1 Vectors

INTRODUCTION

In class XI, we have read about scalars and vectors; various types of vectors—null vector, proper vector, unit vector, like and unlike vectors, equal vectors, collinear and non-collinear vectors, free and localised vectors, co-initial vectors, coplanar and non-coplanar vectors, negative of a vector; multiplication of a vector by a scalar, sum and difference of vectors; position vector of a point; triangle inequality; section formula; vectors in two and three dimensions; expressing a vector in terms of unit vectors along x, y and z axes (*i.e.* $\hat{i}$, $\hat{j}$, $\hat{k}$), and applications of vector algebra in geometry.

In the present chapter, we shall study :

(*i*) Scalar (dot) product of vectors — its properties

(*ii*) Vector (cross) product of vectors — its properties, area of a triangle, collinear vectors

(*iii*) Scalar triple product — volume of a parallelopiped, coplanarity

(*iv*) Proofs of formulae (using vectors) and applications of vectors in some geometrical problems.

1.1 SCALAR (OR DOT) PRODUCT OF TWO VECTORS

Since vectors have directions, these cannot be multiplied like numbers. We shall see that there are two types of products of vectors — one of these yield a scalar quantity and the other a vector quantity. Moreover, as the origin of this branch of Mathematics lies in physical problems, the definitions of products of vectors should be so framed that are helpful in applications to physical sciences.

Angle between two vectors

Given two (non-zero) vectors $\vec{a}$ and $\vec{b}$, we can shift them parallel to themselves so that they intersect at a point. Then the angle, say θ, between them is called the **angle between the vectors** $\vec{a}$ and $\vec{b}$. Note that $0 \le \theta \le \pi$.

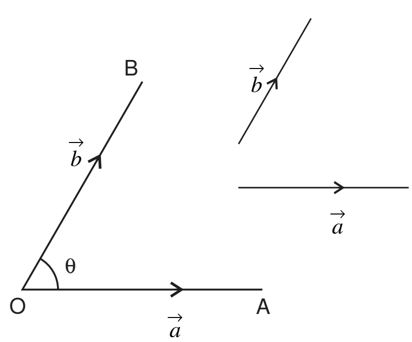

Fig. 1.1.

1.1.1 Scalar or dot product of two vectors

*The **scalar** (or **dot**) **product** of two (non-zero) vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \bullet \vec{b}$ (read as $\vec{a}$ dot $\vec{b}$), is defined as*

$$\vec{a} \bullet \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = ab\cos\theta,$$

where $a = |\vec{a}|$, $b = |\vec{b}|$ and $\theta\,(0 \le \theta \le \pi)$ is the angle between $\vec{a}$ and $\vec{b}$.

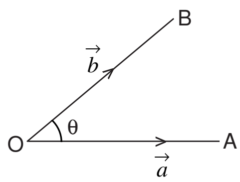

Fig. 1.2.

Remarks

1. If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or both $\vec{a} = \vec{0}$ and $\vec{b} = \vec{0}$, we define $\vec{a} \bullet \vec{b} = 0$.

2. The vectors $\vec{a}$ and $\vec{b}$ are known as factors of $\vec{a} \bullet \vec{b}$.

3. The dot product of two vectors is a ***scalar*** quantity.

4. If $\theta = 0$, $\vec{a} \bullet \vec{b} = ab$ (as $\cos 0 = 1$).

5. If $\theta = \pi$, $\vec{a} \bullet \vec{b} = -ab$ (as $\cos\pi = -1$).

In particular, $\vec{a} \bullet (-\vec{a}) = |\vec{a}|\,|-\vec{a}| \cos\pi$ $\quad(\because$ angle between $\vec{a}$ and $-\vec{a}$ is $\pi)$

$= |\vec{a}|\,|-\vec{a}|\,(-1)$ $\quad(\because \cos\pi = -1)$

$= -a^2$ $\quad(\because |-\vec{a}| = a)$

6. Condition of perpendicularity of two vectors

If $\vec{a}$ and $\vec{b}$ are perpendicular, the angle between them is $\frac{\pi}{2}$ and we obtain $\vec{a} \bullet \vec{b} = ab\cos\frac{\pi}{2} = 0$.

Conversely, if $\vec{a} \bullet \vec{b} = 0$ *i.e.* if $ab\cos\theta = 0$, then either $a = 0$ or $b = 0$ or $\cos\theta = 0$; it follows that either (or both) of the vectors is a zero vector or else they are perpendicular.

Therefore, ***two non-zero (proper) vectors $\vec{a}$ and $\vec{b}$ are perpendicular iff $\vec{a} \bullet \vec{b} = 0$, which is the required condition of perpendicularity.***

Thus, $\vec{a} \bullet \vec{b} = 0$ iff $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$.

7. Note that $\vec{a} \bullet \vec{b} = \begin{cases} > 0 \text{ if } 0 \le \theta < \frac{\pi}{2} \text{ i.e. if angle between vectors is acute} \\ 0 \text{ if } \theta = \frac{\pi}{2} \text{ i.e. if } \vec{a} \text{ and } \vec{b} \text{ are perpendicular} \\ < 0 \text{ if } \frac{\pi}{2} < \theta \le \pi \text{ i.e. if angle between vectors is obtuse.} \end{cases}$

8. Square of a vector

The scalar product of a vector $\vec{a}$ with itself is called the square of the vector $\vec{a}$, and is written as $\vec{a} \bullet \vec{a}$ or $(\vec{a})^2$.

$$(\vec{a})^2 = \vec{a} \bullet \vec{a} = aa\cos 0 = a \cdot a \cdot 1 = a^2 = |\vec{a}|^2.$$

Thus, the square of a vector is equal to the square of its modulus.

The **length** of a vector can be found by using

$$|\vec{a}| = \sqrt{\vec{a} \bullet \vec{a}} \qquad (\because |\vec{a}|^2 = \vec{a} \bullet \vec{a})$$

9. Squares and scalar products of $\hat{i}, \hat{j}, \hat{k}$

Since $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along the coordinate axes *i.e.* along three mutually perpendicular lines, we have

$$\hat{i} \bullet \hat{i} = 1.1 \cos 0 = 1.$$

Similarly, $\hat{j} \bullet \hat{j} = 1,\ \hat{k} \bullet \hat{k} = 1.$

Also $\hat{i} \bullet \hat{j} = 1.1 \cos 90° = 1.1.0 = 0.$

Similarly, $\hat{j} \bullet \hat{i} = 0,\ \hat{j} \bullet \hat{k} = 0,$

$\hat{k} \bullet \hat{j} = 0,\ \hat{k} \bullet \hat{i} = 0,\ \hat{i} \bullet \hat{k} = 0.$

1.1.2 Properties of scalar (or dot) product

Theorem 1. *The scalar product is commutative i.e.* $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$.

Proof. If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then by definition, $\vec{a} \bullet \vec{b} = 0$, $\vec{b} \bullet \vec{a} = 0$ so that $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$.

If $\vec{a}, \vec{b}$ are two non-zero vectors, let θ be the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$. Then angle between $\vec{b}$ and $\vec{a}$ is also θ.

$\therefore \quad \vec{a} \bullet \vec{b} = a\,b \cos\theta = b\,a \cos\theta = \vec{b} \bullet \vec{a}.$

Thus, $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$.

Theorem 2. *If $\vec{a}, \vec{b}$ are any vectors and m is any real number (scalar), then*

$$(m\vec{a}) \bullet \vec{b} = m(\vec{a} \bullet \vec{b}) = \vec{a} \bullet (m\vec{b}).$$

Corollary 1. $\vec{a} \bullet (-\vec{b}) = -(\vec{a} \bullet \vec{b}) = (-\vec{a}) \bullet (\vec{b})$.

Corollary 2. $(-\vec{a}) \bullet (-\vec{b}) = \vec{a} \bullet \vec{b}$.

Theorem 3. *The scalar product is distributive w.r.t. addition i.e.*

$$\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}.$$

Corollary 1. $\vec{a} \bullet (\vec{b} + \vec{c} + \vec{d}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c} + \vec{a} \bullet \vec{d}$ and, in general

$$\vec{a} \bullet (\vec{b} + \vec{c} + \vec{d} + \ldots) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c} + \vec{a} \bullet \vec{d} + \ldots$$

Corollary 2. $\vec{a} \bullet (\vec{b} - \vec{c}) = \vec{a} \bullet \vec{b} - \vec{a} \bullet \vec{c}$.

We have $\vec{a} \bullet (\vec{b} - \vec{c}) = \vec{a} \bullet (\vec{b} + (-\vec{c})) = \vec{a} \bullet \vec{b} + \vec{a} \bullet (-\vec{c})$

$= \vec{a} \bullet \vec{b} + (-(\vec{a} \bullet \vec{c})) = \vec{a} \bullet \vec{b} - \vec{a} \bullet \vec{c}$.

Corollary 3. Scalar product of sums of vectors

We have $(\vec{a} + \vec{b}) \bullet (\vec{c} + \vec{d}) = (\vec{a} + \vec{b}) \bullet \vec{c} + (\vec{a} + \vec{b}) \bullet \vec{d}$

$= \vec{c} \bullet (\vec{a} + \vec{b}) + \vec{d} \bullet (\vec{a} + \vec{b})$

($\because$ the scalar product is commutative)

$$= \vec{c} \bullet \vec{a} + \vec{c} \bullet \vec{b} + \vec{d} \bullet \vec{a} + \vec{d} \bullet \vec{b}$$

$$= \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c} + \vec{a} \bullet \vec{d} + \vec{b} \bullet \vec{d}.$$

Corollary 4. (*i*) $(\vec{a} + \vec{b})^2 = (\vec{a})^2 + 2\,\vec{a} \bullet \vec{b} + (\vec{b})^2$

(*ii*) $(\vec{a} - \vec{b})^2 = (\vec{a})^2 - 2\,\vec{a} \bullet \vec{b} + (\vec{b})^2$

(*iii*) $(\vec{a} + \vec{b}) \bullet (\vec{a} - \vec{b}) = (\vec{a})^2 - (\vec{b})^2$

(*iv*) $(\vec{a} + \vec{b})^2 + (\vec{a} - \vec{b})^2 = 2[(\vec{a})^2 + (\vec{b})^2]$

(*v*) $(\vec{a} + \vec{b})^2 - (\vec{a} - \vec{b})^2 = 4\,\vec{a} \bullet \vec{b}$.

Proof. (*i*) $(\vec{a} + \vec{b})^2 = (\vec{a} + \vec{b}) \bullet (\vec{a} + \vec{b}) = \vec{a} \bullet \vec{a} + \vec{a} \bullet \vec{b} + \vec{b} \bullet \vec{a} + \vec{b} \bullet \vec{b}$

$= \vec{a} \bullet \vec{a} + \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{b} + \vec{b} \bullet \vec{b}$ $(\because \vec{b} \bullet \vec{a} = \vec{a} \bullet \vec{b})$

$= (\vec{a})^2 + 2\,\vec{a} \bullet \vec{b} + (\vec{b})^2$.

Proofs of (*ii*), (*iii*), (*iv*) and (*v*) are left for the reader.

Corollary 5. If $\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c}$, then either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a}$ is perpendicular to $(\vec{b} - \vec{c})$.

Proof. $\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c} \Rightarrow \vec{a} \bullet \vec{b} - \vec{a} \bullet \vec{c} = 0 \Rightarrow \vec{a} \bullet (\vec{b} - \vec{c}) = 0$

$\Rightarrow$ either $\vec{a} = \vec{0}$ or $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a}$ is perpendicular to $(\vec{b} - \vec{c})$

$\Rightarrow$ either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a}$ is perpendicular to $(\vec{b} - \vec{c})$.

1.1.3 Scalar product of two vectors in terms of their rectangular components

Let $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$,

then $\vec{a} \bullet \vec{b} = (a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}) \bullet (b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k})$

$= a_1 b_1\,\hat{i} \bullet \hat{i} + a_1 b_2\,\hat{i} \bullet \hat{j} + a_1 b_3\,\hat{i} \bullet \hat{k} + a_2 b_1\,\hat{j} \bullet \hat{i} + a_2 b_2\,\hat{j} \bullet \hat{j}$

$+\; a_2 b_3\,\hat{j} \bullet \hat{k} + a_3 b_1\,\hat{k} \bullet \hat{i} + a_3 b_2\,\hat{k} \bullet \hat{j} + a_3 b_3\,\hat{k} \bullet \hat{k}$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$ $(\because (\hat{i})^2 = (\hat{j})^2 = (\hat{k})^2 = 1$ and $\hat{i} \bullet \hat{j} = \hat{j} \bullet \hat{k} = \hat{k} \bullet \hat{i} = 0)$

Thus, $\boldsymbol{\vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$.

Therefore, the scalar product of two vectors is equal to the sum of the products of their corresponding rectangular components.

Corollary. Let $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$, $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$, then $\vec{a}$ and $\vec{b}$ are perpendicular iff $\vec{a} \bullet \vec{b} = \vec{0}$ *i.e.* iff $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$.

Thus, $\vec{a}$ *and* $\vec{b}$ *are* ***perpendicular*** *iff* $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$.

1.1.4 Angle between two vectors

Angle θ between two vectors $\vec{a}$ and $\vec{b}$ can be found by using

$$\vec{a} \bullet \vec{b} = ab \cos\theta \;\Rightarrow\; \cos\theta = \frac{\vec{a} \bullet \vec{b}}{ab} = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}|\,|\vec{b}|}.$$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

1.1.5 Geometrical interpretation of scalar product

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent vectors $\vec{a}$ and $\vec{b}$ respectively, then

$$a = |\vec{a}| = |\overrightarrow{OA}| = |OA| = OA \text{ and}$$

$$b = |\vec{b}| = |\overrightarrow{OB}| = |OB| = OB.$$

Let ∠AOB = θ and M, N be the feet of perpendiculars from A, B on OB, OA respectively.

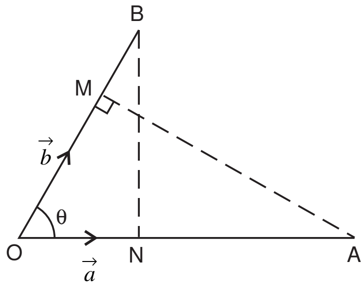

Fig. 1.3.

From right angled ΔOAM,

$$\cos\theta = \frac{OM}{OA} \Rightarrow OM = OA\cos\theta \qquad \ldots(i)$$

Now, projection of $\vec{a}$ on $\vec{b}$ = OM = OA cos θ = $a\cos\theta$,

$\therefore \quad \vec{a}\bullet\vec{b} = ab\cos\theta = b\,(a\cos\theta) = b$. projection of $\vec{a}$ on $\vec{b}$.

Similarly, projection of $\vec{b}$ on $\vec{a}$ = ON = OB cos θ = $b\cos\theta$,

$\therefore \quad \vec{a}\bullet\vec{b} = ab\cos\theta = a\,(b\cos\theta) = a$. projection of $\vec{b}$ on $\vec{a}$.

Thus, $\vec{a}\cdot\vec{b}$ can be defined as the product of the modulus of one vector and the projection of the other vector upon it.

Also, we find that projection of $\vec{a}$ on $\vec{b}$ = $a\cos\theta = a \cdot \dfrac{\vec{a}\bullet\vec{b}}{ab} = \dfrac{\vec{a}\bullet\vec{b}}{|\vec{b}|}$.

Hence, the (scalar) projection of vector $\vec{a}$ on vector $\vec{b}$ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Similarly, the (scalar) projection of vector $\vec{b}$ on vector $\vec{a}$ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

1.1.6 Direction cosines of a given vector

*Let $\overrightarrow{AB}$ be any vector in space and $\overrightarrow{OP}$ be a vector passing through origin O and parallel to the vector $\overrightarrow{AB}$. If the vector $\overrightarrow{OP}$ makes angles α, β and γ with the three rectangular axes respectively as shown in fig. 1.4, then α, β and γ are called the **direction angles** of the vector $\overrightarrow{AB}$ and cos α, cos β and cos γ are called **direction cosines** of the vector $\overrightarrow{AB}$.*

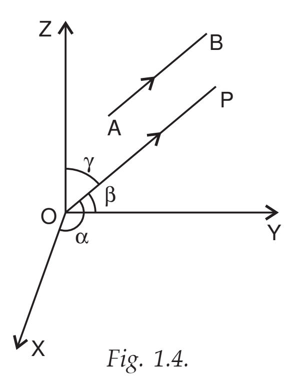

Fig. 1.4.

If α, β and γ are the direction angles of a (non-zero) vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then its direction cosines are given as

$$\cos\alpha = \frac{\vec{a}\bullet\hat{i}}{|\vec{a}|\,|\hat{i}|} = \frac{\vec{a}\bullet\hat{i}}{|\vec{a}|},\ \cos\beta = \frac{\vec{a}\bullet\hat{j}}{|\vec{a}|} \text{ and } \cos\gamma = \frac{\vec{a}\bullet\hat{k}}{|\vec{a}|}$$

i.e. $\cos\alpha = \dfrac{a_1}{|\vec{a}|}$, $\cos\beta = \dfrac{a_2}{|\vec{a}|}$ and $\cos\gamma = \dfrac{a_3}{|\vec{a}|}$

i.e. $|\vec{a}|\cos\alpha = a_1$, $|\vec{a}|\cos\beta = a_2$ and $|\vec{a}|\cos\gamma = a_3$.

It follows that the scalar components a_1, a_2 and a_3 of the vector $\vec{a}$ are the projections of $\vec{a}$ along x-axis, y-axis and z-axis respectively.

ILLUSTRATIVE EXAMPLES

Example 1. *If $\vec{a}$ is a unit vector and $(2\vec{a}+\vec{b})\cdot(2\vec{a}-\vec{b}) = 2$, then find $|\vec{b}|$.*

Solution. Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

Given $(2\vec{a}+\vec{b})\cdot(2\vec{a}-\vec{b}) = 2 \Rightarrow 2\vec{a}\cdot 2\vec{a} - 2\vec{a}\cdot\vec{b} + \vec{b}\cdot 2\vec{a} - \vec{b}\cdot\vec{b} = 2$

$\Rightarrow 4|\vec{a}|^2 - |\vec{b}|^2 = 2$ $(\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$

$\Rightarrow 4.1 - |\vec{b}|^2 = 2$ $(\because |\vec{a}| = 1)$

$\Rightarrow |\vec{b}|^2 = 2 \Rightarrow |\vec{b}| = \sqrt{2}$ (as magnitude of a vector is non-negative)

Example 2. *If $(\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$, find $|\vec{a}|$ and $|\vec{b}|$.*

Solution. Given $(\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b}) = 8 \Rightarrow \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b} = 8$

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$ $(\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$

$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8$ $(\because |\vec{a}| = 8|\vec{b}|$, given)

$\Rightarrow 63|\vec{b}|^2 = 8 \Rightarrow |\vec{b}| = \sqrt{\dfrac{8}{63}}$.

$\therefore |\vec{a}| = 8|\vec{b}| = 8\sqrt{\dfrac{8}{63}}$.

Example 3. *If the angle between two vectors $\vec{a}$ and $\vec{b}$ of equal magnitude is 30° and their scalar product is $2\sqrt{3}$, find their magnitudes.*

Solution. Given $|\vec{a}| = |\vec{b}|$, θ (angle between $\vec{a}$ and $\vec{b}$) = 30° and $\vec{a}\cdot\vec{b} = 2\sqrt{3}$

$\Rightarrow |\vec{a}||\vec{b}|\cos 30^\circ = 2\sqrt{3} \Rightarrow |\vec{a}|^2 . \dfrac{\sqrt{3}}{2} = 2\sqrt{3}$

$\Rightarrow |\vec{a}|^2 = 4 \Rightarrow |\vec{a}| = 2$

Hence, $|\vec{a}| = 2 = |\vec{b}|$.

Example 4. *Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively and $\vec{a}\cdot\vec{b} = \sqrt{6}$.*

Solution. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

Given $\vec{a}\cdot\vec{b} = \sqrt{6} \Rightarrow |\vec{a}||\vec{b}|\cos\theta = \sqrt{6}$

$\Rightarrow \sqrt{3}\,.\,2\cos\theta = \sqrt{6}$ $(\because |\vec{a}| = \sqrt{3}$ and $|\vec{b}| = 2$ given)

$\Rightarrow \cos\theta = \dfrac{\sqrt{6}}{2\sqrt{3}} = \dfrac{1}{\sqrt{2}} \Rightarrow \cos\theta = \cos 45^\circ \Rightarrow \theta = 45^\circ$.

Hence, the angle between the given vectors = 45°.

Example 5. *Find the angle between two vectors* $\hat{i} - 2\hat{j} + 3\hat{k}$ *and* $3\hat{i} - 2\hat{j} + \hat{k}$.

Solution. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

Then $\vec{a} \bullet \vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k}) \bullet (3\hat{i} - 2\hat{j} + \hat{k})$

$= 1 \times 3 + (-2)(-2) + 3 \times 1 = 3 + 4 + 3 = 10$

$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$ and $|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$.

Let θ be the angle between the given vectors, then θ is given by

$$\cos\theta = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|} = \frac{10}{\sqrt{14}\sqrt{14}} = \frac{10}{14} = \frac{5}{7} \Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right).$$

Example 6. *If* $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ *and* $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, *show that the vectors* $\vec{a} + \vec{b}$ *and* $\vec{a} - \vec{b}$ *are perpendicular to each other.*

Solution. Given $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$,

$\therefore \quad \vec{a} + \vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k}) + (3\hat{i} - \hat{j} + 2\hat{k}) = 4\hat{i} + \hat{j} - \hat{k}$ and

$\vec{a} - \vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = -2\hat{i} + 3\hat{j} - 5\hat{k}$.

$\therefore \quad (\vec{a} + \vec{b}) \bullet (\vec{a} - \vec{b}) = 4(-2) + 1 \times 3 + (-1)(-5)$

$= -8 + 3 + 5 = 0.$

Thus, the dot product of two non-zero vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is zero, therefore, the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular to each other.

Example 7. *If* $\vec{a} = 4\hat{i} + 2\hat{j} - \hat{k}$ *and* $\vec{b} = 5\hat{i} + 2\hat{j} - 3\hat{k}$, *find the angle between the vectors* $\vec{a} + \vec{b}$ *and* $\vec{a} - \vec{b}$.

Solution. Given $\vec{a} = 4\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 5\hat{i} + 2\hat{j} - 3\hat{k}$

$\therefore \quad \vec{a} + \vec{b} = (4\hat{i} + 2\hat{j} - \hat{k}) + (5\hat{i} + 2\hat{j} - 3\hat{k}) = 9\hat{i} + 4\hat{j} - 4\hat{k}$ and

$\vec{a} - \vec{b} = (4\hat{i} + 2\hat{j} - \hat{k}) - (5\hat{i} + 2\hat{j} - 3\hat{k}) = -\hat{i} + 0\hat{j} + 2\hat{k}$

$\therefore \quad (\vec{a} + \vec{b}) \bullet (\vec{a} - \vec{b}) = 9 \times (-1) + 4 \times 0 + (-4) \times 2 = -17$

$|\vec{a} + \vec{b}| = \sqrt{9^2 + 4^2 + (-4)^2} = \sqrt{113}$ and $|\vec{a} - \vec{b}| = \sqrt{(-1)^2 + 0^2 + 2^2} = \sqrt{5}$.

Let θ be the angle between the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is given by

$$\cos\theta = \frac{(\vec{a}+\vec{b}) \bullet (\vec{a}-\vec{b})}{|\vec{a}+\vec{b}||\vec{a}-\vec{b}|} = \frac{-17}{\sqrt{113}\sqrt{5}} = -\frac{17}{\sqrt{565}} \Rightarrow \theta = \cos^{-1}\left(-\frac{17}{\sqrt{565}}\right).$$

Example 8. *Prove that the three vectors* $3\hat{i} + \hat{j} + 2\hat{k}$, $\hat{i} - \hat{j} - \hat{k}$ *and* $\hat{i} + 5\hat{j} - 4\hat{k}$ *are at right angles to one another.*

Solution. Let $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + 5\hat{j} - 4\hat{k}$.

We note that all the three vectors are non-zero.

$$\text{Now} \quad \vec{a} \cdot \vec{b} = (3\hat{i} + \hat{j} + 2\hat{k}) \cdot (\hat{i} - \hat{j} - \hat{k})$$
$$= (3)(1) + (1)(-1) + (2)(-1) = 0.$$

Thus, the dot product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is zero, therefore, these vectors are perpendicular to each other.

$$\text{Again } \vec{b} \cdot \vec{c} = (\hat{i} - \hat{j} - \hat{k}) \cdot (\hat{i} + 5\hat{j} - 4\hat{k})$$
$$= (1)(1) + (-1)(5) + (-1)(-4) = 0$$

$$\text{and} \quad \vec{c} \cdot \vec{a} = (\hat{i} + 5\hat{j} - 4\hat{k}) \cdot (3\hat{i} + \hat{j} + 2\hat{k})$$
$$= (1)(3) + (5)(1) + (-4)(2) = 0.$$

As above, it follows that $\vec{b}, \vec{c}$ are perpendicular and $\vec{c}, \vec{a}$ are perpendicular. Hence, all the three given vectors are perpendicular to one another.

Example 9. *Find the value of λ so that the two vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $-4\hat{i} - 6\hat{j} + \lambda\hat{k}$ are*

(i) parallel *(ii) perpendicular to each other.*

Solution. Let $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = -4\hat{i} - 6\hat{j} + \lambda\hat{k}$.

(*i*) $\vec{a}$ and $\vec{b}$ are parallel to each other iff

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \text{ i.e. iff } \frac{2}{-4} = \frac{3}{-6} = \frac{-1}{\lambda} \Rightarrow \lambda = 2.$$

(*ii*) $\vec{a}$ and $\vec{b}$ are perpendicular to each other iff $\vec{a} \cdot \vec{b} = 0$

i.e. if $(2)(-4) + (3)(-6) + (-1)(\lambda) = 0 \Rightarrow \lambda = -8 - 18 = -26$.

Example 10. *If $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + \lambda\hat{j} + 3\hat{k}$, find the value of λ so that $\vec{a} + \vec{b}$ is perpendicular to $\vec{a} - \vec{b}$.*

Solution. Given $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + \lambda\hat{j} + 3\hat{k}$,

$\therefore \quad \vec{a} + \vec{b} = 4\hat{i} + (2 + \lambda)\hat{j} + 12\hat{k}$ and $\vec{a} - \vec{b} = 2\hat{i} + (2 - \lambda)\hat{j} + 6\hat{k}$.

As the vector $\vec{a} + \vec{b}$ is perpendicular to the vector $\vec{a} - \vec{b}$,

we have $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$

$\Rightarrow \quad 4 \times 2 + (2 + \lambda)(2 - \lambda) + 12 \times 6 = 0 \Rightarrow 8 + 4 - \lambda^2 + 72 = 0$

$\Rightarrow \quad \lambda^2 = 84 \Rightarrow \lambda = \pm 2\sqrt{21}$.

Example 11. *If $\vec{a}$ and $\vec{b}$ are unit vectors such that $2\vec{a} - 4\vec{b}$ and $10\vec{a} + 8\vec{b}$ are perpendicular to each other, find the angle between vectors $\vec{a}$ and $\vec{b}$.* (*I.S.C. 2005*)

Solution. Since $\vec{a}$ and $\vec{b}$ are unit vectors, $|\vec{a}| = 1$ and $|\vec{b}| = 1$.

As the vectors $2\vec{a} - 4\vec{b}$ and $10\vec{a} + 8\vec{b}$ are perpendicular to each other,

$$(2\vec{a} - 4\vec{b}) \cdot (10\vec{a} + 8\vec{b}) = 0$$

$\Rightarrow \quad 20(\vec{a})^2 + 16\,\vec{a} \cdot \vec{b} - 40\,\vec{a} \cdot \vec{b} - 32(\vec{b})^2 = 0$

$\Rightarrow \quad 20\,|\vec{a}|^2 - 24\,\vec{a} \cdot \vec{b} - 32\,|\vec{b}|^2 = 0$

$\Rightarrow \quad 20 \,.\, 1^2 - 24 \,|\, \vec{a} \,|\, |\, \vec{b} \,|\, \cos\theta - 32 \,.\, 1^2 = 0$ where θ is the angle between vectors $\vec{a}$ and $\vec{b}$

$\Rightarrow \quad 20 - 24 \,.\, 1 \,.\, 1 \,.\, \cos\theta - 32 = 0 \;\Rightarrow -12 - 24\cos\theta = 0$

$\Rightarrow \quad \cos\theta = -\frac{1}{2} \Rightarrow \theta = 120°.$

Hence, the angle between vectors $\vec{a}$ and $\vec{b}$ is 120°.

Example 12. *Let* $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ *and* $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$, *find :*

(*i*) *the projection of* $\vec{a}$ *on* $\vec{b}$ (*ii*) *projection of* $\vec{b}$ *on* $\vec{a}$.

Solution. Given $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$,

$\therefore \quad \vec{a} \bullet \vec{b} = 1 \times 7 + 3 \times (-1) + 7 \times 8 = 60.$

(*i*) $|\vec{b}| = \sqrt{7^2 + (-1)^2 + 8^2} = \sqrt{114}$.

The projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a} \bullet \vec{b}}{|\vec{b}|} = \dfrac{60}{\sqrt{114}}$.

(*ii*) $|\vec{a}| = \sqrt{1^2 + 3^2 + 7^2} = \sqrt{59}$.

The projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a} \bullet \vec{b}}{|\vec{a}|} = \dfrac{60}{\sqrt{59}}$.

Example 13. *Find* λ *when the scalar projection of* $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ *on* $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ *is 4 units.*

Solution. The scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \bullet \vec{b}}{|\vec{b}|} = 4$ (given) ...(*i*)

Here $\vec{a} \bullet \vec{b} = \lambda \times 2 + 1 \times 6 + 4 \times 3 = 2\lambda + 18$

and $|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7.$

From (*i*), we get $\dfrac{2\lambda + 18}{7} = 4 \Rightarrow \lambda = 5.$

Example 14. *The vectors* $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ *and* $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ *are mutually perpendicular. Given that* $|\vec{a}| = |\vec{b}|$, *find the values of x and y.* (*I.S.C. 2002*)

Solution. Since $\vec{a} \perp \vec{b}$, $\vec{a} \bullet \vec{b} = 0$

$\Rightarrow \quad 3 \times 2 + x \times 1 + (-1) \times y = 0 \;\Rightarrow\; y - x = 6$ ...(*i*)

Given $|\vec{a}| = |\vec{b}| \;\Rightarrow\; |\vec{a}|^2 = |\vec{b}|^2$

$\Rightarrow \quad 3^2 + x^2 + (-1)^2 = 2^2 + 1^2 + y^2 \;\Rightarrow y^2 - x^2 = 5$

$\Rightarrow \quad (y - x)(y + x) = 5$

$\Rightarrow \quad 6(y + x) = 5$ (using (*i*))

$\Rightarrow \quad y + x = \frac{5}{6}$ ...(*ii*)

Solving (*i*) and (*ii*), we get $x = -\frac{31}{12}$ and $y = \frac{41}{12}$.

$\Rightarrow \quad \vec{a}\bullet\vec{a} + \vec{a}\bullet\vec{b} + \vec{a}\bullet\vec{c} + \vec{b}\bullet\vec{a} + \vec{b}\bullet\vec{b} + \vec{b}\bullet\vec{c} + \vec{c}\bullet\vec{c} + \vec{c}\bullet\vec{b} + \vec{c}\bullet\vec{a} = 0$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\bullet\vec{b} + \vec{b}\bullet\vec{c} + \vec{c}\bullet\vec{a}) = 0$

$\Rightarrow \quad 1^2 + 4^2 + 2^2 + 2\mu = 0 \Rightarrow 21 + 2\mu = 0$

$\Rightarrow \quad \mu = -\frac{21}{2}.$

EXERCISE 1.1

1. Evaluate the scalar product $(3\vec{a} - 5\vec{b})\bullet(2\vec{a}+7\vec{b})$.

2. (*i*) If $\vec{a}$ is a unit vector and $(\vec{x}+\vec{a})\bullet(\vec{x}-\vec{a}) = 12$, find $|\vec{x}|$.

(*ii*) If $\vec{a}$ is a unit vector and $(2\vec{a}+\vec{b})\bullet(2\vec{a}-\vec{b}) = 2$, then find $|\vec{b}|$.

3. If $(\vec{a}+\vec{b})\bullet(\vec{a}-\vec{b}) = 3$ and $|\vec{a}| = 2|\vec{b}|$, find $|\vec{a}|$ and $|\vec{b}|$.

4. (*i*) If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\vec{a}\bullet\vec{b} = 1$, find $(\vec{a}-\vec{b})^2$.

(*ii*) If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $\vec{a}\bullet\vec{b} = 8$, find $|\vec{a}-\vec{b}|$.

(*iii*) If $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}\bullet\vec{b} = -8$, find $|\vec{a}+3\vec{b}|$.

5. If the angle between two vectors $\vec{a}$ and $\vec{b}$ of equal magnitude is 60° and their scalar product is $\frac{1}{2}$, find their magnitudes.

6. (*i*) Find the angle between two vectors $\vec{a}$ and $\vec{b}$, given that $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\vec{a}\bullet\vec{b} = 6$.

(*ii*) Find the angle between vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = |\vec{b}| = 3$ and $\vec{a}\bullet\vec{b} = 1$.

(*iii*) Find the angle between two vectors if they have same length $\sqrt{2}$ and their scalar product is -1.

7. Find the angle between the vectors $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$.

8. If $\vec{a} = 5\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{b} = 6\hat{i}-8\hat{j}$, then find

(*i*) $|\vec{a}|$ (*ii*) $|\vec{b}|$ (*iii*) $|\vec{a}+\vec{b}|$ (*iv*) $|\vec{a}-\vec{b}|$ (*v*) $\vec{a}\bullet\vec{b}$

(*vi*) $\vec{b}\bullet\vec{a}$ (*vii*) the angle θ between $\vec{a}$ and $\vec{b}$

(*viii*) the projection of $\vec{a}$ on $\vec{b}$ (*ix*) the projection of $\vec{b}$ on $\vec{a}$.

9. (*i*) Find $(\vec{b}-\vec{a})\bullet(3\vec{a}+\vec{b})$ where $\vec{a} = \hat{i}-2\hat{j}+5\hat{k}$, $\vec{b} = 2\hat{i}+\hat{j}-3\hat{k}$.

(*ii*) Find $(\vec{a}+3\vec{b})\bullet(2\vec{a}-\vec{b})$ where $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$, $\vec{b} = 3\hat{i}+2\hat{j}-\hat{k}$.

10. (*i*) Find the projection of $\vec{a} = 2\hat{i}+3\hat{j}+2\hat{k}$ on the vector $\vec{b} = \hat{i}+2\hat{j}+\hat{k}$

(*ii*) Find the projection of $\hat{i}-\hat{j}$ in the direction of $\hat{i}+\hat{j}$.

(*iii*) Find λ when the projection of $\hat{i}+\lambda\hat{j}+\hat{k}$ on $\hat{i}+\hat{j}$ is $\sqrt{2}$ units.

21. If the points A, B, C and D have position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ respectively, show that ABCD is a rectangle. Also, find its area.

22. If the volume of the parallelopiped whose coterminus edges are represented by the vectors $5\hat{i}-4\hat{j}+\hat{k}$, $4\hat{i}+3\hat{j}+\lambda\hat{k}$ and $\hat{i}-2\hat{j}+7\hat{k}$ is 216 cubic units, find the value of λ.

23. Using scalar triple product, prove that the points $(-1, 4, -3)$, $(3, 2, -5)$, $(-3, 8, -5)$ and $(-3, 2, 1)$ are coplanar.

24. The position vectors of the points A, B, C and D are $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$ respectively. If the points A, B, C and D lie in a plane, find the value of λ.

25. Find λ so that the four points with position vectors $\hat{i}+2\hat{j}+3\hat{k}$, $3\hat{i}-\hat{j}+2\hat{k}$, $-2\hat{i}+\lambda\hat{j}+\hat{k}$, $6\hat{i}-4\hat{j}+2\hat{k}$ are coplanar.

26. Three concurrent edges OA, OB, OC of a parallelopiped are represented by three vectors $2\hat{i}+\hat{j}-\hat{k}$, $\hat{i}+2\hat{j}+3\hat{k}$, $-3\hat{i}-\hat{j}+\hat{k}$. Find the area of the face made by OA and OB, and the volume of the solid. Also find the surface area of the parallelopiped and the combined length of all its edges.

Hint. Surface area $= 2\,(|\vec{a}\times\vec{b}|+|\vec{b}\times\vec{c}|+|\vec{c}\times\vec{a}|)$,

and length of edges $= 4\,(|\vec{a}|+|\vec{b}|+|\vec{c}|)$.

ANSWERS

EXERCISE 1.1

1. $6|\vec{a}|^2+11\,\vec{a}\cdot\vec{b}-35|\vec{b}|^2$. **2.** (*i*) $\sqrt{13}$ (*ii*) $\sqrt{2}$.

3. $|\vec{a}|=2$, $|\vec{b}|=1$. **4.** (*i*) 23 (*ii*) $\sqrt{13}$ (*iii*) $\sqrt{37}$.

5. 1, 1. **6.** (*i*) 60° (*ii*) $\cos^{-1}\left(\frac{1}{9}\right)$ (*iii*) 120°. **7.** $\cos^{-1}\left(-\frac{1}{3}\right)$.

8. (*i*) $5\sqrt{2}$ (*ii*) 10 (*iii*) $9\sqrt{2}$ (*iv*) $\sqrt{138}$ (*v*) 6

(*vi*) 6 (*vii*) $\cos^{-1}\left(\frac{3}{25\sqrt{2}}\right)$ (*viii*) $\frac{3}{5}$ (*ix*) $\frac{6}{5\sqrt{2}}$.

9. (*i*) -106 (*ii*) -15. **10.** (*i*) $\frac{5}{3}\sqrt{6}$ units (*ii*) 0 (*iii*) 1.

11. (*i*) 3 (*ii*) $\frac{7}{5}$.

13. 120°. **14.** 30°. **15.** $\sqrt{7}$. **16.** 4. **18.** 90°.

19. 2 units. **20.** $\pm\sqrt{73}$. **21.** (*i*) -15 (*ii*) $\frac{2}{3}$. **22.** 8.

30. $\vec{b}$ is any vector. **32.** $\cos^{-1}\left(\frac{3}{7}\right)$, $\cos^{-1}\left(-\frac{6}{7}\right)$, $\cos^{-1}\left(\frac{2}{7}\right)$. **33.** $\frac{5}{2}\hat{i}-\frac{1}{2}\hat{j}$.

34. (*i*) $3\hat{i}+2\hat{k}$ (*ii*) $\hat{i}+2\hat{j}+2\hat{k}$. **35.** $\pm\frac{100}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$. **37.** $\cos^{-1}\left(\frac{31}{50}\right)$.