$ and $\left(0, \frac{5}{2}\right)$. The line divides the plane into two parts.



Further, as O (0, 0) does not satisfy the inequation $3x + 2y > 5$ $(\because 3.0 + 2.0 = 0 < 5)$, therefore, the graph of (*i*) consists of that part of the plane divided by the line $3x + 2y = 5$ which does not contain the origin.

Similarly, draw the graph of the inequation $x + y \geq 1$. The point E is the intersection of equations $x + y = 1$ and $3x + 2y = 5$. Hence, E is (3, – 2).

Shade the common part of the graphs of both the given inequations (*i*) and (*ii*).

The solution region of the given inequations consists of all points in the shaded part of the co-ordinate plane shown in the above figure. The points on the ray EC of the line DC are included in the solution.

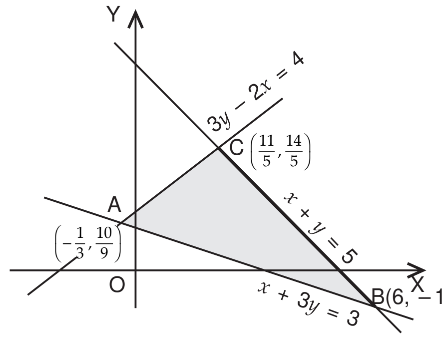
Example 2. *Solve the following inequations simultaneously :*
$3y - 2x < 4$, $x + 3y > 3$ *and* $x + y \leq 5$.

Solution. The given inequations are

$$3y - 2x < 4 \quad \ldots(i)$$
$$x + 3y > 3 \quad \ldots(ii)$$
and $$x + y \leq 5 \quad \ldots(iii)$$

To draw the graph of $3y - 2x < 4$:

Draw the straight line $3y - 2x = 4$ which passes through the points $(-2, 0)$ and $\left(0, \frac{4}{3}\right)$. The line divides the plane into two parts. Further as O (0, 0) satisfies the inequation $3y - 2x < 4$, ($\because 3.0 - 2.0 = 0 < 4$), therefore, the graph consists of that part of the plane divided by the line $3y - 2x = 4$ which contains the origin.



Similarly, draw the graphs of other two inequations $x + 3y > 3$ and $x + y \leq 5$.

The corner points are A $\left(-\frac{1}{3}, \frac{10}{9}\right)$, B $(6, -1)$ and C $\left(\frac{11}{5}, \frac{14}{5}\right)$.

Shade the common part of the graphs of all the three given inequations (*i*), (*ii*) and (*iii*).

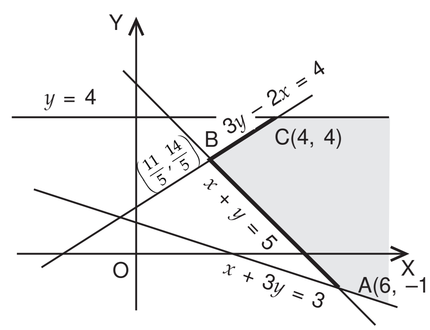
The solution region consists of all the points in the shaded part of the co-ordinate plane shown in the figure. The points on the line segment BC are included in the solution.

Example 3. *Solve the following inequations simultaneously :*
$3y - 2x \leq 4$, $x + 3y > 3$, $x + y \geq 5$, $y < 4$.

Solution. The given inequations are

$$3y - 2x \leq 4 \quad \ldots(i)$$
$$x + 3y > 3 \quad \ldots(ii)$$
$$x + y \geq 5 \quad \ldots(iii)$$
and $$y < 4 \quad \ldots(iv)$$

Draw the graphs of all the given inequations. The corner points are A $(6, -1)$, B $\left(\frac{11}{5}, \frac{14}{5}\right)$ and C $(4, 4)$.

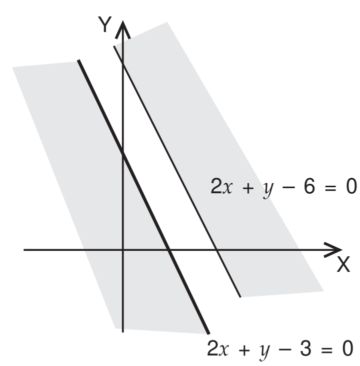


Shade the common part of the graphs of all the four given inequations.

The solution region consists of all the points in the shaded part of the coordinate plane shown in the figure. The points on the line segments AB and BC are included in the solution.

Example 4. *Solve the following linear equations simultaneously :*
$2x + y - 3 \leq 0$, $2x + y - 6 > 0$.

Solution. The adjoining figure shows the regions corresponding to inequations $2x + y - 3 \leq 0$ and $2x + y - 6 > 0$. As the two lines are parallel, they never meet, and so the two shaded areas never overlap. Hence, there is *no solution* of the given simultaneous linear equations. In other words, the *solution set is empty*. We also say that the given system of simultaneous linear inequations is **infeasible**.



EXERCISE 3.1

Solve the following systems of linear inequations :

1. $3x + 2y > 5$ and $y > 2$.
2. $3x + 2y < 6$ and $x + 2y > 4$.
3. $2x + 3y < 12$, $x \geq 2$ and $y \geq 1$.
4. $x - 2y + 11 > 0$, $2x - 3y + 18 \geq 0$ and $y \geq 0$.
5. $x + 2y \leq 8$, $x - y \leq 2$, $x > 0$ and $y > 0$.
6. $3x + 2y - 6 < 0$, $3x + 2y \geq 18$

3.2 LINEAR PROGRAMMING

In a military operation, the effort is to inflict maximum damage to enemy at minimum cost and loss. In an industry, the management always tries to utilise its resources in the best possible manner. The industrialist would like to have unlimited profits, but he is constrained by limited manpower, capital and market demand. A salaried person tries to make investments in such a manner that the returns on investment are high but at the same time income tax liability is also kept low.

In all the above cases, if constraints are represented by linear equations/inequations (in one, two or more variables), and a particular plan of action from several alternatives is to be chosen, we use *linear programming*. The word *linear* means that all inequations used and the function to be maximized or minimized are linear, and the word *programming* refers to *planning* (choosing amongst alternatives) rather than the computer programming sense. American economist Dantzig is credited with developing the subject of linear programming.

Thus, linear programming is a method for determining optimum values of a linear function subject to constraints expressed as linear equations or inequalities.

A practical problem may involve dozens of variables, and is usually solved by using Simplex Method and a computer. In this chapter, we will restrict ourselves to linear functions involving two variables, so that they can be solved by drawing graph on *xy*-plane.

Three Classical Linear Programming Problems (L.P.P.)

(*i*) *Manufacturing problems*

When a firm has the choice of manufacturing a mix of different products, and each product requires a fixed manpower, machine hours, labour hours per unit of the product, warehouse space per unit of the output etc., it has to choose which products should be produced in what numbers so as to maximize the profits.

(*ii*) *Diet problems*

Different kinds of foods have different amounts of nutrients (vitamins, minerals etc). In diet problems, we have to determine the amounts of different kinds of foods which should be included in the diet so as to minimize the cost of the diet such that it contains a certain (minimum) amount of each nutrient.

(*iii*) *Transportation problems*

A factory may have different godowns at different locations, and it has to send its products to many places from the stocks at these godowns. The problem is to find out what amounts should be transported from what godowns to what places so that total transportation cost is minimized.

Some Definitions

Objective Function. If $a_1, a_2, \ldots, a_n$ are constants and $x_1, x_2, \ldots, x_n$ are variables (called *decision variables*), then the linear function $Z = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ which is be to optimized (maximized or minimized) is called the *objective function*. It is always *non-negative*. In business applications, objective function of total profit or volume of production is to be maximized while the objective function of total production cost or production time is to be minimized.

Non-negative Restrictions. The values of the variables $x_1, x_2, \ldots, x_n$ in an L.P.P. are always non-negative (≥ 0). Thus,

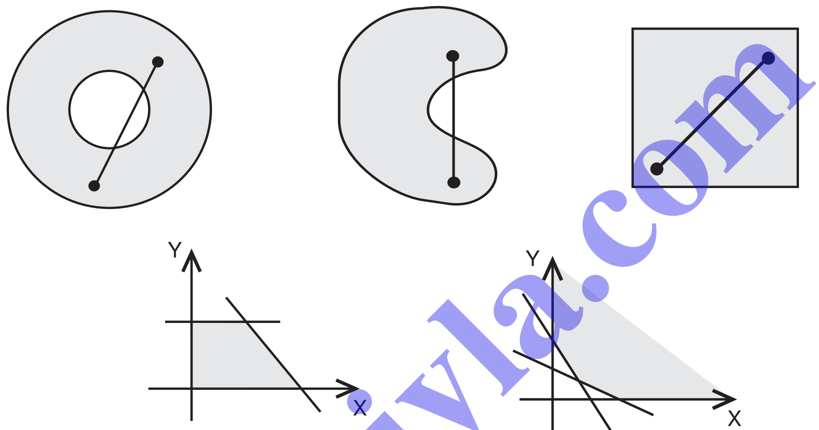
$$x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_n \geq 0.$$

For two variables x and y, it means that we will be working only in the first quadrant of the co-ordinate plane.

Constraints. The inequations or equations on the variables of an L.P.P. are called constraints. They may be of $=, >, \geq, <$ or $\leq$ type.

Feasible Region and Feasible Solution

The common region determined by all the constraints of an L.P.P. is called a *feasible region* and every point in this region is called a *feasible solution* to L.P.P. The feasible region is always a *convex* set, meaning that if any two points in this set are joined by a straight line segment, all the points on this segment also lie in the set. Note that first two figures below are not convex sets, but the next three are convex sets. Also note that feasible region may be bounded or unbounded, but it is always convex.



Optimal Solution. A feasible solution which maximizes or minimizes the objective function is called an *optimal solution*. An L.P.P. may have zero, one or more than one optimal solution.

Basic Requirements of Linear Programming

(*i*) There must be a well defined *objective* to achieve (to maximize or minimize).

(*ii*) There must be *limited availability of the resources i.e.* there are *constraints* or restrictions on the usage/allocation of limited resources amongst the competing activities.

(*iii*) There are a *finite* number of *decision variables* (*activities* or *products*) and a finite number of constraints.

(*iv*) All the elements of an L.P.P. should be *quantifiable*.

(*v*) Both the objective function and constraints must be expressed in terms of *linear* equations or inequations.

(*vi*) Programming/planning requires that these must be *alternative courses of line of action i.e.* there are alternatives available to which resources can be put.

(*vii*) All decision variables should assume non-negative values.

Advantages of Linear Programming

We have already mentioned a few application areas of linear programming - military operations, choosing optimal product line, diet problems and transportation problems. It helps in choosing the best alternative from a set of feasible alternatives, so that profit is maximized, or costs are minimized. Some other application areas are :

(*i*) Choosing the media mix (radio, TV, newspapers, hoardings, magazines, internet) to maximize the advertising effectiveness, within given publicity budget.

(*ii*) Determining shortest routes for travelling salesmen.
(*iii*) Helping a farmer decide best crop mix so as to minimize risk and maximize profit.

Thus, the linear programming helps in

(*i*) choosing the best alternative amongst a set of alternatives so that profit is maximized or cost is minimized.
(*ii*) taking into consideration not only internal factors like manpower, machine, budget, storage availability etc. but also external factors like market demand, purchasing power of the customer etc.
(*iii*) identifying bottlenecks in the production process.
(*iv*) choosing best production policy and inventory policy so that seasonal fluctuations in demand can be handled.

Limitations of Linear Programming

(*i*) It deals with optimizing a single objective. In practice, a number of objectives may be there.
(*ii*) The assumption that input and output variables are directly proportional is not strictly true. *Economies of scale* usually ensure that the more you produce, lesser is the average cost.
(*iii*) The linearity of variables assumes that resources required for multiple activities are sum total of resources required for individual activities. However, *synergies* of product mix usually mean that the requirement is less than the sum.
(*iv*) In practice, many decision variables assume *integral values*, *e.g.*, number of workers. L.P. deals with variables having continuous values.

3.3 FORMULATION OF AN L.P.P. IN TWO VARIABLES x AND y

There are *three* steps in the mathematical formulation of an L.P.P. and solving it. These are :

(*i*) To identify the objective function as a linear combination of variables (x and y) and to construct all constraints *i.e.* linear equations and inequations involving these variables. Thus, an L.P.P. can be stated mathematically as

Maximize (or minimize) $Z = ax + by$

subject to the conditions

$a_i x + b_i y \leq$ (*or* $\geq$ *or* $=$ *or* $>$ *or* $<$) c_i, *where* $i = 1$ *to* n

and the non-negative restrictions

$x \geq 0,\ y \geq 0.$

(*ii*) To find the solutions (*feasible region*) of these equations and inequations by some mathematical method. Here we will study only the graphical method.
(*iii*) To find the *optimal solution i.e.* to select particular values of the variables x and y that give the desired value (maximum/minimum) of the objective function.

ILLUSTRATIVE EXAMPLES

Example 1. *A shopkeeper deals in two items—wall hangings and artificial plants. He has ₹ 15000 to invest and a space to store atmost 80 pieces. A wall hanging costs him ₹ 300 and an artificial plant ₹ 150. He can sell a wall hanging at a profit of ₹ 50 and an artificial plant at a profit of ₹ 18. Assuming that he can sell all the items that he buys, formulate a linear programming problem in order to maximize his profit.*

Solution. Let x be the number of wall hangings and y be the number of artificial plants that the dealer buys and sells. Then the profit of the dealer is $Z = 50x + 18y$, which is the objective function.

As a wall hanging costs ₹ 300 and an artificial plant costs ₹ 150, the cost of x wall hangings and y artificial plants is $300x + 150y$. We are given that the dealer can invest atmost ₹ 15000. Hence, the investment constraint is

$$300x + 150y \leq 15000$$

i.e. $$2x + y \leq 100$$

As the dealer has space to store atmost 80 pieces, we have another constraint (space constraint) :

$$x + y \leq 80$$

Also the number of wall hangings and artificial plants can't be negative. Thus we have the non-negativity constraints :

$$x \geq 0, y \geq 0.$$

Thus the mathematical formulation of the L.P.P. is :

Maximize $Z = 50x + 18y$ *subject to the constraints*

$2x + y \leq 100, \quad x + y \leq 80, \quad x \geq 0, \quad y \geq 0.$

Example 2. *(Diet problem) A diet is to contain atleast 80 units of vitamin A and 100 units of minerals. Two foods* F_1 *and* F_2 *are available. Food* F_1 *costs ₹ 4 per kg and* F_2 *costs ₹ 5 per kg. One kg of food* F_1 *contains 3 units of vitamin A and 4 units of minerals. One kg of food* F_2 *contains 6 units of vitamin A and 3 units of minerals. We wish to find the minimum cost for diet that consists of mixture of these two foods and also meets the minimum nutritional requirements. Formulate this as a linear programming problem.*

Solution. Let the mixture consist of x kg of food F_1 and y kg of food F_2.

We make the following table from given data :

Resources	*Food (in kg)*		*Requirement (in units)*
	F_1 (x)	F_2 (y)	
Vitamin A (units/kg)	3	6	80
Minerals (units/kg)	4	3	100
Cost (₹/kg)	4	5	

Minimum requirement of vitamin A is 80 units,

therefore, $$3x + 6y \geq 80$$

Similarly, the minimum requirement of minerals is 100 units,

therefore, $$4x + 3y \geq 100$$

Also $$x \geq 0, y \geq 0$$

As cost of food F_1 is ₹ 4 per kg, and the cost of food F_2 is ₹ 5 per kg, the total cost of purchasing x kg of food F_1 and y kg of food F_2 is

$$Z = 4x + 5y,$$

which is the objective function.

Hence, the mathematical formulation of the L.P.P. is :

Minimize $Z = 4x + 5y$ *subject to the constraints*

$3x + 6y \geq 80, \quad 4x + 3y \geq 100, \quad x \geq 0, y \geq 0.$

Example 3. *(Manufacturing problem) A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts, while it takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts.*

He earns a profit of ₹ 2·50 per package on nuts and ₹ 1 per package on bolts. Form a linear programming problem to maximize his profit, if he operates each machine for atmost 12 hours.

Solution. Suppose that x packages of nuts and y packages of bolts are produced. The objective of the manufacturer is to maximize the profit, $2{\cdot}50x + 1.y$

Time required on machine A to produce x packages of nuts and y packages of bolts is $1.x + 3.y = x + 3y$, while time required on machine B is $3.x + 1.y = 3x + y$.

From given data, we can formulate the L.P.P. as :

Maximize $Z = 2{\cdot}50x + y$ *subject to the constraints*

$x + 3y \leq 12$ *(Machine A constraint)*

$3x + y \leq 12$ *(Machine B constraint)*

$x \geq 0,\ y \geq 0$ *(Non-negativity constraints)*

Example 4. *(Transportation problem) A brick manufacturer has two depots A and B, with stocks of 30000 and 20000 bricks respectively. He receives orders from three builders P, Q and R for 15000, 20000 and 15000 bricks respectively. The cost (in ₹) of transporting 1000 bricks to the builders from the depots are given below :*

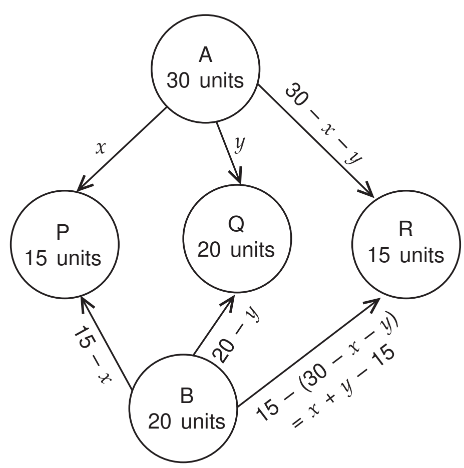
To / *From*	*Transportation cost per 1000 bricks (in ₹)*		
	P	*Q*	*R*
A	40	20	20
B	20	60	40

The manufacturer wishes to find how to fulfil the order so that transportation cost is minimum. Formulate the L.P.P.

Solution. To simplify, assume that 1 unit = 1000 bricks

Suppose that depot A supplies x units to P and y units to Q, so that depot A supplies $30 - x - y$ bricks to builder R.

Now as P requires a total of 15000 bricks, it requires $(15 - x)$ units from depot B. Similarly Q requires $(20 - y)$ units from B and R requires $15 - (30 - x - y) = (x + y - 15)$ units from B.



Using the transportation cost given in table, total transportation cost is

$$Z = 40x + 20y + 20(30 - x - y) + 20(15 - x) + 60(20 - y) + 40(x + y - 15)$$
$$= 40x - 20y + 1500$$

Obviously the constraints are that all quantities of bricks supplied from A and B to P, Q, R are non-negative,

i.e. $x \geq 0,\ y \geq 0,\ \ 30 - x - y \geq 0,\ \ 15 - x \geq 0,\ \ 20 - y \geq 0,\ \ x + y - 15 \geq 0$

Hence the problem can be formulated as L.P.P. as

Minimize $Z = 40x - 20y + 1500$ *subject to the constraints*

$x + y \geq 15,\ x + y \leq 30,\ \ x \leq 15,\ y \leq 20,\ \ x \geq 0,\ y \geq 0.$

EXERCISE 3.2

1. A furniture dealer deals in only two items—tables and chairs. He has ₹ 20000 to invest and a space to store atmost 80 pieces. A table costs him ₹ 800 and a chair costs him ₹ 200. He can sell a table for ₹ 950 and a chair for ₹ 280. Assume that he can sell all the items that he buys. Formulate this problem as an L.P.P. so that he can maximize his profit.

2. A diet for a sick person must contain atleast 3000 units of vitamins, 60 units of minerals and 1600 units of calories. Two foods F_1 and F_2 are available at a cost of ₹ 6 and ₹ 5 per unit respectively. One unit of F_1 contains 200 units of vitamins, 1 unit of mineral and 40 units of calories, while one unit of F_2 contains 100 units of vitamins, 2 units of minerals and 40 units of calories. The aim is to formulate a mixture of foods F_1 and F_2 which provides these minimum levels of nutrients and also costs the least. Formulate it as L.P.P.

3. A retired person wants to invest an amount of upto ₹ 20000. His broker recommends investing in two types of bonds A and B, bond A yielding 10% return on the amount invested and bond B yielding 15% return on the amount invested. After some consideration, he decides to invest atleast ₹ 5000 in bond A and no more than ₹ 8000 in bond B. He also wants to invest atleast as much in bond A as in bond B. Formulate as L.P.P. to maximize his return on investments.

4. A manufacturer has three machines M_1, M_2 and M_3 installed in his factory. Machines M_1 and M_2 are capable of being operated for at the most 12 hours, whereas machine M_3 must be operated for atleast 5 hours a day. The manufacturer produces only two items, each requiring the use of these three machines. The following table gives the number of hours required on these machines for producing 1 unit of A or B.

Item	*Number of hours required on the machines*		
	M_1	M_2	M_3
A	1	2	1
B	2	1	$\frac{5}{4}$

He makes a profit of ₹ 60 on item A and ₹ 40 on item B. He wishes to find out how many of each item he should produce to have maximum profit. Formulate it as L.P.P.

5. There is a factory located at two places P and Q. From these locations, a certain commodity is delivered to each of three depots situated at A, B and C. The weekly requirements of the depots are respectively, 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below :

To / From	*Cost (₹/unit)*		
	A	*B*	*C*
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that transportation cost is minimum? Formulate the above linear programming problem mathematically.

3.4 GRAPHICAL METHOD OF SOLVING AN L.P.P.

The following theorems are fundamental in solving L.P.P.

Theorem 1. Let R be the feasible region for an L.P.P. and Z = $ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), subject to the constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2. Let R be the feasible region for an L.P.P. and Z = $ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.

Remark. If R is unbounded, then objective function may or may not have a maximum or minimum value. However, if it (maximum/minimum) exists, then by Theorem 1, it must occur at corner point of R.

There are two methods of finding the optimal solution in the feasible region.

(*i*) **Corner Point Method**

To solve an L.P.P. graphically, we follow these steps :

1. Convert the given linear constraints into equalities and then draw their graphs which will be straight lines.
2. Find the feasible region of the L.P.P. and find the corner points.
3. Evaluate the objective function Z = $ax + by$ at each corner point, and let M and m be respectively the largest and smallest values.
4. If feasible region R is bounded, then M and m are the maximum and minimum values of Z.
5. If feasible region R is unbounded, then
   (*a*) M is the maximum value of Z if the open half plane $ax + by > M$ has no point in common with the feasible region. Otherwise Z has no maximum value.
   (*b*) m is the minimum value of Z if the open half plane $ax + by < m$ has no point in common with the feasible region. Otherwise Z has no minimum value.
6. If two corner points of the feasible region are both optimal solutions of the same type *i.e.* both give the same maximum/minimum values, then every point on the line segment joining these points gives the same optimal solution.

(*ii*) **Iso-profit or Iso-cost Method**

In this method, we choose a fixed value k of the objective function such that $Z = ax + by = k$ is a straight line, called *iso-profit* or *iso-cost line* as every point on this line will yield same (*iso*) profit or cost (= k). Now move this line parallel to itself over the feasible region so that it passes through all corner points.

For maximization, the corner corresponding to iso-profit line *farthest from the origin* gives maximum value. For minimization, the corner corresponding to iso-cost line *closest to the origin* gives minimum value. If the objective function has same slope as one of the constraints, the iso-profit or iso-cost line will coincide with one of the outer lines of the feasible region and we *may* have *infinite number of optimal solutions*.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following linear programming problem graphically :*

Maximize and minimize $Z = 60x + 15y$ *subject to the constraints*

$x + y \le 50,\ 3x + y \le 90,\ x, y \ge 0.$

13. Two godowns A and B have a grain storage capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops D, E and F, whose requirements are 60, 50 and 40 quintals respectively. The costs of transportation per quintal from the godowns to the shops are given in the following table :

Transportation costs per quintal (in ₹)		
From / *To*	*A*	*B*
D	6·00	4·00
E	3·00	2·00
F	2·50	3·00

How should the supplies be transported in order that the transportation cost is minimum?

14. An oil company has two depots A and B with capacities of 7000 litre and 4000 litre respectively. The company is to supply oil to three petrol pumps D, E and F, whose requirements are 4500 litre, 3000 litre and 3500 litre respectively. The distance (in km) between the depots and the petrol pumps is given in the following table :

Distance from the depot (in km)		
From / *To*	*A*	*B*
D	7	3
E	6	4
F	3	2

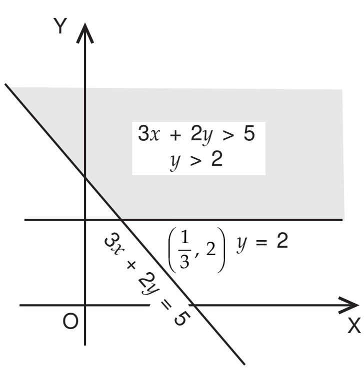
Assuming that the transportation cost per km is ₹ 1 per litre, how should the delivery be scheduled in order that the transportation cost is minimum?

ANSWERS

EXERCISE 3.1

In answers 1 to 5, the solution set (region) consists of all points in the shaded part of the co-ordinate plane in their respective diagrams.

1.



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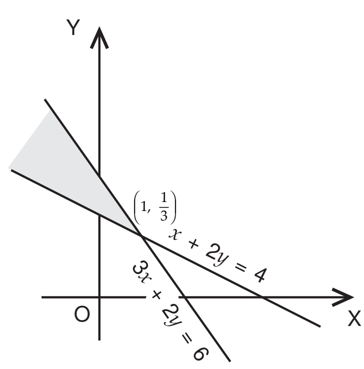

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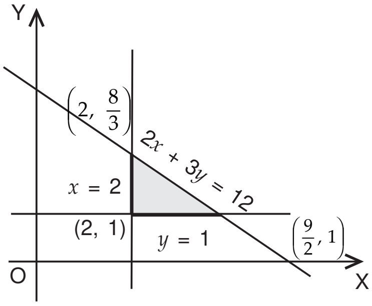

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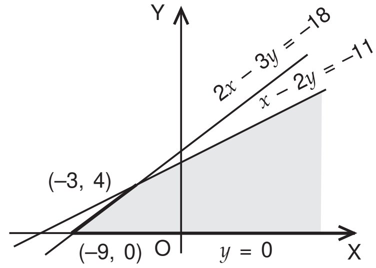

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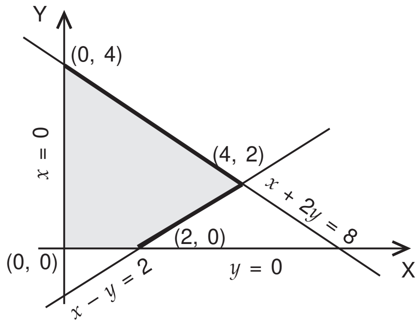

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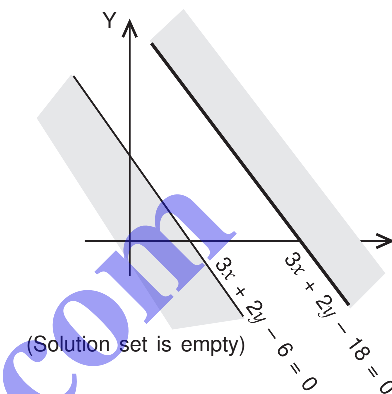

EXERCISE 3.2

1. Maximize Z = $150x + 80y$, subject to the constraints
   $4x + y \le 100$, $x + y \le 80$, $x \ge 0$, $y \ge 0$.
2. Minimize Z = $6x + 5y$, subject to the constraints
   $2x + y \ge 30$, $x + 2y \ge 60$, $x + y \ge 40$, $x \ge 0$, $y \ge 0$.
3. Maximize Z = $0{\cdot}10x + 0{\cdot}15y$, subject to the constraints
   $x + y \le 20000$, $x \ge 5000$, $y \le 8000$,
   $x \ge y$, $x \ge 0$, $y \ge 0$.
4. Maximize Z = $60x + 40y$, subject to the constraints
   $x + 2y \le 12$, $2x + y \le 12$, $x + \frac{5}{4}y \ge 5$, $x \ge 0$, $y \ge 0$.
5. Minimize Z = $x - 7y + 190$ subject to the constraints
   $x + y \le 8$, $x \le 5$, $y \le 5$, $x + y \ge 4$, $x \ge 0$, $y \ge 0$.

EXERCISE 3.3

1. Maximum 1800 at $x = 30$, $y = 0$ and minimum 0 at $x = 0$, $y = 0$.
2. Minimum –12 at $x = 4$, $y = 0$.
3. Maximum 18 at $x = 4$, $y = 3$.
4. Minimum 7 at $x = \frac{3}{2}$, $y = \frac{1}{2}$.
5. Minimum 22 at $x = 4$, $y = 2$.
6. Maximum 325·5 at $x = \frac{21}{2}$, $y = \frac{69}{2}$.
7. Minimum 2300 at $x = 4$, $y = 3$.
8. Minimum 155 at $x = 0$, $y = 5$.
9. Maximum 96 at $x = 12$, $y = 16$.
10. Minimum 6 at the corner points (6, 0) and (0, 3). In fact, all points on the line segment joining the points (6, 0) and (0, 3) yield the minimum value 6.
11. Minimum 60 at (5, 5). Maximum 180 occurs at two corner points (15, 15) and (0, 20). In fact, all points on the line segment joining the points (15, 15) and (0, 20) yield the maximum value 180.
12. No feasible solution.
13. Z has no maximum value.