# 2 Straight Lines in Space

## INTRODUCTION

In class XI, we studied Analytical Geometry in two dimensions and introduced 3-dimensional geometry, and restricted our study to cartesian method only. Now we shall use vector algebra to 3-dimensional geometry which makes its study simple and elegant.

In this chapter, we shall study :

(*i*) the vector and cartesian equations of a line in different forms
(*ii*) angle between two lines when their equations are known
(*iii*) coplanar and skew lines
(*iv*) shortest distance between two lines
(*v*) conditions for intersection of two lines
(*vi*) condition for coplanarity of two lines.

## 2.1 STRAIGHT LINE (IN SPACE)

We know that in space a straight line is uniquely determined if

(*i*) it passes through a given point and has a given direction or
(*ii*) it passes through two given points.

We shall obtain the vector equation and cartesian equation of a line (in space).

### 2.1.1 Equation of a straight line passing through a given point and parallel to a given vector

Let the line pass through the given point A with position vector $\vec{a}$ and let it be parallel to the given vector $\vec{b}$ (as shown in fig. 2.1).

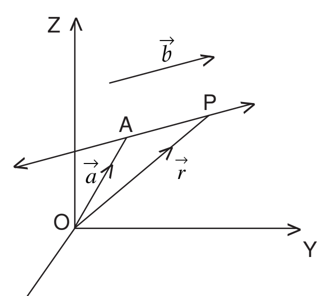

*Fig. 2.1.*

Let P with position vector $\vec{r}$ be any point, then P lies on the line iff $\overrightarrow{AP}$ is parallel to the vector $\vec{b}$ *i.e.* iff $\overrightarrow{AP} = \lambda \vec{b}$, where $\lambda$ is a real number (called *parameter*)

*i.e.* iff $\overrightarrow{OP} - \overrightarrow{OA} = \lambda \vec{b}$

*i.e.* iff $\vec{r} - \vec{a} = \lambda \vec{b}$

*i.e.* iff $\vec{r} = \vec{a} + \lambda \vec{b}$ ... (1)

which is the required ***vector equation*** of the line.

### Cartesian form

Let the given point be A $(x_1, y_1, z_1)$ and $< a, b, c >$ be the direction numbers of the line and P be $(x, y, z)$, then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \text{ and}$$

$$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}.$$

On substituting these values of $\vec{r}$, $\vec{a}$ and $\vec{b}$ in (1), we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1 + a\lambda)\hat{i} + (y_1 + b\lambda)\hat{j} + (z_1 + c\lambda)\hat{k}$$

On equating the coefficients of the base vectors $\hat{i}, \hat{j}, \hat{k}$, we get

$$\left.\begin{aligned} x &= x_1 + a\lambda \\ y &= y_1 + b\lambda \\ z &= z_1 + c\lambda \end{aligned}\right\} \quad \ldots(2)$$

*These are called **parametric (or one point) form** of the line.*

Any point on the line is P $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$. The point A is called the ***base point.***

On eliminating $\lambda$ from equations (2), we get

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z_1 - z_1}{c} \quad \ldots(3)$$

*These are called **symmetrical form** of the line.*

## 2.1.2 Equation of a straight line passing through two given points

Let the line pass through two given points A and B with position vectors $\vec{a}$ and $\vec{b}$ respectively (as shown in fig 2.2).

Let P with position vector $\vec{r}$ be any point, then P lies on the line AB iff

$\overrightarrow{AP} = \lambda\, \overrightarrow{AB}$, where $\lambda$ is real number (called *parameter*)

*i.e.* iff $\overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

*i.e.* iff $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ …(1)

which is the required ***vector equation*** of the line.

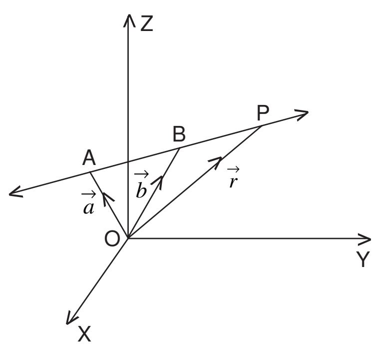


*Fig. 2.2.*

### Cartesian form

Let the given points be A $(x_1, y_1, z_1)$, B $(x_2, y_2, z_2)$ and the point P be $(x, y, z)$, then from (1), we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + t[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}].$$

On equating the coefficients of base vectors, we get

$$\left.\begin{aligned} x &= x_1 + \lambda(x_2 - x_1) \\ y &= y_1 + \lambda(y_2 - y_1) \\ z &= z_1 + \lambda(z_2 - z_1) \end{aligned}\right\} \quad \ldots(2)$$

*These are called **parametric (or two points) form** of the line.*

On eliminating $\lambda$ from the equations (2), we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad \ldots(3)$$

*These are called **symmetrical form.***

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find the vector equation of the line which is parallel to the vector* $2\hat{i} - \hat{j} + 3\hat{k}$ *and which passes through the point (5, –2, 4). Also find its cartesian equations.*

**Solution.** Let $\vec{r}$ be the position vector of any point P $(x, y, z)$ on the given line, then the vector equation of the line is

$$\vec{r} = 5\hat{i} - 2\hat{j} + 4\hat{k} + \lambda(2\hat{i} - \hat{j} + 3\hat{k}).$$

Now as $\vec{r}$ is the position vector of P $(x, y, z)$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (5 + 2\lambda)\hat{i} - (2 + \lambda)\hat{j} + (4 + 3\lambda)\hat{k}.$$

On equating coefficients of the base vectors, we get

$$x = 5 + 2\lambda,\ y = -2 - \lambda,\ z = 4 + 3\lambda.$$

On eliminating the parameter $\lambda$ from these equations, we get

$\dfrac{x-5}{2} = \dfrac{y+2}{-1} = \dfrac{z-4}{3}$, which are the required cartesian equations.

**Example 2.** *If the cartesian equations of a line are* $\dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{z-5}{-1}$, *find its vector equation.*

**Solution.** The given line passes through the point (1, –2, 5) and is parallel to the vector $2\hat{i} + 3\hat{j} - \hat{k}$.

Let $\vec{r}$ be the position vector of any point on the given line, then the vector equation of the line is

$\vec{r} = \hat{i} - 2\hat{j} + 5\hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$ where $\lambda$ is a parameter.

**Example 3.** *Find the vector and the cartesian equations of the line which passes through the point (– 2, 4, – 5) and is parallel to the line given by* $\dfrac{x+3}{3} = \dfrac{y-4}{5} = \dfrac{z+8}{6}$.

**Solution.** Direction ratios of the given line are < 3, 5, 6 >. The required line passes through the point A(– 2, 4, – 5) and is parallel to the vector $3\hat{i} + 5\hat{j} + 6\hat{k}$, therefore, its vector equation is

$\vec{r} = -2\hat{i} + 4\hat{j} - 5\hat{k} + \lambda(3\hat{i} + 5\hat{j} + 6\hat{k})$, where $\lambda$ is a parameter.

Cartesian equations of the required line are

$$\frac{x-(-2)}{3} = \frac{y-4}{5} = \frac{z-(-5)}{6} \qquad \left|\ \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}\right.$$

*i.e.* $\dfrac{x+2}{3} = \dfrac{y-4}{5} = \dfrac{z+5}{6}$.

**Example 4.** *Find the vector equation of a line passing through the point with position vector* $2\hat{i} - \hat{j} + \hat{k}$ *and parallel to the line joining the points* $-\hat{i} + 4\hat{j} + \hat{k}$ *and* $\hat{i} + 2\hat{j} + 2\hat{k}$. *Also find cartesian equations of the line.*

**Solution.** Let A, B and C be the points with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

$$\therefore \quad \overrightarrow{BC} = \text{P.V. of C} - \text{P.V. of B}$$
$$= (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) = 2\hat{i} - 2\hat{j} + \hat{k}.$$

We have to find the vector equation of a line through A and parallel to $\overrightarrow{BC}$.

$\therefore$ The vector equation of the required line is

$\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ where $\lambda$ is a parameter.

Now, as $\vec{r}$ is the position vector of any point P $(x, y, z)$ on the line, we get

$x\hat{i} + y\hat{j} + z\hat{k} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$.

On equating the coefficients of the base vectors, we get

$x = 2 + 2\lambda,\ y = -1 - 2\lambda,\ z = 1 + \lambda$.

On eliminating the parameter $\lambda$ from these equations, we get

$\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$, which are the required cartesian equations.

**Example 5.** *Find the vector and the cartesian equations of the line that passes through the points A (3, – 2, – 5) and B (3, – 2, 6).*

**Solution.** Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A (3, – 2, – 5) and B (3, – 2, 6) respectively. Then

$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k} \Rightarrow \vec{b} - \vec{a} = 11\hat{k}$.

Let $\vec{r}$ be the position vector of any point P $(x, y, z)$ on the line passing through the points A and B. The vector equation of the line is

$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ *i.e.* $\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda(11\hat{k})$

*i.e.* $\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + 11\lambda\hat{k}$, where $\lambda$ is a parameter.

Now as $\vec{r}$ is the position vector of P $(x, y, z)$, we get

$x\hat{i} + y\hat{j} + z\hat{k} = 3\hat{i} - 2\hat{j} - 5\hat{k} + 11\lambda\hat{k}$

*i.e.* $x\hat{i} + y\hat{j} + z\hat{k} = 3\hat{i} - 2\hat{j} + (11\lambda - 5)\hat{k}$.

On equating the coefficients of base vectors, we get

$x = 3,\ y = -2,\ z = 11\lambda - 5$, which are the required cartesian equations (in parametric form).

**Example 6.** *The points A (1, 2, 3), B (– 1, –2, –1) and C (2, 3, 2) are three vertices of a parallelogram ABCD. Find vector and cartesian equations of the sides AB and BC. Also find the coordinates of D.*

**Solution.** Given A (1, 2, 3), B (–1, –2, –1) and C (2, 3, 2).

Direction numbers of the line AB are $< -1 - 1,\ -2 - 2,\ -1 - 3 >$ *i.e.* $< 1, 2, 2 >$.

$\therefore$ The vector equation of the side AB is

$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$, $\lambda$ is a parameter,

and the cartesian equations of this line are $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{2}$.

Direction numbers of the side BC are $< 2 + 1,\ 3 + 2,\ 2 + 1 >$ *i.e.* $< 3, 5, 3 >$.

$\therefore$ The vector equation of the side BC is

$\vec{r} = -\hat{i} - 2\hat{j} - \hat{k} + \mu(3\hat{i} + 5\hat{j} + 3\hat{k})$, $\mu$ is a parameter,

and the cartesian equations of this line are $\frac{x+1}{3} = \frac{y+2}{5} = \frac{z+1}{3}$.

Let D be ($\alpha$, $\beta$, $\gamma$). Since ABCD is a parallelogram, diagonals AC and BD bisect each other *i.e.* mid-point of segment AC is same as mid-point of segment BD

$$\Rightarrow \left(\frac{1+2}{2}, \frac{2+3}{2}, \frac{3+2}{2}\right) = \left(\frac{\alpha-1}{2}, \frac{\beta-2}{2}, \frac{\gamma-1}{2}\right)$$

$\Rightarrow$ $\alpha - 1 = 3$, $\beta - 2 = 5$, $\gamma - 1 = 5$ $\Rightarrow$ $\alpha = 4$, $\beta = 7$, $\gamma = 6$.

Hence, the point D is (4, 7, 6).

**Example 7.** *Find the coordinates of the point where the line through the points (5, 1, 6) and (3, 4, 1) crosses the yz-plane.*

**Solution.** Direction numbers of the line passing through the points (5, 1, 6) and (3, 4, 1) are

$< 3 - 5, 4 - 1, 1 - 6 >$ *i.e.* $< -2, 3, -5 >$ *i.e.* $< 2, -3, 5 >$.

$\therefore$ The equations of the line through the given points are

$$\frac{x-5}{2} = \frac{y-1}{-3} = \frac{z-6}{5} \qquad \ldots(i)$$

Any point on the line ($i$) is $(5 + 2\lambda, 1 - 3\lambda, 6 + 5\lambda)$.

If this point lies on $yz$-plane, then its $x$-coordinate is zero

$\Rightarrow$ $5 + 2\lambda = 0$ $\Rightarrow$ $\lambda = -\frac{5}{2}$.

$\therefore$ The required point is $\left(0, 1+3.\frac{5}{2}, 6-5.\frac{5}{2}\right)$ *i.e.* $\left(0, \frac{17}{2}, -\frac{13}{2}\right)$.

**Example 8.** *Show that the points whose position vectors are given by* $-2\hat{i} + 3\hat{j} + 5\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$ *and* $7\hat{i} - \hat{k}$ *are collinear.*

**Solution.** Let A, B and C be the points whose position vectors are $-2\hat{i} + 3\hat{j} + 5\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$ and $7\hat{i} - \hat{k}$ respectively.

Equation of the line passing through points A and B is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

*i.e.* $\vec{r} = -2\hat{i} + 3\hat{j} + 5\hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k} - (-2\hat{i} + 3\hat{j} + 5\hat{k}))$

*i.e.* $\vec{r} = -2\hat{i} + 3\hat{j} + 5\hat{k} + \lambda(3\hat{i} - \hat{j} - 2\hat{k}) \qquad \ldots(i)$

The points A, B and C are collinear if the point C lies on ($i$)

*i.e.* if $7\hat{i} - \hat{k} = -2\hat{i} + 3\hat{j} + 5\hat{k} + \lambda(3\hat{i} - \hat{j} - 2\hat{k})$ is consistent

*i.e.* if $7 = -2 + 3\lambda$, $0 = 3 - \lambda$ and $-1 = 5 - 2\lambda$ are consistent

*i.e.* if $\lambda = 3$, $\lambda = 3$ and $\lambda = 3$, which are true.

Hence, the given points are collinear.

***Alternative method (cartesian approach)***

The coordinates of the points A, B and C are

(– 2, 3, 5), (1, 2, 3) and (7, 0, – 1) respectively.

Direction numbers of the line AB are $< 1 + 2, 2 - 3, 3 - 5 >$ *i.e.* $< 3, -1, -2 >$.

The equations of the line AB are

$$\frac{x+2}{3} = \frac{y-3}{-1} = \frac{z-5}{-2} \qquad \ldots(ii)$$

Now the points A, B and C are collinear if the point C (7, 0, – 1) lies on ($ii$)

*i.e.* if $\frac{7+2}{3} = \frac{0-3}{-1} = \frac{-1-5}{-2}$ are consistent

*i.e.* if $3 = 3 = 3$, which is true.

Hence, the given points are collinear.

**Example 9.** *Find the values of p and q by using vector method such that the points A (5, 0, 5), B (2, 1, 3) and C (–4, p, q) are collinear.*

**Solution.** Direction numbers of the line AB are < 2 – 5, 1 – 0, 3 – 5 > *i.e.* < –3, 1, –2 > *i.e.* < 3, –1, 2 >.

∴ The vector equation of the line AB is

$$\vec{r} = 5\hat{i} + 0\hat{j} + 5\hat{k} + \lambda(3\hat{i} - \hat{j} + 2\hat{k}).$$

The position vector of the point C is $-4\hat{i} + p\hat{j} + q\hat{k}$.

Since the points A, B and C are collinear, C lies on the line AB

$$\Rightarrow \quad -4\hat{i} + p\hat{j} + q\hat{k} = 5\hat{i} + 5\hat{k} + \lambda(3\hat{i} - \hat{j} + 2\hat{k}).$$

Equating the coefficients of the base vectors, we get

$$-4 = 5 + 3\lambda, \; p = -\lambda \; \text{ and } \; q = 5 + 2\lambda$$

$$\Rightarrow \lambda = -3, \; p = -(-3) = 3, \; q = 5 + 2(-3) = -1.$$

Hence, $p = 3$ and $q = -1$.

**Example 10.** *The cartesian equations of a line are $2x - 3 = 3y + 1 = 5 - 6z$. Find the direction ratios of the line and write down the vector and cartesian equations of the line through (7, –5, 0) which is parallel to the given line.*

**Solution.** The equations of the line are

$$2x - 3 = 3y + 1 = 5 - 6z \; i.e. \; 2\left(x - \frac{3}{2}\right) = 3\left(y + \frac{1}{3}\right) = -6\left(z - \frac{5}{6}\right)$$

$$i.e. \quad \frac{x - \frac{3}{2}}{\frac{1}{2}} = \frac{y + \frac{1}{3}}{\frac{1}{3}} = \frac{z - \frac{5}{6}}{-\frac{1}{6}} \quad i.e. \quad \frac{x - \frac{3}{2}}{3} = \frac{y + \frac{1}{3}}{2} = \frac{z - \frac{5}{6}}{-1}.$$

Therefore, direction ratios of the given line are < 3, 2, – 1 >. The vector equation of the line through (7, –5, 0) and parallel to the given line is

$$\vec{r} = 7\hat{i} - 5\hat{j} + \lambda(3\hat{i} + 2\hat{j} - \hat{k}), \text{ where } \lambda \text{ is a parameter}$$

and the cartesian equations of this line are

$$\frac{x-7}{3} = \frac{y+5}{2} = \frac{z}{-1}.$$

**Example 11.** *Find the points on the line through the points A (1, 2, 3) and B (3, 5, 9) at a distance of 14 units from the mid-point of segment AB.*

**Solution.** Direction numbers of the line AB are < 3 – 1, 5 –2, 9 – 3 > *i.e.* < 2, 3, 6 >.

The equations of the line AB are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$.

Any point on the line AB is P(1 + 2λ, 2 + 3λ, 3 + 6λ).

Mid-point of AB is $M\left(\frac{1+3}{2}, \frac{2+5}{2}, \frac{3+9}{2}\right)$ *i.e.* $M\left(2, \frac{7}{2}, 6\right)$.

According to given | MP | = 14

$$\Rightarrow \quad \sqrt{(2\lambda + 1 - 2)^2 + \left(3\lambda + 2 - \frac{7}{2}\right)^2 + (6\lambda + 3 - 6)^2} = 14$$

$$\Rightarrow \quad (2\lambda - 1)^2 + \left(3\lambda - \frac{3}{2}\right)^2 + (6\lambda - 3)^2 = 196$$

$$\Rightarrow \quad 4\lambda^2 - 4\lambda + 1 + 9\lambda^2 - 9\lambda + \frac{9}{4} + 36\lambda^2 - 36\lambda + 9 = 196$$

$$\Rightarrow \quad 49\lambda^2 - 49\lambda - \frac{735}{4} = 0$$

$\Rightarrow\ 4\lambda^2 - 4\lambda - 15 = 0 \Rightarrow (2\lambda - 5)(2\lambda + 3) = 0$

$\Rightarrow\ \lambda = \frac{5}{2}, -\frac{3}{2}.$

$\therefore$ The required points are $\left(1+2.\frac{5}{2}, 2+3.\frac{5}{2}, 3+6.\frac{5}{2}\right)$ and $\left(1-2.\frac{3}{2}, 2-3.\frac{3}{2}, 3-6.\frac{3}{2}\right)$

*i.e.* $\left(6, \frac{19}{2}, 18\right)$ and $\left(-2, -\frac{5}{2}, -6\right)$.

## EXERCISE 2.1

**1.** Find the vector equation of the line passing through the point (2, –3, 5) and parallel to the vector $3\hat{i} + 2\hat{j} - 7\hat{k}$.

**2.** Find the cartesian equation of the line which passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction of the vector $\hat{i} + 2\hat{j} - \hat{k}$.

**3.** Find the vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$.

**4.** Find the cartesian equations of the line which passes through the point (– 2, 4, – 5) and parallel to the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

**5.** Find the vector equation of a line passing through (1, – 2, 3) and parallel to the line $\frac{x-1}{3} = \frac{y+2}{-5} = \frac{z+5}{2}$.

**6.** Find the vector equation of the line passing through the points (– 1, 0, 2) and (3, 4, 6).

**7.** Find the vector and the cartesian equations of the line that passes through origin and the point (5, – 2, 3).

**8.** The cartesian equations of a line AB are $\frac{2x-1}{2} = \frac{4-y}{7} = \frac{z+1}{2}$. Write the direction ratios of a line parallel to AB.

**9.** The cartesian equations of a line are $2x - 3 = 3y + 1 = 5 - 6z$. Write the direction ratios of the line.

**10.** Write the vector equation of the $x$-axis.

**11.** Find the vector and the cartesian equations of the line through the point (5, 2, – 4) and which is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.

**12.** Find the vector equation of the line passing through the point A (2, – 1, 1) and parallel to the line joining the points B (– 1, 4, 1) and C (1, 2, 2). Also find the cartesian equation of the line.

**13.** Find the vector equation of a line passing through the point with position vector $\hat{i} - 2\hat{j} - 3\hat{k}$ and parallel to the line joining the points with position vectors $\hat{i} - \hat{j} + 4\hat{k}$ and $2\hat{i} + \hat{j} + 2\hat{k}$. Also find the cartesian equations of the line.

**14.** Find the vector equation of the line joining the points whose position vectors are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} - 3\hat{k}$. Also find its cartesian equations.

**15.** The cartesian equations of a line are $6x - 2 = 3y + 1 = 2z - 2$. Find direction ratios of the line and write down the vector equation of the line through (2, – 1, – 1) which is parallel to the given line.

**16.** Find the equations of a line through A(1, –1, 5) and parallel to the line $\frac{x-2}{3} = \frac{y-5}{-2}, z = -1$.

**17.** The cartesian equations of a line are $x = ay + b$, $z = cy + d$. Find direction numbers of the line, also find its equation in vector form.

**18.** The points A (4, 5, 10), B (2, 3, 4) and C (1, 2, –1) are three vertices of a parallelogram ABCD. Find vector and cartesian equations of the sides AB and BC. Also find the coordinates of D.

**19.** (*i*) Find the coordinates of the point where the line through A (3, 4, 1) and B (5, 1, 6) crosses the *xy*-plane.

(*ii*) Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the *zx*-plane.

**20.** (*i*) Find the equation of the line passing through the points A (–2, 4, 7), B (3, – 6, –8). Hence show that the points A, B and C (1, –2, –2) are collinear.

(*ii*) If the points (– 1, 3, 2), (– 4, 2, – 2) and (5, 5, λ) are collinear, then find the value of λ.

**21.** By using vector method, find the values of *p* and *q* so that the points (*p*, *q*, 1), (– 1, 4, –2) and (0, 2, –1) may be collinear.

**22.** Find the coordinates of the points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which are at a distance of 3 units from the point (1, –2, 3).

**23.** Find the point on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance $3\sqrt{2}$ from the point (1, 2, 3).

**24.** Find points on the line through the points A(3, 4, –1) and B(5, 1, 5) at a distance of 7 units from the mid-point of [AB].

## 2.2 ANGLE BETWEEN TWO LINES

***Angle*** *between two lines is the same as the angle between the two vectors along their directions.* Observe that angle between a pair of straight lines depends only on their directions and not on their positions.

In this section, we shall find angle between two lines when their equations are known.

### 2.2.1 To find angle between two lines

Let $\vec{r} = \vec{a} + \lambda \vec{b}$ ...(1) and $\vec{r} = \vec{a}' + \mu \vec{c}$ ...(2)

be the vector equations of the two lines.

These lines are in the directions of $\vec{b}$ and $\vec{c}$.

If θ is the angle between these lines, then θ is the angle between the vectors $\vec{b}$ and $\vec{c}$.

$$\therefore \quad \cos\theta = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}|\,|\vec{c}|}. \qquad ...(3)$$

**Cartesian form**

Let the equations of the given lines in cartesian form be

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \qquad ...(4)$$

**6.** Find the value of $\lambda$ so that the line joining the points A(3, 2, 1) and B(4, $\lambda$, 5) may intersect the line joining the points C(4, 2, –2) and D(6, 5, –1).

**7.** Find the equations of the line joining the points (–2, 1, 3) and (5, 1, –1) and show that it is perpendicular to $y$-axis.

**8.** A (1, 0, 4), B (0, –11, 3), C (2, –3, 1) are three points and D is the foot of perpendicular from A on BC. Find the co-ordinates of D.

**9.** Find the equations of the perpendicular drawn from the point A (2, 4, –1) to the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$. Also obtain the length of perpendicular.

**10.** Find the image of the point (5, 9, 3) in the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.

**11.** A line with direction numbers < 2, 7, –5 > is drawn to intersect the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$.

Find the coordinates of the points of intersection and the length intercepted on it.

**12.** Find whether or not the two lines given below intersect :

$\vec{r} = (2\lambda+1)\hat{i} - (\lambda+1)\hat{j} + (\lambda+1)\hat{k}$, $\vec{r} = (3\mu+2)\hat{i} - (5\mu+5)\hat{j} + (2\mu+1)\hat{k}$.

**13.** Find the shortest distance between the lines

$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$.

Also find the equations of the line of shortest distance.

## ANSWERS

### EXERCISE 2.1

**1.** $\vec{r} = 2\hat{i} - 3\hat{j} + 5\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 7\hat{k})$, where $\lambda$ is a real number.

**2.** $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$.

**3.** $\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$.

**4.** $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$.

**5.** $\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} - 5\hat{j} + 2\hat{k})$.

**6.** $\vec{r} = -\hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$.

**7.** $\vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$; $\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$.

**8.** < 1, –7, 2 >. **9.** < 3, 2, –1 >. **10.** $\vec{r} = \lambda\hat{i}$.

**11.** $\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$; $\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$.

**12.** $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$; $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$.

**13.** $\vec{r} = \hat{i} - 2\hat{j} - 3\hat{k} + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$; $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z+3}{-2}$.

**14.** $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(-\hat{i} + 3\hat{j} - 4\hat{k})$; $\frac{x-2}{-1} = \frac{y+1}{3} = \frac{z-1}{-4}$.

**15.** < 1, 2, 3 >; $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$.

**16.** $\frac{x-1}{3} = \frac{y+1}{-2}$, $z = 5$.

**17.** < $a$, 1, $c$ >; $\vec{r} = (b\hat{i} + d\hat{k}) + \lambda(a\hat{i} + \hat{j} + c\hat{k})$.

**18.** $\vec{r} = 4\hat{i} + 5\hat{j} + 10\hat{k} + \lambda(\hat{i} + \hat{j} + 3\hat{k})$, $\frac{x-4}{1} = \frac{y-5}{1} = \frac{z-10}{3}$;

$\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + \mu(\hat{i} + \hat{j} + 5\hat{k})$, $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{5}$; D (3, 4, 5).

**19.** (*i*) $\left(\frac{13}{5},\ \frac{23}{5},\ 0\right)$ (*ii*) $\left(\frac{17}{3},\ 0,\ \frac{23}{3}\right)$.

**20.** (*i*) $\frac{x+2}{1} = \frac{y-4}{-2} = \frac{z-7}{-3}$ (*ii*) 10.

**21.** $p = 2,\ q = -2.$ **22.** $\left(\frac{13}{7}, -\frac{5}{7}, \frac{39}{7}\right), \left(\frac{1}{7}, -\frac{23}{7}, \frac{3}{7}\right).$

**23.** $(-2, -1, 3), \left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17}\right).$ **24.** $\left(6, -\frac{1}{2}, 8\right), \left(2, \frac{11}{2}, -4\right).$

## EXERCISE 2.2

**1.** (*i*) 0 (*ii*) $\cos^{-1}\left(\frac{19}{21}\right)$. **2.** (*i*) $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$ (*ii*) $\cos^{-1}\left(\frac{2}{3}\right)$.

**4.** (*i*) $-\frac{10}{7}$ (*ii*) 1.

**6.** (*i*) $\cos^{-1}\left(\frac{3}{\sqrt{182}}\right)$ (*ii*) $\cos^{-1}\left(\frac{3}{\sqrt{182}}\right)$ (*iii*) $\cos^{-1}\left(\frac{4}{\sqrt{150}}\right)$ (*iv*) $\cos^{-1}\left(\frac{1}{5}\right)$.

**7.** (*i*) (2, 3, –1) (*ii*) (3, 5, 9); $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$, 7 units.

**8.** $\frac{x}{1} = \frac{y-6}{4} = \frac{z+9}{-4}$; $(-1, 2, -5)$. **9.** (0, 5, 1). **10.** (0, 5, 1).

**11.** $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$. **12.** $\frac{x-1}{4} = \frac{y-1}{-5} = \frac{z+3}{1}$.

**13.** $\vec{r} = (\hat{i} + \hat{j} - 3\hat{k}) + \lambda(4\hat{i} - 5\hat{j} + \hat{k})$. **14.** $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(-6\hat{i} - 3\hat{j} + 6\hat{k})$.

## EXERCISE 2.3

**1.** (*i*) $\frac{3\sqrt{2}}{2}$ units (*ii*) $\frac{2}{\sqrt{5}}$ units (*iii*) $\frac{10}{\sqrt{59}}$ units (*iv*) $\frac{8}{\sqrt{29}}$ units.

**2.** $3\sqrt{30}$ units; $\frac{x-3}{2} = \frac{y-8}{6} = \frac{z-3}{-1}$. **3.** $\frac{1}{\sqrt{6}}$ units; $\frac{x-\frac{5}{3}}{1} = \frac{y-3}{-2} = \frac{z-\frac{13}{3}}{1}$.

**4.** (*i*) No (*ii*) Yes (*iii*) Yes. **6.** $\sqrt{26}$ units.

**7.** $\vec{r} = -\hat{i} + 2\hat{j} + \hat{k} + \lambda(\hat{i} - 2\hat{j} + \hat{k})$; $\sqrt{\frac{83}{6}}$ units. **8.** $\sqrt{21}$ units.

**9.** $\vec{r} = 3\hat{i} - 4\hat{k} + \lambda(5\hat{i} - 2\hat{j} + 4\hat{k})$; $\sqrt{\frac{974}{45}}$ units.

**10.** (1, 3, 2). **11.** (10, 14, 4). **13.** (*i*) Yes (*ii*) No.

## CHAPTER TEST

**1.** D(1, 10, – 15); $\frac{x-2}{1} = \frac{y+3}{-13} = \frac{z-4}{19}$. **2.** $\lambda = 1,\ \mu = -13.$

**3.** < 2, –3, 6 >; $\vec{r} = -\hat{i} - 2\hat{j} + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$.

**4.** (1, 8, 4), (1, –4, –12). **5.** (0, 0, –4).

**6.** 5. **7.** $\frac{x+2}{7} = \frac{z-3}{-4}$, $y = 0$. **8.** $\left(\frac{22}{9}, -\frac{11}{9}, \frac{5}{9}\right)$.

**9.** $\frac{x-2}{6} = \frac{y-4}{3} = \frac{z+1}{2}$; 7 units. **10.** (1, 1, 11).

**11.** (2, 8, – 3) and (0, 1, 2); $\sqrt{78}$ units. **12.** Do not intersect.

**13.** $2\sqrt{29}$ units; $\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$.