27. Direction Cosines and Directions Ratios

Exercise 27.1

1. Question

If a line makes angles of 90°, 60° and 30° with the positive direction of x, y, and z-axis respectively, find its direction cosines.

Answer

Let us assume the angles that made with the positive direction of x, y, and z-axes be α, β, γ.

Then we get,

$\Rightarrow \alpha = 90^0$

$\Rightarrow \beta = 60^0$

$\Rightarrow \gamma = 30^0$

We know that if a line makes angles of α, β, γ with the positive x, y, and z-axes then the direction cosines of that line is the cosine of that angles made by that line with the axes.

Let us assume that l, m, n are the direction cosines of the line. Then,

$\Rightarrow l = \cos\alpha$

$\Rightarrow m = \cos\beta$

$\Rightarrow n = \cos\gamma$

We substitute the values of α, β, γ in the above equations for the values of l, m, n.

$\Rightarrow l = \cos(90^0)$

$\Rightarrow l = 0$

$\Rightarrow m = \cos(60^0)$

$\Rightarrow m = \frac{1}{2}$

$\Rightarrow n = \cos(30^0)$

$\Rightarrow n = \frac{\sqrt{3}}{2}$

∴ The direction cosines of the given line is $\mathbf{0, \frac{1}{2}, \frac{\sqrt{3}}{2}}$.

2. Question

If a line has direction ratios 2, –1, –2, determine its cosines.

Answer

Let us assume the direction ratios of the line be r_1, r_2, r_3.

Then:

$\Rightarrow r_1 = 2$

$\Rightarrow r_2 = -1$

$\Rightarrow r_3 = -2$

Let us assume the direction cosines for the line be l, m, n

We know that for a line of direction ratios r_1, r_2, r_3 and having direction cosines l, m, n has the following property.

$\Rightarrow l = \frac{r_1}{\sqrt{r_1^2+r_2^2+r_3^2}}$

$\Rightarrow m = \frac{r_2}{\sqrt{r_1^2+r_2^2+r_3^2}}$

$\Rightarrow n = \frac{r_3}{\sqrt{r_1^2+r_2^2+r_3^2}}$

Let us substitute the values of r_1, r_2, r_3 to find the values of l, m, n.

$\Rightarrow l = \frac{2}{\sqrt{2^2+(-1)^2+(-2)^2}}$

$\Rightarrow l = \frac{2}{\sqrt{4+1+4}}$

$\Rightarrow l = \frac{2}{\sqrt{9}}$

$\Rightarrow l = \frac{2}{3}$

$\Rightarrow m = \frac{-1}{\sqrt{2^2+(-1)^2+(-2)^2}}$

$\Rightarrow m = \frac{-1}{\sqrt{4+1+4}}$

$\Rightarrow m = \frac{-1}{\sqrt{9}}$

$\Rightarrow m = \frac{-1}{3}$

$\Rightarrow n = \frac{-2}{\sqrt{2^2+(-1)^2+(-2)^2}}$

$\Rightarrow n = \frac{-2}{\sqrt{4+1+4}}$

$\Rightarrow n = \frac{-2}{\sqrt{9}}$

$\Rightarrow n = \frac{-2}{3}$

∴ The direction cosines for the given line is $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$.

3. Question

Find the direction cosines of the line passing through two points (–2,4,–5) and (1,2,3).

Answer

Let us assume the given two points of line be X(–2,4,–5) and Y(1,2,3).

Let us also assume the direction ratios for the given line be (r_1, r_2, r_3).

We know that direction ratios for a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is (x_2–x_1, y_2–y_1, z_2–z_1).

So, using this property the direction ratios for the given line is, ⇒ (r_1, r_2, r_3) = (1–(–2), 2–4, 3–(–5))

⇒ (r_1, r_2, r_3) = (1+2, 2–4, 3+5)

⇒ (r_1, r_2, r_3) = (3, –2, 8)

Let us assume l, m, n be the direction cosines of the given line.

We know that for a line of direction ratios r_1, r_2, r_3 and having direction cosines l, m, n has the following property.

$\Rightarrow l = \frac{r_1}{\sqrt{r_1^2+r_2^2+r_3^2}}$

$\Rightarrow m = \frac{r_2}{\sqrt{r_1^2+r_2^2+r_3^2}}$

$\Rightarrow n = \frac{r_3}{\sqrt{r_1^2+r_2^2+r_3^2}}$

Let us substitute the values of r_1, r_2, r_3 to find the values of l, m, n.

$\Rightarrow l = \frac{3}{\sqrt{3^2+(-2)^2+8^2}}$

$\Rightarrow l = \frac{3}{\sqrt{9+4+64}}$

$\Rightarrow l = \frac{3}{\sqrt{77}}$

$\Rightarrow m = \frac{-2}{\sqrt{3^2+(-2)^2+8^2}}$

$\Rightarrow m = \frac{-2}{\sqrt{9+4+64}}$

$\Rightarrow m = \frac{-2}{\sqrt{77}}$

$\Rightarrow n = \frac{8}{\sqrt{3^2+(-2)^2+8^2}}$

$\Rightarrow n = \frac{8}{\sqrt{9+4+64}}$

$\Rightarrow n = \frac{8}{\sqrt{77}}$

∴ The Direction Cosines for the given line is $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$.

4. Question

Using direction ratios show that the points A(2,3,–4), B(1,–2,3), C(3,8,–11) are collinear.

Answer

Given points are:

⇒ A = (2,3,–4)

⇒ B = (1,–2,3)

⇒ C = (3,8,–11)

We know that for points D, E, F to be collinear the direction ratios of any two lines from DE, DF, EF are to be proportional;

We know that direction ratios for a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is (x_2–x_1, y_2–y_1, z_2–z_1).

Let us assume direction ratios for AB is (r_1, r_2, r_3) and BC is (r_4, r_5, r_6).

The proportional condition can be stated as $\frac{r_1}{r_4} = \frac{r_2}{r_5} = \frac{r_3}{r_6} = k(\text{constant})$.

Let us find the direction ratios of AB

⇒ (r_1, r_2, r_3) = (1–2, –2–3, 3–(–4))

⇒ $(r_1, r_2, r_3) = (1–2, –2–3, 3+4)$

⇒ $(r_1, r_2, r_3) = (–1, –5, 7)$

Let us find the direction ratios of BC

⇒ $(r_4, r_5, r_6) = (3–1, 8–(–2), –11–3)$

⇒ $(r_4, r_5, r_6) = (3–1, 8+2, –11–3)$

⇒ $(r_4, r_5, r_6) = (2, 10, –14)$

Now

⇒ $\frac{r_1}{r_4} = \frac{-1}{2}$ ......(1)

⇒ $\frac{r_2}{r_5} = \frac{-5}{10}$

⇒ $\frac{r_2}{r_5} = -\frac{1}{2}$ ......(2)

⇒ $\frac{r_3}{r_6} = \frac{7}{-14}$

⇒ $\frac{r_3}{r_6} = -\frac{1}{2}$ ......(3)

From (1),(2),(3) we get,

⇒ $\frac{r_1}{r_4} = \frac{r_2}{r_5} = \frac{r_3}{r_6} = -\frac{1}{2}$

So, from the above relational we can say that points A, B , C are collinear.

5. Question

Find the directional cosines of the sides of the triangle whose vertices are (3,5,–4), (–1,1,2), (–5,–5,–2).

Answer

Let us write the given points as:

⇒ A = (3,5,–4)

⇒ B = (–1,1,2)

⇒ C = (–5,–5,–2)

Let us assume the direction ratios of sides AB be (r_1,r_2,r_3), BC be (r_4,r_5,r_6) and CA be (r_7,r_8,r_9)

We know that direction ratios for a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $(x_2–x_1, y_2–y_1, z_2–z_1)$.

Let us find the direction ratios for the side AB

⇒ $(r_1,r_2,r_3) = (–1–3, 1–5, 2–(–4))$

⇒ $(r_1,r_2,r_3) = (–1–3, 1–5, 2+4)$

⇒ $(r_1,r_2,r_3) = (–4,–4,6)$

Let us find the direction ratios for the side BC

⇒ $(r_4,r_5,r_6) = (–5–(–1), –5–1, –2–2)$

⇒ $(r_4,r_5,r_6) = (–5+1, –5–1, –2–2)$

⇒ $(r_4,r_5,r_6) = (–4,–6,–4)$

Let us find the direction ratios for the side CA

⇒ $(r_7,r_8,r_9) = (3-(-5), 5-(-5), -4-(-2))$

⇒ $(r_7,r_8,r_9) = (3+5, 5+5, -4+2)$

⇒ $(r_7,r_8,r_9) = (8,10,-2)$

Let us assume l_1, m_1, n_1 be the direction cosines of line AB, l_2, m_2, n_2 be the direction cosines of line BC and l_3, m_3, n_3 be the direction cosines of line CA.

We know that for a line of direction ratios r_1, r_2, r_3 and having direction cosines l, m, n has the following property.

⇒ $l = \frac{r_1}{\sqrt{r_1^2+r_2^2+r_3^2}}$

⇒ $m = \frac{r_2}{\sqrt{r_1^2+r_2^2+r_3^2}}$

⇒ $n = \frac{r_3}{\sqrt{r_1^2+r_2^2+r_3^2}}$

Let us follow the above property and find the direction cosines of each side.

Now, let's find the direction cosines of side AB,

⇒ $l_1 = \frac{-4}{\sqrt{(-4)^2+(-4)^2+6^2}}$

⇒ $l_1 = \frac{-4}{\sqrt{16+16+36}}$

⇒ $l_1 = \frac{-4}{\sqrt{64}}$

⇒ $l_1 = \frac{-4}{\sqrt{4\times17}}$

⇒ $l_1 = \frac{-4}{2\times\sqrt{17}}$

⇒ $l_1 = \frac{-2}{\sqrt{17}}$

⇒ $m_1 = \frac{-4}{\sqrt{(-4)^2+(-4)^2+6^2}}$

⇒ $m_1 = \frac{-4}{\sqrt{16+16+36}}$

⇒ $m_1 = \frac{-4}{\sqrt{68}}$

⇒ $m_1 = \frac{-4}{\sqrt{4\times17}}$

⇒ $m_1 = \frac{-4}{2\times\sqrt{17}}$

⇒ $m_1 = \frac{-2}{\sqrt{17}}$

⇒ $n_1 = \frac{6}{\sqrt{(-4)^2+(-4)^2+6^2}}$

⇒ $n_1 = \frac{6}{\sqrt{16+16+36}}$

⇒ $n_1 = \frac{6}{\sqrt{68}}$

⇒ $n_1 = \frac{6}{\sqrt{4\times17}}$

⇒ $n_1 = \frac{6}{2\times\sqrt{17}}$

$\Rightarrow n_1 = \frac{3}{\sqrt{17}}$

The direction cosines for the side AB is $\left(\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$.

Let’s find the directional cosines for the side BC,

$\Rightarrow l_2 = \frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$

$\Rightarrow l_2 = \frac{-4}{\sqrt{16+36+16}}$

$\Rightarrow l_2 = \frac{-4}{\sqrt{68}}$

$\Rightarrow l_2 = \frac{-4}{2\times\sqrt{17}}$

$\Rightarrow l_2 = \frac{-2}{\sqrt{17}}$

$\Rightarrow m_2 = \frac{-6}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$

$\Rightarrow m_2 = \frac{-6}{\sqrt{16+36+16}}$

$\Rightarrow m_2 = \frac{-6}{\sqrt{68}}$

$\Rightarrow m_2 = \frac{-6}{\sqrt{4\times17}}$

$\Rightarrow m_2 = \frac{-6}{2\times\sqrt{17}}$

$\Rightarrow m_2 = \frac{-3}{\sqrt{17}}$

$\Rightarrow n_2 = \frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$

$\Rightarrow n_2 = \frac{-4}{\sqrt{16+36+16}}$

$\Rightarrow n_2 = \frac{-4}{\sqrt{68}}$

$\Rightarrow n_2 = \frac{-4}{2\times\sqrt{17}}$

$\Rightarrow n_2 = \frac{-2}{\sqrt{17}}$

The direction cosines for the sides BC is $\left(\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right)$.

Let’s find the direction cosines for the side CA,

$\Rightarrow l_3 = \frac{8}{\sqrt{8^2+10^2+(-2)^2}}$

$\Rightarrow l_3 = \frac{8}{\sqrt{64+100+4}}$

$\Rightarrow l_3 = \frac{8}{\sqrt{168}}$

$\Rightarrow l_3 = \frac{8}{\sqrt{4\times42}}$

$\Rightarrow l_3 = \frac{8}{2\times\sqrt{42}}$

$\Rightarrow l_3 = \frac{4}{\sqrt{42}}$

$\Rightarrow m_3 = \frac{10}{\sqrt{8^2+10^2+(-2)^2}}$

$\Rightarrow m_3 = \frac{10}{\sqrt{64+100+4}}$

$\Rightarrow m_3 = \frac{10}{\sqrt{168}}$

$\Rightarrow m_3 = \frac{10}{\sqrt{4\times 42}}$

$\Rightarrow m_3 = \frac{10}{2\times\sqrt{42}}$

$\Rightarrow m_3 = \frac{5}{\sqrt{42}}$

$\Rightarrow n_3 = \frac{-2}{\sqrt{8^2+10^2+(-2)^2}}$

$\Rightarrow n_3 = \frac{-2}{\sqrt{64+100+4}}$

$\Rightarrow n_3 = \frac{-2}{\sqrt{168}}$

$\Rightarrow n_3 = \frac{-2}{\sqrt{4\times 42}}$

$\Rightarrow n_3 = \frac{-2}{2\times\sqrt{42}}$

$\Rightarrow n_3 = \frac{-1}{\sqrt{42}}$

The direction cosines for the sides CA is $\left(\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}\right)$.

6. Question

Find the angle between the vectors with direction ratios proportional to 1,–2,1 and 4,3,2.

Answer

Let us assume the direction ratios of vectors be (r_1,r_2,r_3) and (r_4,r_5,r_6).

Then,

$\Rightarrow (r_1,r_2,r_3) = (1,-2,1)$

$\Rightarrow (r_4,r_5,r_6) = (4,3,2)$

We know that the angle between the vectors with direction ratios proportional to (a_1,b_1,c_1) and (a_2,b_2,c_2) is given by:

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Using the above formula we calculate the angle between the vectors.

Let α be the angle between the two vectors given in the problem.

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{(1\times 4)+(-2\times 3)+(1\times 2)}{\sqrt{1^2+(-2)^2+1^2}\sqrt{4^2+3^2+2^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{4-6+2}{\sqrt{1+4+1}\sqrt{16+9+4}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{0}{\sqrt{6}\sqrt{29}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}(0)$$

$\Rightarrow \alpha = \frac{\pi}{2}$

∴ The angle between two given vectors is $\frac{\pi}{2}$ **or 90⁰**.

7. Question

Find the angle between the vectors with direction ratios proportional to 2,3,–6 and 3,–4,5.

Answer

Let us assume the direction ratios of vectors be (r_1,r_2,r_3) and (r_4,r_5,r_6).

Then,

$\Rightarrow (r_1,r_2,r_3) = (2,3,-6)$

$\Rightarrow (r_4,r_5,r_6) = (3,-4,5)$

We know that the angle between the vectors with direction ratios proportional to (a_1,b_1,c_1) and (a_2,b_2,c_2) is given by:

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Using the above formula we calculate the angle between the vectors.

Let α be the angle between the two vectors given in the problem.

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{(2\times3)+(3\times-4)+(-6\times5)}{\sqrt{2^2+3^2+(-6)^2}\sqrt{3^2+(-4)^2+5^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{6-12-30}{\sqrt{4+9+36}\sqrt{9+16+25}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{-36}{\sqrt{49}\sqrt{50}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{-36}{7\sqrt{2\times25}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{-36}{7\times5\times\sqrt{2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{-18\sqrt{2}}{35}\right)$$

∴ The angle between two given vectors is $\cos^{-1}\left(\frac{-18\sqrt{2}}{35}\right)$.

8. Question

Find the acute angle between the lines whose direction ratios are proportional to 2:3:6 and 1:2:2.

Answer

Given that the direction ratios of the lines are proportional to 2:3:6 and 1:2:2.

Let us denote the lines in the form of vectors as **A** and **B**.

Let's write the vectors:

⇒ **A** = 2**i** + 3**j** + 6**k**

⇒ **B** = 1**i** + 2**j** + 2**k**

We know that the angle between the vectors a_1**i** + b_1**j** + c_1**k** and a_2**i** + b_2**j** + c_2**k** is given by:

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Let’s assume the angle between the vectors **A** and **B** be α,

Using the given formula we find the value of α.

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{(2\times1)+(3\times2)+(6\times2)}{\sqrt{2^2+3^2+6^2}\sqrt{1^2+2^2+2^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{2+6+12}{\sqrt{4+9+36}\sqrt{1+4+4}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{20}{\sqrt{49}\sqrt{9}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{20}{7\times3}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{20}{21}\right)$$

The acute angle between the two vectors is given by $\mathbf{\cos^{-1}\left(\frac{20}{21}\right)}$.

9. Question

Show that the points (2,3,4), (–1,–2,1), (5,8,7) are collinear.

Answer

Let us indicate given points with A, B and C.

⇒ A = (2,3,4)

⇒ B = (–1,–2,1)

⇒ C = (5,8,7)

We know that for points D, E, F to be collinear the direction ratios of any two lines from DE, DF, EF are to be proportional;

We know that direction ratios for a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $(x_2–x_1, y_2–y_1, z_2–z_1)$.

Let us assume direction ratios for AB is (r_1, r_2, r_3) and BC is (r_4, r_5, r_6).

The proportional condition can be stated as $\frac{r_1}{r_4} = \frac{r_2}{r_5} = \frac{r_3}{r_6} = k(\text{constant})$.

Let us find the direction ratios of AB

⇒ (r_1, r_2, r_3) = (–1–2, –2–3, 1–4)

⇒ (r_1, r_2, r_3) = (–3,–5,–3)

Let us find the direction ratios of BC

⇒ (r_4, r_5, r_6) = (5–(–1), 8–(–2), 7–1)

⇒ (r_4, r_5, r_6) = (5+1, 8+2, 7–1)

⇒ (r_4, r_5, r_6) = (6, 10, 6)

Now

$\Rightarrow \frac{r_1}{r_4} = \frac{-3}{6}$

$\Rightarrow \frac{r_1}{r_4} = \frac{-1}{2}$ ……(1)

$\Rightarrow \frac{r_2}{r_5} = \frac{-5}{10}$

$\Rightarrow \frac{r_2}{r_5} = -\frac{1}{2}$ ......(2)

$\Rightarrow \frac{r_3}{r_6} = \frac{6}{-12}$

$\Rightarrow \frac{r_3}{r_6} = -\frac{1}{2}$ ......(3)

From (1),(2),(3) we get,

$\Rightarrow \frac{r_1}{r_4} = \frac{r_2}{r_5} = \frac{r_3}{r_6} = -\frac{1}{2}$

So, from the above relational we can say that points (2,3,4), (–1,–2,1) , (5,8,7) are collinear.

10. Question

Show that the line through points (4,7,8) and (2,3,4) is parallel to the line through the points (–1,–2,1) and (1,2,5).

Answer

Let us denote the points as follows:

⇒ A = (4,7,8)

⇒ B = (2,3,4)

⇒ C = (–1,–2,1)

⇒ D = (1,2,5)

If two lines are said to be parallel the directional ratios of two lines need to be proportional.

Let us assume the direction ratios for line AB be (r_1,r_2,r_3) and CD be (r_4,r_5,r_6)

We know that direction ratios for a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $(x_2–x_1, y_2–y_1, z_2–z_1)$.

Let's find the direction ratios for the line AB

⇒ (r_1,r_2,r_3) = (2–4, 3–7, 4–8)

⇒ (r_1,r_2,r_3) = (–2,–4,–4)

Let's find the direction ratios for the line CD

⇒ (r_4,r_5,r_6) = (1–(–1), 2–(–2), 5–1)

⇒ (r_4,r_5,r_6) = (1+1, 2+2, 5–1)

⇒ (r_4,r_5,r_6) = (2,4,4)

The proportional condition can be stated as $\frac{r_1}{r_4} = \frac{r_2}{r_5} = \frac{r_3}{r_6} = k(\text{constant})$.

Let check whether the directional ratios are proportional or not,

$\Rightarrow \frac{r_1}{r_4} = \frac{-2}{2}$

$\Rightarrow \frac{r_1}{r_4} = -1$ ......(1)

$\Rightarrow \frac{r_2}{r_5} = \frac{-4}{4}$

$\Rightarrow \frac{r_2}{r_5} = -1$ ......(2)

$\Rightarrow \frac{r_3}{r_6} = \frac{-4}{4}$

$\Rightarrow \frac{r_3}{r_6} = -1$......(3)

From (1),(2),(3) we can say that the direction ratios of the lines are proportional. So, the lines are parallel to each other.

11. Question

Show that the line through points (1,–1,2) and (3,4,–2) is perpendicular to the line through the points (0,3,2) and (3,5,6).

Answer

Let us denote the points as follows:

⇒ A = (1,–1,2)

⇒ B = (3,4,–2)

⇒ C = (0,3,2)

⇒ D = (3,5,6)

If two lines of direction ratios (a_1,b_1,c_1) and (a_2,b_2,c_2) are said to be perpendicular to each other. Then the following condition is need to be satisfied:

$\Rightarrow a_1.a_2+b_1.b_2+c_1.c_2=0$ ......(1)

Let us assume the direction ratios for line AB be (r_1,r_2,r_3) and CD be (r_4,r_5,r_6)

We know that direction ratios for a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $(x_2–x_1, y_2–y_1, z_2–z_1)$.

Let's find the direction ratios for the line AB

$\Rightarrow (r_1,r_2,r_3) = (3–1, 4–(–1), –2–2)$

$\Rightarrow (r_1,r_2,r_3) = (3–1, 4+1, –2–2)$

$\Rightarrow (r_1,r_2,r_3) = (2,5,–4)$

Let's find the direction ratios for the line CD

$\Rightarrow (r_4,r_5,r_6) = (3–0, 5–3, 6–2)$

$\Rightarrow (r_4,r_5,r_6) = (3,2,4)$

Let us check whether the lines are perpendicular or not using (1)

$\Rightarrow r_1.r_4+r_2.r_5+r_3.r_6 = (2\times3)+(5\times2)+(–4\times4)$

$\Rightarrow r_1.r_4+r_2.r_5+r_3.r_6 = 6+10–16$

$\Rightarrow r_1.r_4+r_2.r_5+r_3.r_6 = 0$

Since the condition is clearly satisfied, we can say that the given lines are perpendicular to each other.

12. Question

Show that the line joining the origin to the point (2,1,1) is perpendicular to the line determined by the points (3,5,–1) and (4,3,–1).

Answer

Let us denote the points as follows:

⇒ O = (0,0,0)

⇒ A = (2,1,1)

$\Rightarrow B = (3,5,-1)$

$\Rightarrow C = (4,3,-1)$

If two lines of direction ratios (a_1,b_1,c_1) and (a_2,b_2,c_2) are said to be perpendicular to each other. Then the following condition is need to be satisfied:

$\Rightarrow a_1.a_2+b_1.b_2+c_1.c_2=0$ ......(1)

Let us assume the direction ratios for line OA be (r_1,r_2,r_3) and BC be (r_4,r_5,r_6)

We know that direction ratios for a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $(x_2-x_1, y_2-y_1, z_2-z_1)$.

Let's find the direction ratios for the line OA

$\Rightarrow (r_1,r_2,r_3) = (2-0, 1-0, 1-0)$

$\Rightarrow (r_1,r_2,r_3) = (2,1,1)$

Let's find the direction ratios for the line BC

$\Rightarrow (r_4,r_5,r_6) = (4-3, 3-5, -1-(-1))$

$\Rightarrow (r_4,r_5,r_6) = (4-3, 3-5, -1+1)$

$\Rightarrow (r_4,r_5,r_6) = (1,-2,0)$

Let us check whether the lines are perpendicular or not using (1)

$\Rightarrow r_1.r_4+r_2.r_5+r_3.r_6 = (2\times1)+(1\times-2)+(1\times0)$

$\Rightarrow r_1.r_4+r_2.r_5+r_3.r_6 = 2-2+0$

$\Rightarrow r_1.r_4+r_2.r_5+r_3.r_6 = 0$

Since the condition is clearly satisfied, we can say that the given lines are perpendicular to each other.

13. Question

Find the angle between the lines whose direction ratios are proportional to a,b,c and b–c, c–a, a–b.

Answer

Let us assume the direction ratios of vectors be (r_1,r_2,r_3) and (r_4,r_5,r_6).

Then,

$\Rightarrow (r_1,r_2,r_3) = (a,b,c)$

$\Rightarrow (r_4,r_5,r_6) = (b-c, c-a, a-b)$

We know that the angle between the lines with direction ratios proportional to (a_1,b_1,c_1) and (a_2,b_2,c_2) is given by:

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Using the above formula we calculate the angle between the lines.

Let α be the angle between the two lines given in the problem.

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{(a\times(b-c))+(b\times(c-a))+(c\times(a-b))}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{ab-ac+bc-ab+ac-bc}{\sqrt{a^2+b^2+c^2}\sqrt{b^2+c^2-2bc+c^2+a^2-2ac+a^2+b^2-2ab}}\right)$$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{0}{\sqrt{a^2+b^2+c^2}\sqrt{2a^2+2b^2+2c^2-2ac-2bc-2ca}}\right)$

$\Rightarrow \alpha = \cos^{-1}(0)$

$\Rightarrow \alpha = \frac{\pi}{2}$

∴ The angle between two given vectors is $\frac{\pi}{2}$ **or** $\mathbf{90^0}$.

14. Question

If the coordinates of the points A, B, C, D are (1,2,3), (4,5,7),(–4,3,–6),(2,9,2), then find the angle between AB and CD.

Answer

Given points are:

⇒ A = (1,2,3)

⇒ B = (4,5,7)

⇒ C = (–4,3,–6)

⇒ D = (2,9,2)

Let us assume the direction ratios for line AB be (r_1,r_2,r_3) and CD be (r_4,r_5,r_6)

We know that direction ratios for a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $(x_2–x_1, y_2–y_1, z_2–z_1)$.

Let's find the direction ratios for the line AB

⇒ (r_1,r_2,r_3) = (4–1, 5–2, 7–3)

⇒ (r_1,r_2,r_3) = (3,3,4)

Let's find the direction ratios for the line CD

⇒ (r_4,r_5,r_6) = (2–(–4), 9–3, 2–(–6))

⇒ (r_4,r_5,r_6) = (2+4, 9–3, 2+6)

⇒ (r_4,r_5,r_6) = (6,6,8)

We know that the angle between the vectors with direction ratios proportional to (a_1,b_1,c_1) and (a_2,b_2,c_2) is given by:

$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$

Using the above formula we calculate the angle between the vectors.

Let α be the angle between the two vectors given in the problem.

$\Rightarrow \alpha = \cos^{-1}\left(\frac{(3.6)+(3.6)+(4.8)}{\sqrt{3^2+3^2+4^2}\sqrt{6^2+6^2+8^2}}\right)$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{18+18+32}{\sqrt{9+9+16}\sqrt{36+36+64}}\right)$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{68}{\sqrt{34}\sqrt{136}}\right)$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{68}{\sqrt{34\times136}}\right)$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{68}{\sqrt{4624}}\right)$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{68}{68}\right)$

$\Rightarrow \alpha = \cos^{-1}(1)$

$\Rightarrow \alpha = 0^0$

∴ The angle between the given two vectors is $\mathbf{0^0}$.

15. Question

Find the direction cosines of the lines, connected by the relations: l+ m+ n = 0 and 2lm+ 2ln– mn =0.

Answer

Given relations are:

⇒ 2lm+ 2ln– mn =0 ……(1)

⇒ l+ m+ n =0

⇒ l = (–m– n) ……(2)

Substituting (2) in (1) we get,

⇒ 2(–m–n)m + 2(–m–n)n – mn =0

⇒ $2(-m^2-mn) + 2(-mn-n^2) - mn = 0$

⇒ $-2m^2 - 2mn - 2mn - 2n^2 - mn = 0$

⇒ $-2m^2 - 5mn - 2n^2 = 0$

⇒ $2m^2+5mn+2n^2=0$

⇒ $2m^2+4mn+mn+2n^2=0$

⇒ 2m(m+2n)+n(m+2n)=0

⇒ (2m+n)(m+2n)=0

⇒ 2m+n=0 or m+2n=0

⇒ 2m=–n or m=–2n

$\Rightarrow m = \frac{-n}{2}$ or $m = -2n$ ……(3)

Substituting the values of (3) in eq(2), we get

For 1st line:

$\Rightarrow l = -\left(\frac{-n}{2}\right) - n$

$\Rightarrow l = \frac{n}{2} - n$

$\Rightarrow l = \frac{-n}{2}$

The direction ratios for the first line is $\left(\frac{-n}{2}, \frac{-n}{2}, n\right)$.

Let us assume l_1, m_1, n_1 be the direction cosines of 1st line.

We know that for a line of direction ratios r_1, r_2, r_3 and having direction cosines l, m, n has the following property.

$\Rightarrow l = \frac{r_1}{\sqrt{r_1^2+r_2^2+r_3^2}}$

$\Rightarrow m = \frac{r_2}{\sqrt{r_1^2+r_2^2+r_3^2}}$

$\Rightarrow n = \frac{r_3}{\sqrt{r_1^2+r_2^2+r_3^2}}$

Using the above formulas we get,

$\Rightarrow l_1 = \frac{\frac{-n}{2}}{\sqrt{\left(\frac{-n}{2}\right)^2+\left(\frac{-n}{2}\right)^2+n^2}}$

$\Rightarrow l_1 = \frac{\frac{-n}{2}}{\sqrt{\frac{n^2}{4}+\frac{n^2}{4}+n^2}}$

$\Rightarrow l_1 = \frac{\frac{-n}{2}}{\sqrt{\frac{3n^2}{2}}}$

$\Rightarrow l_1 = \frac{-1}{\sqrt{6}}$

$\Rightarrow m_1 = \frac{\frac{-n}{2}}{\sqrt{\left(\frac{-n}{2}\right)^2+\left(\frac{-n}{2}\right)^2+n^2}}$

$\Rightarrow m_1 = \frac{\frac{-n}{2}}{\sqrt{\frac{n^2}{4}+\frac{n^2}{4}+n^2}}$

$\Rightarrow m_1 = \frac{\frac{-n}{2}}{\sqrt{\frac{3n^2}{2}}}$

$\Rightarrow m_1 = \frac{-1}{\sqrt{6}}$

$\Rightarrow n_1 = \frac{n}{\sqrt{\left(\frac{-n}{2}\right)^2+\left(\frac{-n}{2}\right)^2+n^2}}$

$\Rightarrow n_1 = \frac{n}{\sqrt{\frac{n^2}{4}+\frac{n^2}{4}+n^2}}$

$\Rightarrow n_1 = \frac{n}{\sqrt{\frac{3n^2}{2}}}$

$\Rightarrow n_1 = \sqrt{\frac{2}{3}}$

The Direction cosines for the 1^{st} line is $\left(\frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right)$

For 2^{nd} line:

$\Rightarrow l = -(-2n) - n$

$\Rightarrow l = 2n - n$

$\Rightarrow l = n$

The direction ratios for the second line is $(n, -2n, n)$.

Let us assume l_2, m_2, n_2 be the direction cosines of 1^{st} line.

We know that for a line of direction ratios r_1, r_2, r_3 and having direction cosines l, m, n has the following property.

$\Rightarrow l=\frac{r_1}{\sqrt{r_1^2+r_2^2+r_3^2}}$

$\Rightarrow m=\frac{r_2}{\sqrt{r_1^2+r_2^2+r_3^2}}$

$\Rightarrow n=\frac{r_3}{\sqrt{r_1^2+r_2^2+r_3^2}}$

Using the above formulas we get,

$\Rightarrow l_2=\frac{n}{\sqrt{n^2+(-2n)^2+n^2}}$

$\Rightarrow l_2=\frac{n}{\sqrt{n^2+4n^2+n^2}}$

$\Rightarrow l_2=\frac{n}{\sqrt{6n^2}}$

$\Rightarrow l_2=\frac{n}{(\sqrt{6}n)}$

$\Rightarrow l_2=\frac{1}{\sqrt{6}}$

$\Rightarrow m_2=\frac{-2n}{\sqrt{n^2+(-2n)^2+n^2}}$

$\Rightarrow m_2=\frac{-2n}{\sqrt{n^2+4n^2+n^2}}$

$\Rightarrow m_2=\frac{-2n}{\sqrt{6n^2}}$

$\Rightarrow m_2=\frac{-2n}{(\sqrt{6}n)}$

$\Rightarrow m_2=\frac{-2}{\sqrt{6}}$

$\Rightarrow n_2=\frac{n}{\sqrt{n^2+(-2n)^2+n^2}}$

$\Rightarrow n_2=\frac{n}{\sqrt{n^2+4n^2+n^2}}$

$\Rightarrow n_2=\frac{n}{\sqrt{6n^2}}$

$\Rightarrow n_2=\frac{n}{(\sqrt{6}n)}$

$\Rightarrow n_2=\frac{1}{\sqrt{6}}$

The Direction Cosines for the 2nd line is $\left(\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$.

16 A. Question

Find the angle between the lines whose direction cosines are given by the equations:

$l+m+n=0$ and $l^2+m^2-n^2=0$

Answer

Given relations are:

$\Rightarrow l^2+m^2-n^2=0$ ......(1)

$\Rightarrow l+m+n=0$

$\Rightarrow l=-m-n$......(2)

Substituting (2) in (1) we get,

⇒ $(-m-n)^2+m^2-n^2=0$

⇒ $m^2+n^2+2mn+m^2-n^2=0$

⇒ $2m^2+2mn=0$

⇒ $2m(m+n)=0$

⇒ $2m=0$ or $m+n=0$

⇒ $m=0$ or $m=-n$ ......(3)

Substituting value of m from(3) in (2)

For the 1st line:

⇒ $l=-0-n$

⇒ $l=-n$

The Direction Ratios for the first line is (–n,0,n)

For the 2nd line:

⇒ $l=-(-n)-n$

⇒ $l=n-n$

⇒ $l=0$

The Direction Ratios for the second line is (0,–n,n)

We know that the angle between the lines with direction ratios proportional to (a_1,b_1,c_1) and (a_2,b_2,c_2) is given by:

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Using the above formula we calculate the angle between the lines.

Let α be the angle between the two lines given in the problem.

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{(-n.0)+(0.-n)+(n.n)}{\sqrt{(-n)^2+0^2+n^2}\sqrt{0^2+(-n)^2+n^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{0+0+n^2}{\sqrt{2n^2}\sqrt{2n^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{n^2}{2n^2}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \alpha = \frac{\pi}{3}$$

∴ The angle between given two lines is $\frac{\pi}{3}$ or 60^0.

16 B. Question

Find the angle between the lines whose direction cosines are given by the equations:

$2l-m+2n=0$ and $mn+nl+lm=0$

Answer

Given relations are:

$\Rightarrow$ mn+nl+lm=0 ......(1)

$\Rightarrow$ 2l–m+2n=0

$\Rightarrow$ m=2l+2n ......(2)

Substituting (2) in (1) we get,

$\Rightarrow$ (2l+2n)n+nl+l(2l+2n)=0

$\Rightarrow$ $2ln+2n^2+nl+2l^2+2ln=0$

$\Rightarrow$ $2n^2+5ln+2l^2=0$

$\Rightarrow$ $2n^2+4ln+ln+2l^2=0$

$\Rightarrow$ 2n(n+2l)+l(n+2l)=0

$\Rightarrow$ (2n+l)(n+2l)=0

$\Rightarrow$ 2n+l=0 or n+2l=0

$\Rightarrow$ l=–2n or 2l=–n ......(3)

Substituting the values of(3) in (2) we get,

For the 1st line:

$\Rightarrow$ m = 2(–2n)+2n

$\Rightarrow$ m=–4n+2n

$\Rightarrow$ m=–2n

The direction ratios for the 1st line is (–2n,–2n,n)

For the 2nd line:

$\Rightarrow$ m=–n+2n

$\Rightarrow$ m=n

The direction ratios for the 2nd line is $\left(\frac{-n}{2}, n, n\right)$

We know that the angle between the lines with direction ratios proportional to (a_1,b_1,c_1) and (a_2,b_2,c_2) is given by:

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Using the above formula we calculate the angle between the lines.

Let α be the angle between the two lines given in the problem.

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{\left(-2n\times\frac{-n}{2}\right)+(-2n\times n)+(n\times n)}{\sqrt{(-2n)^2+(-2n)^2+n^2}\sqrt{\left(\frac{n}{2}\right)^2+(n)^2+n^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{n^2-2n^2+n^2}{\sqrt{4n^2+4n^2+n^2}\sqrt{\frac{n^2}{4}+n^2+n^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{0}{\sqrt{9n^2}\sqrt{\frac{9n^2}{4}}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}(0)$$

$\Rightarrow \alpha = \frac{\pi}{2}$

∴ the angle between two lines is $\frac{\pi}{2}$ or 90^0.

16 C. Question

Find the angle between the lines whose direction cosines are given by the equations:

l+2m+3n=0 and 3lm–4ln+mn=0

Answer

Given relations are:

⇒ 3lm–4ln+mn=0 ……(1)

⇒ l+2m+3n=0

⇒ l=–2m–3n ……(2)

Substituting (2) in (1) we get,

⇒ 3(–2m–3n)m –4(–2m–3n)n +mn =0

⇒ $3(-2m^2-3mn) -4(-2mn-3n^2) +mn=0$

⇒ $-6m^2-9mn+8mn+12n^2+mn=0$

⇒ $12n^2-6m^2=0$

⇒ $m^2-2n^2=0$

$\Rightarrow (m-\sqrt{2}n)(m+\sqrt{2}n) = 0$

$\Rightarrow m-\sqrt{2}n = 0 \text{ or } m+\sqrt{2}n = 0$

$\Rightarrow m = \sqrt{2}n \text{ or } m = -\sqrt{2}n$ ……(3)

Substituting the values of (3) in (2) we get,

For the 1st line:

$\Rightarrow l = -2(\sqrt{2}n) - 3n$

$\Rightarrow l = -(3+2\sqrt{2})n$

The Direction Ratios for the 1st line is $(-(3+2\sqrt{2})n, \sqrt{2}n, n)$.

For the 2nd line:

$\Rightarrow l = -2(-\sqrt{2}n) - 3n$

$\Rightarrow l = (2\sqrt{2}-3)n$

The Direction Ratios for the 2nd line is $((2\sqrt{2}-3)n, -\sqrt{2}n, n)$.

We know that the angle between the lines with direction ratios proportional to (a_1,b_1,c_1) and (a_2,b_2,c_2) is given by:

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Using the above formula we calculate the angle between the lines.

Let α be the angle between the two lines given in the problem.

$\Rightarrow \alpha = \cos^{-1}\left(\frac{\left(\left(-(3+2\sqrt{2})n\right)\times\left((2\sqrt{2}-3)n\right)\right)+(\sqrt{2}n\times-\sqrt{2}n)+(n\times n)}{\sqrt{\left(-(3+2\sqrt{2})n\right)^2+(\sqrt{2}n)^2+n^2}\sqrt{\left((2\sqrt{2}-3)n\right)^2+(-\sqrt{2}n)^2+n^2}}\right)$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{n^2(9-8-2+1)}{\sqrt{9n^2+8n^2+12\sqrt{2}n^2+2n^2+n^2}\sqrt{9n^2+8n^2-12\sqrt{2}n^2+2n^2+n^2}}\right)$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{0n^2}{\sqrt{20n^2+12\sqrt{2}n^2}\sqrt{20n^2-12\sqrt{2}n^2}}\right)$

$\Rightarrow \alpha = \cos^{-1}(0)$

$\Rightarrow \alpha = \frac{\pi}{2}$

∴ The angle between two lines is $\frac{\pi}{2}$ **or** 90^0.

16 D. Question

Find the angle between the lines whose direction cosines are given by the equations:

2l+2m–n=0 and mn+ln+lm=0

Answer

Given relations are:

⇒ mn+ln+lm=0 ……(1)

⇒ 2l+2m–n=0

⇒ n=2l+2m ……(2)

Substituting (2) in (1) we get,

⇒ m(2l+2m)+l(2l+2m)+lm=0

⇒ $2lm+2m^2+2l^2+2lm+lm=0$

⇒ $2m^2+5lm+2l^2=0$

⇒ $2m^2+4lm+lm+2l^2=0$

⇒ 2m(m+2l)+l(m+2l)=0

⇒ (2m+l)(m+2l)=0

⇒ 2m+l=0 or m+2l=0

⇒ 2m=–l or 2l=–m ……(3)

Substituting the values of (3) in (2), we get

For the 1st line:

⇒ n=2l–l

⇒ n=l

The Direction Ratios for the first line is $\left(l,-\frac{l}{2},l\right)$

For the 2nd line:

⇒ n=–m+2m

⇒ n=m

The Direction Ratios for the second line is $\left(\frac{-m}{2},m,m\right)$

We know that the angle between the lines with direction ratios proportional to (a_1,b_1,c_1) and (a_2,b_2,c_2) is given by:

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Using the above formula we calculate the angle between the lines.

Let α be the angle between the two lines given in the problem.

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{\left(l\times\frac{-m}{2}\right)+\left(\frac{-l}{2}\times m\right)+(l\times m)}{\sqrt{l^2+\left(\frac{-l}{2}\right)^2+l^2}\sqrt{\left(\frac{-m}{2}\right)^2+m^2+m^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{\frac{-lm}{2}-\frac{lm}{2}+lm}{\sqrt{l^2+\frac{l^2}{4}+l^2}\sqrt{\frac{m^2}{4}+m^2+m^2}}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{0}{\sqrt{\frac{9l^2}{4}}\sqrt{\frac{9m^2}{4}}}\right)$$

$$\Rightarrow \alpha = \frac{\pi}{2}$$

∴ the angle between two lines is $\frac{\pi}{2}$ **or** 90^0.

Very Short Answer

1. Question

Define direction cosines of a directed line.

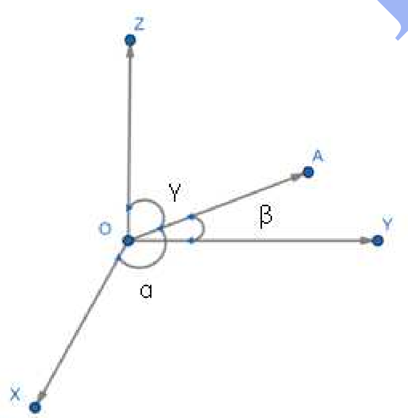
Answer

The direction cosines of a directed line can be defined as cosine values of the angles made by the directed line with the x-axis, y-axis and z-axis respectively.

Explanation:

Consider a directed line $\overrightarrow{OA}$, in the three dimensional space.

If α, β and γ be the angles made by the directed line $\overrightarrow{OA}$ with the x-axis, y-axis and z-axis respectively.



In the above figure, the direction cosines of line OA are:

Cos α = cosine of the angle between x-axis (OX) and the directed line $\overrightarrow{OA}$.

Cos β = cosine of the angle between y-axis (OY) and the directed line $\overrightarrow{OA}$.

Cos γ = cosine of the angle between z-axis (OZ) and the directed line $\overrightarrow{OA}$.

2. Question

What are the direction cosines of X-axis?

Answer

As per the definition of direction cosines, the cosine values of the angles formed by the directed line with the x-axis, y-axis and z-axis.

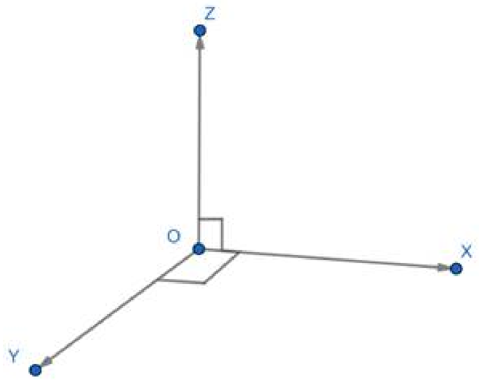
Here we consider the directed line to be the x-axis.

So from the below figure, we can say,

α = the angle formed by the x-axis with x-axis = 0°

β = the angle formed by the x-axis with y-axis = 90°

γ = the angle formed by the x-axis with y-axis = 90°



Therefore,

cos α = cos 0° = 1

cos β = cos 90° = 0

cos γ = cos 90° = 0

Hence the direction cosines of x-axis are 1, 0, 0.

3. Question

What are the direction cosines of Y-axis?

Answer

As per the definition of direction cosines, the cosine values of the angles formed by the directed line with the x-axis, y-axis and z-axis.

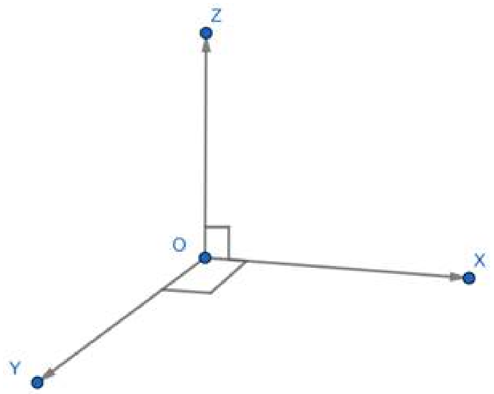
Here we consider the directed line to be the y-axis.

So from the below figure, we can say,

α = the angle formed by the y-axis with x-axis = 90°

β = the angle formed by the y-axis with y-axis = 0°

γ = the angle formed by the y-axis with y-axis = 90°



Therefore,

cos α = cos 90° = 0

cos β = cos 0° = 1

cos γ = cos 90° = 0

Hence the direction cosines of y-axis are 0, 1, 0.

4. Question

What are the direction cosines of Z-axis?

Answer

As per the definition of direction cosines, the cosine values of the angles formed by the directed line with the x-axis, y-axis and z-axis.

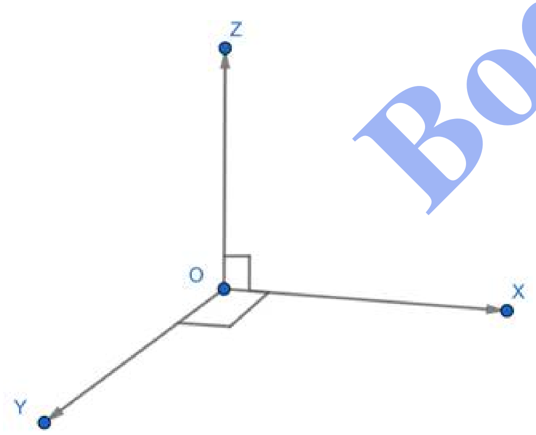
Here we consider the directed line to be the z-axis.

So from the below figure, we can say,

α = the angle formed by the z-axis with x-axis = 90°

β = the angle formed by the z-axis with y-axis = 90°

γ = the angle formed by the x-axis with y-axis = 0°



Therefore,

cos α = cos 90° = 0

cos β = cos 90° = 0

cos γ = cos 0° = 1

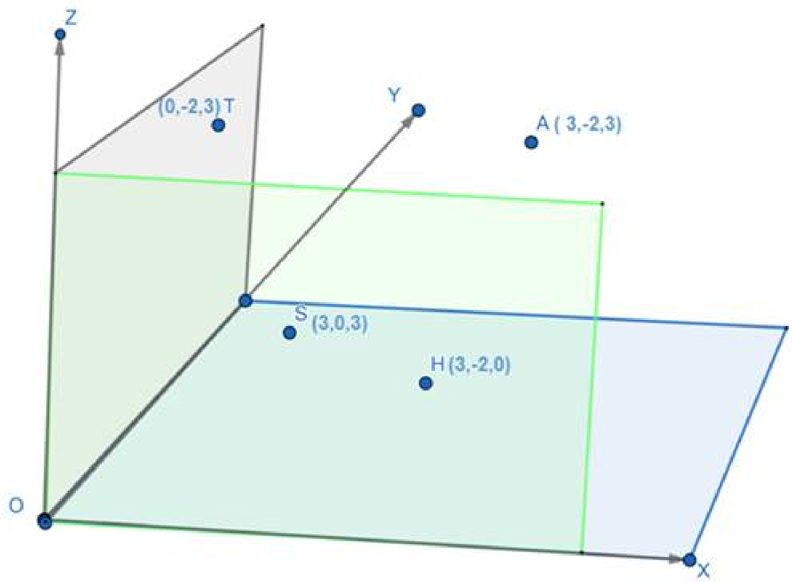
Hence the direction cosines of y-axis are 0, 0, 1.

5. Question

Write the distance of the point (3, –2, 3) from XY, YZ and XZ planes.

Answer

From the given information, A is a point with co-ordinates (3,-2, 3).



If you consider the projection of A(3,-2,3) on the XY-plane is H(3,-2,0) where the z-coordinate will not exist on XY-plane.

Similarly projection of A(3,-2,3) on the YZ-plane is T(0,-2,3) where the x-coordinate will not exist on YZ-plane.

The projection of A(3,-2,3) on the XZ-plane is T(3,0,3) where the x-coordinate will not exist on XZ-plane.

Now, the distance between A and XY-plane = Distance between points A&H

Distance between two points is given by $\sqrt{(a_2-a_1)^2+(b_2-b_1)^2+(c_2-c_1)^2}$

Using this formula,

Distance of point A from XY $=\sqrt{(3-3)^2+(-2-(-2))^2+(0-3)^2}$

$=\sqrt{(0)^2+(0)^2+(0-3)^2}$

$= \sqrt{3^2}$

= 3

Distance of point A from YZ $=\sqrt{(3-0)^2+(-2-(-2))^2+(3-3)^2}$

$=\sqrt{(3)^2+(0)^2+(0)^2}$

$= \sqrt{3^2}$

= 3

Distance of point A from XZ $=\sqrt{(3-3)^2+(-2-0)^2+(3-3)^2}$

$=\sqrt{(0)^2+(-2)^2+(0)^2}$

$= \sqrt{2^2}$

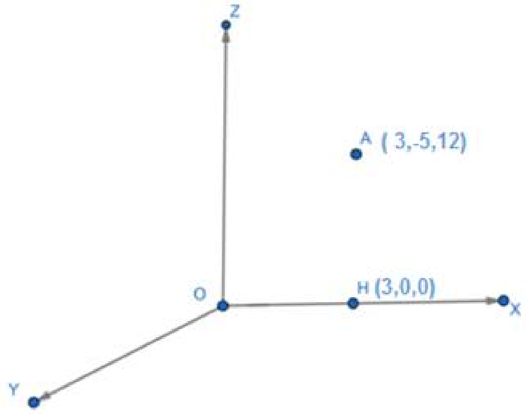
= 2

6. Question

Write the distance of the point (3, –5, 12) from X-axis?

Answer

From the given information, A is a point with co-ordinates (3, -5, 12).



From the figure, we can say that the projection of point A on x-axis will be point H(3,0,0) as the y-coordinate and z-coordinate will be zeros.

Distance between two points is given by $\sqrt{(a_2-a_1)^2+(b_2-b_1)^2+(c_2-c_1)^2}$

Using this formula,

Distance of point A from x-axis (point H)

$= \sqrt{(3-3)^2+(0-(-5))^2+(0-12)^2}$

$= \sqrt{(0)^2+(5)^2+(12)^2}$

$= \sqrt{0+25+144}$

$=\sqrt{169}$

=13

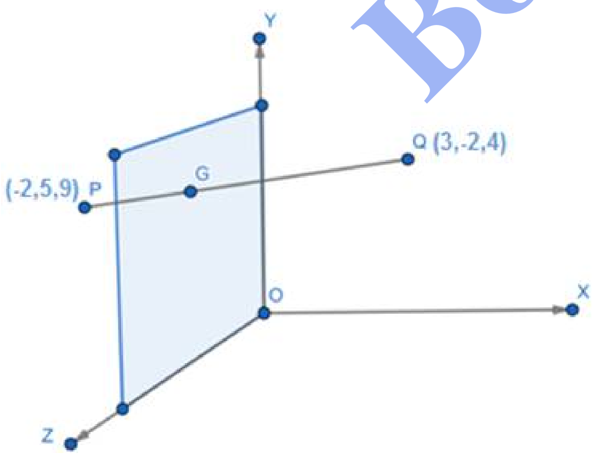
7. Question

Write the ratio in which YZ-plane divides the segment joining P(–2, 5, 9) and Q(3, –2, 4).

Answer

Given the points P(-2,5,9) and Q(3,-2,4)

Let the plane YZ-plane divide line segment PQ at point G(0,y,z) in the ratio m:n.



The coordinates of the point G which divides the line joining points A(x_1,y_1,z_1) and B(x_2,y_2,z_2) in the ratio m:n is given by

$= \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$

Here, we have m:n

x_1 = -2 y_1 = 5 z_1 = 9

$x_2 = 3$ $y_2 = -2$ $z_2 = 4$

By using the above formula, we get,

$$= \left(\frac{m \times (3) + n \times (-2)}{m+n}, \frac{m \times (-2) + n \times (5)}{m+n}, \frac{m \times (4) + m \times (9)}{m+n}\right)$$

$$= \left(\frac{3m - 2n}{m+n}, \frac{-2m + 5n}{m+n}, \frac{4m + 9m}{m+n}\right)$$

Now, this is the same point as G(0,y,z),

As the x-coordinate is zero,

$$\frac{3m - 2n}{m+n} = 0$$

[Cross Multiplying]

3m – 2n = 0 × (m + n)

3m – 2n = 0

3m = 2n

$$\frac{m}{n} = \frac{2}{3}$$

Therefore, the ratio in which the plane-YZ divides the line joining A & B is 2:3

8. Question

A line makes an angle of 60° with each of X-axis and Y-axis. Find the acute angle made by the line with Z-axis.

Answer

Given that, the line makes angles

• 60° with the x-axis.

• 60° with the y-axis.

Let the angle made by the line with z-axis be α.

Now, as per the relation between direction cosines of a line, $l^2 + m^2 + n^2 = 1$ where l,m,n are the direction cosines of a line from x-axis, y-axis and z-axis respectively.

From the problem,

$l = \cos 60° = \frac{1}{2}$

$m = \cos 60° = \frac{1}{2}$

$n = \cos \alpha$

By using the formula,

$l^2 + m^2 + n^2 = 1$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2 \alpha = 1$$

[As cos 60° value is $\frac{1}{2}$]

$$\frac{1}{4} + \frac{1}{4} + \cos^2 \alpha = 1$$

$\frac{1}{2} + \cos^2 \alpha = 1$

$\cos^2 \alpha = 1 - \frac{1}{2}$

$\cos^2 \alpha = \frac{1}{2}$

$\cos \alpha = \frac{1}{\sqrt{2}}$

[As cos 45° = $\frac{1}{\sqrt{2}}$]

α = 45°

Therefore, the angle made by the line with z-axis is 45°

9. Question

If a line makes angles α, β and γ with the coordinate axes, find the value of cos 2α + cos 2β + cos 2γ.

Answer

Given, the line makes the angles α, β and γ respectively with x-axis, y-axis and z-axis.

As per the relation between direction cosines of a line, $l^2 + m^2 + n^2 = 1$ where l,m,n are the direction cosines of a line from x-axis, y-axis and z-axis respectively.

So, we can say that,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ ------ (1)

Now, we should find the value for

cos2α + cos2β + cos2γ

cos2α can be written as $2\cos^2\alpha - 1$,

$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = (2\cos^2\alpha - 1) + (2\cos^2\beta - 1) + (2\cos^2\gamma - 1)$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) – 3$

= 2(1) – 3

[From Equation (1)]

= -1

Therefore,

cos2α + cos2β + cos2γ = -1

10. Question

Write the ratio in which the line segment joining (a, b, c) and (-a, -c, -b) is divided by the xy-plane.

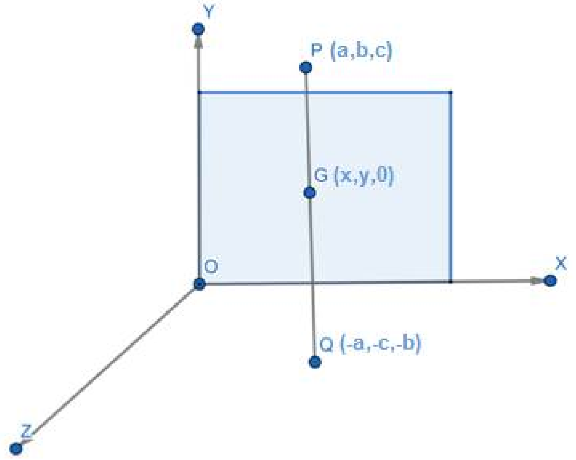
Answer

Given,

The line segment is formed by P and Q points where

Point P = (a,b,c)

Point Q = (-a,-c,-b)



From the figure, we can clearly see that, the line segment joining points P and Q is meeting the plane XY at point G.

Let Point G be (x,y,0) as the z-coordinate on xy plane does not exist.

Also let point G divides the line segment joining P and Q in the ratio m:n.

The coordinates of the point G which divides the line joining points A(x_1,y_1,z_1) and B(x_2,y_2,z_2) in the ratio m:n is given by

$$= \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

Here, we have m:n

$x_1 = a$ $y_1 = b$ $z_1 = c$

$x_2 = -a$ $y_2 = -c$ $z_2 = -b$

By using the above formula, we get,

$$= \left(\frac{m \times (-a) + n \times (a)}{m+n}, \frac{m \times (-c) + n \times (b)}{m+n}, \frac{m \times (-b) + m \times (c)}{m+n}\right)$$

$$= \left(\frac{-am + an}{m+n}, \frac{-cm + bn}{m+n}, \frac{-bm + cm}{m+n}\right)$$

Now, this is the same point as G(x,y,0),

As the x-coordinate is zero,

$$\frac{-bm + cn}{m+n} = 0$$

[Cross Multiplying]

-bm + cn = 0 × (m + n)

-bm + cn = 0

-bm = -cn

$$\frac{m}{n} = \frac{c}{b}$$

Therefore, the ratio in which the plane-XY divides the line joining P & Q is c:b

11. Question

Write the inclination of a line with Z-axis, if its direction ratios are proportional to 0, 1, –1.

Answer

Given, the direction ratios of the line are proportional to (0, 1,-1)

Therefore, consider the direction ratios of the give line can be

a = 0 × k, b = 1 × k, c = (-1) × k

[where k is some proportionality constant]

Now the direction ratios of the line are

a = 0, b = k, c = -k

As we know the direction cosine of z-axis can be given by

cos γ = n = $\frac{c}{\sqrt{a^2+b^2+c^2}}$ where γ is the angle made by the line with the z-axis.

By using the above formula:

$$\cos\gamma = \frac{-k}{\sqrt{0^2 + (k)^2 + (-k)^2}}$$

$$\cos\gamma = \frac{-k}{\sqrt{2k^2}}$$

$$\cos\gamma = \frac{-k}{k\sqrt{2}}$$

$$\cos\gamma = \frac{-1}{\sqrt{2}}$$

[As cosine function is negative, the angle become 135° instead of 45°]

$$\gamma = \frac{3\pi}{4}$$

The inclination of the line with z-axis is $\frac{3\pi}{4}$

12. Question

Write the angle between the lines whose direction ratios are proportional to 1, –2, 1 and 4, 3, 2.

Answer

Given,

• Direction Ratios of Line1 are proportional to (1,-2,1)

• Direction Ratios of Line2 are proportional to (4,3,2)

So we can say that,

Direction ratios of line1

a_1 = 1 × k , b_1 = (-2) × k and c_1 = 1 × k

a_1 = k , b_1 = -2k and c_1 = k

Direction ratios of line2

a_2 = 4 × p , b_2 = 3 × p and c_2 = 2 × p

a_2 = 4p , b_2 = 3p and c_2 = 2p

Now, the angle between the lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is given by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

By using this formula,

$$\cos\theta = \frac{|(k \times 4p) + (-2k \times 3p) + (k \times 2p)|}{\sqrt{k^2 + (-2k)^2 + k^2}\sqrt{(4p)^2 + (3p)^2 + (2p)^2}}$$

$$\cos\theta = \frac{|4kp - 6kp + 2kp|}{\sqrt{k^2 + 4k^2 + k^2}\sqrt{16p^2 + 9p^2 + 4p^2}}$$

$$\cos\theta = \frac{|0|}{\sqrt{k^2 + 4k^2 + k^2}\sqrt{16p^2 + 9p^2 + 4p^2}}$$

cos θ = 0

θ = 90°

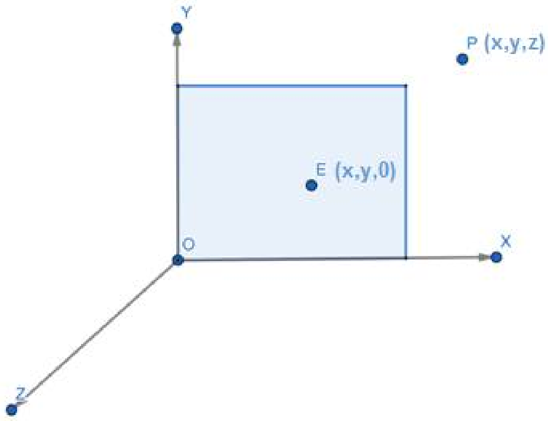
The angle between the lines is 90°.

13. Question

Write the distance of the point P(x, y, z) from XOY plane.

Answer

Given point P(x,y,z)



From the figure, we can say that Point E (x,y,0) is the projection of Point P on the XY-plane (the z-coordinate remains zero on XY-plane).

Distance between two points is given by $\sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}$

Here the distance between Point P & E will give the distance of the point P from the XY-plane.

Here a_1 = x, b_1 =y , c_1 =z

a_2 = x, b_2 = y, c_2 = 0

Distance from P to E =

$\sqrt{(x - x)^2 + (y - y)^2 + (0 - z)^2}$

= √ $(-z)^2$

= √ $(z)^2$

= z

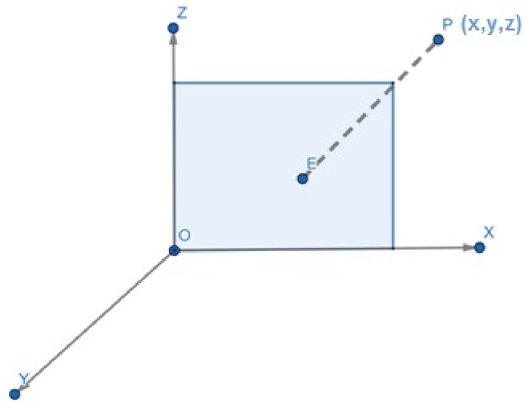
Therefore, the distance between the XY plane and point P is z units.

14. Question

Write the coordinates of the projection of point P(x, y, z) on XOZ-plane.

Answer

Given, point P (x,y,z)



From the figure, we can clearly see the projection of point P on the XOZ plane.

The projection of P on the x-axis will be (x,0,0)

The projection of P on the z-axis will be (0,0,z)

By this we can say that, if we are considering the projection of P on the XOZ plane, the coordinates of Y-axis will be zero,

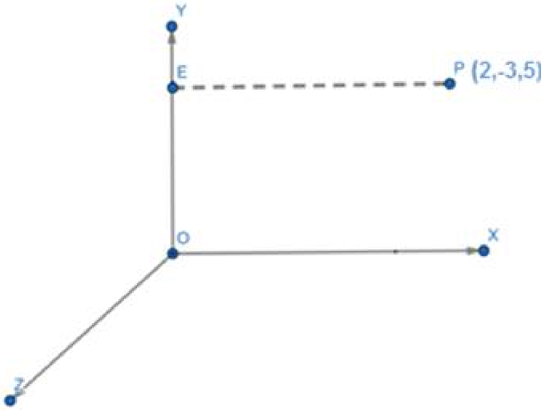
Hence the projection of point P(x,y,z) on the XOZ plane will be point E(x,o,z).

15. Question

Write the coordinates of the projection of the point P(2, -3, 5) on Y-axis.

Answer

Given Point P is (2,-3,5)



From the figure, we can see that Point E is the projection of P (2,-3,5) on the Y-axis.

All the points on the y-axis are of the form (0,y,0).

Hence, the projection of point P on y-axis will be (0,-3,0).

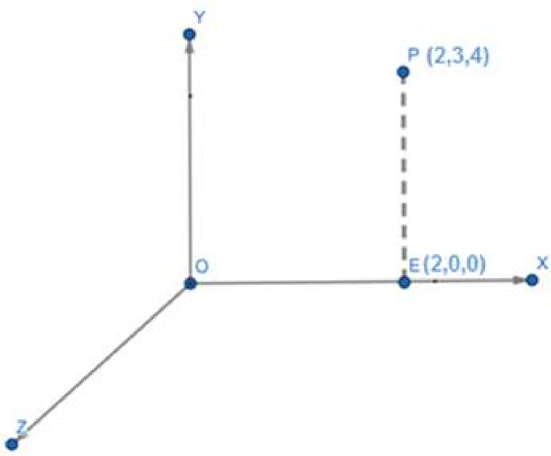
16. Question

Find the distance of the point (2, 3, 4) from the x-axis.

Answer

Given,

The point is (2,3,4). Let this point be P.



From the figure, point E (2,0,0) is the projection of point P(2,3,4) on the x-axis.

The distance between the points P & E will give the distance of the point P from x-axis.

Distance between two points is given by $\sqrt{(a_2-a_1)^2+(b_2-b_1)^2+(c_2-c_1)^2}$

Here

a_1 =2 , b_1 =3 , c_1=4 and a_2 =2 , b_2 =0 , c_2 = 0

Distance between P and x-axis is

$= \sqrt{(2-2)^2+(0-3)^2+(0-4)^2}$

$= \sqrt{(0)^2+(-3)^2+(-4)^2}$

$= \sqrt{9+16}$

= √ 25

= 5

Therefore the distance between, the x-axis and the Point P (2,3,4) is 5 units.

17. Question

If a line has direction ratios proportional to 2, -1, -2, then what are its direction consines?

Answer

Given, the direction ratios of the line are proportional to (2, -1,-2)

Therefore, consider the direction ratios of the give line can be

a = 2 × k, b = (-1) × k, c = (-2) × k

[where k is some proportionality constant]

Now the direction ratios of the line are

a = 2k, b = -k, c = -2k

As we know the direction cosine ae given by

cos α = l $= \frac{a}{\sqrt{a^2+b^2+c^2}}$, cos β = m $= \frac{b}{\sqrt{a^2+b^2+c^2}}$, cos γ = n $= \frac{c}{\sqrt{a^2+b^2+c^2}}$

Where α, β and γ are the angles formed by the line with the three axes.

By using the above formula:

l = cos α =

$$= \frac{2k}{\sqrt{(2k)^2+(-k)^2+(-2k)^2}}$$

$$= \frac{2k}{\sqrt{4k^2+k^2+4k^2}}$$

$$= \frac{2k}{\sqrt{9k^2}}$$

$$= \frac{2k}{3k}$$

Therefore cos α $= \frac{2}{3}$

m =cos β =

$$= \frac{-k}{\sqrt{(2k)^2+(-k)^2+(-2k)^2}}$$

$= \frac{-k}{\sqrt{4k^2 + k^2 + 4k^2}}$

$= \frac{-k}{\sqrt{9k^2}}$

$= \frac{-k}{3k}$

$\cos \beta = \frac{-1}{3}$

$n = \cos \gamma =$

$= \frac{-2k}{\sqrt{(2k)^2 + (-k)^2 + (-2k)^2}}$

$= \frac{-2k}{\sqrt{4k^2 + k^2 + 4k^2}}$

$= \frac{-2k}{\sqrt{9k^2}}$

$= \frac{-2k}{3k}$

$\cos \gamma = -\frac{2}{3}$

Therefore, the direction cosines are $\frac{2}{3}, -\frac{1}{3} - \frac{2}{3}$

18. Question

Write direction cosines of a line parallel to z-axis.

Answer

Given

The line is parallel to z- axis.

So the line would be perpendicular to both x-axis and y-axis.

Hence, the angles formed by the line with x-axis & y-axis are 90° and 90° respectively.

Also the angle formed by the line with z-axis is 0°.

The direction cosines of a line are given by, cos α , cos β , cos γ. Where α,β and γ are angles formed by the line with the x,y and z axes respectively.

Here

α = 90°,β = 90° and γ = 0°

α = cos 90°= 0

β = cos 90°= 0

γ = cos 0° = 1

Therefore the direction cosines of the line parallel to z-axis are (0,0,1).

19. Question

If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the value of θ.

Answer

Given the unit vector makes,

- an angle of $\frac{\pi}{3}$ with x-axis
- an angle of $\frac{\pi}{4}$ with y-axis
- an angle of θ with z-axis
- θ is acute angle

Let the unit vector $\vec{a}$ be: $x\hat{i} + y\hat{j} + z\hat{k}$

As given it is a unit vector,

Therefore $|\vec{a}|$ = 1

As the angle between in $\vec{a}$ and x-axis is $\frac{\pi}{3}$, the scalar product of the vectors can be performed.

The scalar product of the two vectors is given by

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\vec{a}.\hat{i} = |\vec{a}||\hat{i}|\cos\frac{\pi}{3}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).\hat{i} = 1 \times 1 \times \cos\frac{\pi}{3}$$

[as both the vectors are of magnitude 1].

$$(x\hat{i} + y\hat{j} + z\hat{k}).(1\hat{i} + 0\hat{j} + 0\hat{k}) = 1 \times 1 \times \cos\frac{\pi}{3}$$

$$(x \times 1) + (y \times 0) + (z \times 0) = \frac{1}{2}$$

$$x = \frac{1}{2}$$

As the angle between in $\vec{a}$ and y-axis is $\frac{\pi}{4}$, the scalar product of the vectors can be performed.

$$\vec{a}.\hat{j} = |\vec{a}||\hat{j}|\cos\frac{\pi}{4}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).\hat{j} = 1 \times 1 \times \cos\frac{\pi}{4}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).(0\hat{i} + 1\hat{j} + 0\hat{k}) = 1 \times 1 \times \cos\frac{\pi}{4}$$

$$(x \times 0) + (y \times 1) + (z \times 0) = \frac{1}{\sqrt{2}}$$

$$y = \frac{1}{\sqrt{2}}$$

Similarly the angle between in $\vec{a}$ and y-axis is θ , the scalar product of the vectors can be performed.

$$\vec{a}.\hat{k} = |\vec{a}||\hat{k}|\cos\frac{\pi}{4}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).\hat{k} = 1 \times 1 \times \cos\theta$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).(0\hat{i} + 0\hat{j} + 1\hat{k}) = 1 \times 1 \times \cos\theta$$

$$(x \times 0) + (y \times 0) + (z \times 1) = \cos\theta$$

$$z = \cos\theta$$

The magnitude of a vector $x\hat{i} + y\hat{j} + z\hat{k}$ is given by $\sqrt{x^2 + y^2 + z^2}$.

Now consider the magnitude of the vector $\vec{a}$

$$1 = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta}$$

$$1 = \sqrt{\frac{1}{4} + \frac{1}{2} + \cos^2\theta}$$

[Squaring on both sides]

$$1 = \frac{3}{4} + \cos^2\theta$$

$$\cos^2\theta = 1 - \frac{3}{4}$$

$$\cos^2\theta = \frac{1}{4}$$

$$\cos\theta = \pm\sqrt{\frac{1}{4}}$$

$$\cos\theta = \pm\frac{1}{2}$$

As given in the question θ is acute angle, so θ belongs to 1st quadrant and is positive.

Therefore $\theta = \frac{\pi}{3}$

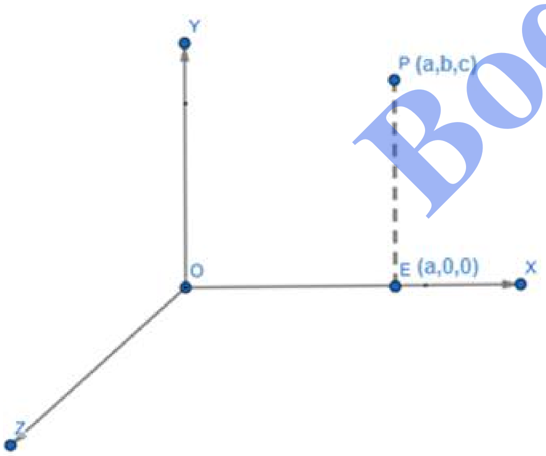
20. Question

Write the distance of a point P(a, b, c) from x-axis.

Answer

Given,

The point is (a,b,c). Let this point be P.



From the figure, point E (a,0,0) is the projection of point P(a,b,c) on the x-axis.

The distance between the points P & E will give the distance of the point P from x-axis.

Distance between two points is given by $\sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}$

Here

$a_1 = a$, $b_1 = b$, $c_1 = c$ and $a_2 = a$, $b_2 = 0$, $c_2 = 0$

Distance between P and x-axis is

$$= \sqrt{(a - a)^2 + (0 - b)^2 + (0 - c)^2}$$

$= \sqrt{(0)^2 + (-b)^2 + (-c)^2}$

$= \sqrt{b^2 + c^2}$

Therefore the distance between, the x-axis and the Point P (a,b,c) is $\sqrt{b^2 + c^2}$ units.

21. Question

If a line makes angle 90° and 60° respectively with positive directions of x an y axe, find the angle which it makes with the positive direction of z-axis.

Answer

Given a line makes,

• an angle of 90° with x-axis

• an angle of 60° with y-axis

So, let the angle made by the line with z-axis is θ

Now, as per the relation between direction cosines of a line, $l^2 + m^2 + n^2 = 1$ where l,m,n are the direction cosines of a line from x-axis, y-axis and z-axis respectively.

From the problem,

$l = \cos 90° = 0$

$m = \cos 60° = \frac{1}{2}$

$n = \cos \theta$

By using the formula,

$l^2 + m^2 + n^2 = 1$

$0^2 + \left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$

$\frac{1}{4} + \cos^2\theta = 1$

$\cos^2\theta = 1 - \frac{1}{4}$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

As the angle made by the line with positive z-axis, so the cosine angle is positive.

Therefore, $\cos \theta = \frac{\sqrt{3}}{2}$

Hence θ = 30°.