25. Vector or Cross Product

Exercise 25.1

1. Question

If $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = -\hat{i} + 3\hat{k}$, find $\left|\vec{a} \times \vec{b}\right|$.

Answer

Given $\vec{a} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{b} = -\hat{\imath} + 3\hat{k}$

We need to find the magnitude of the vector $\vec{a} \times \vec{b}$.

Recall the cross product of two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, 3, -2)$ and $(b_1, b_2, b_3) = (-1, 0, 3)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{\imath}[(3)(3) - (0)(-2)] - \hat{\jmath}[(1)(3) - (-1)(-2)] + \hat{k}[(1)(0) - (-1)(3)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{\imath}[9 - 0] - \hat{\jmath}[3 - 2] + \hat{k}[0 - (-3)]$$

$$\therefore \vec{a} \times \vec{b} = 9\hat{\imath} - \hat{\jmath} + 3\hat{k}$$

Recall the magnitude of the vector $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is

$$\left|x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $\left|\vec{a} \times \vec{b}\right|$.

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{9^2 + (-1)^2 + 3^2}$$

$$\Rightarrow \left|\vec{a} \times \vec{b}\right| = \sqrt{81 + 1 + 9}$$

$$\therefore \left|\vec{a} \times \vec{b}\right| = \sqrt{91}$$

Thus, $\left|\vec{a} \times \vec{b}\right| = \sqrt{91}$

2 A. Question

If $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, find the value of $\left|\vec{a} \times \vec{b}\right|$.

Answer

Given $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{b} = \hat{\imath} + \hat{\jmath} + \hat{k}$

We need to find the magnitude of the vector $\vec{a} \times \vec{b}$.

Recall the cross product of two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (3, 4, 0)$ and $(b_1, b_2, b_3) = (1, 1, 1)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(4)(1) - (1)(0)] - \hat{j}[(3)(1) - (1)(0)] + \hat{k}[(3)(1) - (1)(4)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[4 - 0] - \hat{j}[3 - 0] + \hat{k}[3 - 4]$$

$$\therefore \vec{a} \times \vec{b} = 4\hat{i} - 3\hat{j} - \hat{k}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a} \times \vec{b}|$.

$$|\vec{a} \times \vec{b}| = \sqrt{4^2 + (-3)^2 + (-1)^2}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{16 + 9 + 1}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{26}$$

Thus, $|\vec{a} \times \vec{b}| = \sqrt{26}$

2 B. Question

If $\vec{a} = 2\hat{i} + \hat{j}, \vec{b} = \hat{i} + \hat{j} + \hat{k},$ find the magnitude of $\vec{a} \times \vec{b}$.

Answer

Given $\vec{a} = 2\hat{i} + \hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

We need to find the magnitude of the vector $\vec{a} \times \vec{b}$.

Recall the cross product of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (2, 1, 0)$ and $(b_1, b_2, b_3) = (1, 1, 1)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(1)(1) - (1)(0)] - \hat{j}[(2)(1) - (1)(0)] + \hat{k}[(2)(1) - (1)(1)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[1 - 0] - \hat{j}[2 - 0] + \hat{k}[2 - 1]$$

$$\therefore \vec{a} \times \vec{b} = \hat{i} - 2\hat{j} + \hat{k}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a} \times \vec{b}|$.

$$|\vec{a} \times \vec{b}| = \sqrt{1^2 + (-2)^2 + 1^2}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{1 + 4 + 1}$$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{6}$

Thus, the magnitude of the vector $\vec{a} \times \vec{b} = \sqrt{6}$

3 A. Question

Find a unit vector perpendicular to both the vectors $4\hat{i} - \hat{j} + 3\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$.

Answer

Given two vectors $4\hat{i} - \hat{j} + 3\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$

Let $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$

We need to find a unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

Recall a vector that is perpendicular to two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (4, -1, 3)$ and $(b_1, b_2, b_3) = (-2, 1, -2)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(-1)(-2) - (1)(3)] - \hat{j}[(4)(-2) - (-2)(3)] + \hat{k}[(4)(1) - (-2)(-1)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[2 - 3] - \hat{j}[-8 + 6] + \hat{k}[4 - 2]$$

$$\therefore \vec{a} \times \vec{b} = -\hat{i} + 2\hat{j} + 2\hat{k}$$

Let the unit vector in the direction of $\vec{a} \times \vec{b}$ be $\hat{p}$.

We know unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

$$\Rightarrow \hat{p} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a} \times \vec{b}|$.

$$|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + 2^2 + 2^2}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{1 + 4 + 4}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{9} = 3$$

So, we have $\hat{p} = \frac{\vec{a} \times \vec{b}}{3}$

$$\Rightarrow \hat{p} = \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$$

Thus, the required unit vector that is perpendicular to both $\vec{a}$ and $\vec{b}$ is $\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$.

3 B. Question

Find a unit vector perpendicular to the plane containing the vectors $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Answer

Given two vectors $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

We need to find a unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

Recall a vector that is perpendicular to two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (2, 1, 1)$ and $(b_1, b_2, b_3) = (1, 2, 1)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(1)(1) - (2)(1)] - \hat{j}[(2)(1) - (1)(1)] + \hat{k}[(2)(2) - (1)(1)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[1 - 2] - \hat{j}[2 - 1] + \hat{k}[4 - 1]$$

$$\therefore \vec{a} \times \vec{b} = -\hat{i} - \hat{j} + 3\hat{k}$$

Let the unit vector in the direction of $\vec{a} \times \vec{b}$ be $\hat{p}$.

We know unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

$$\Rightarrow \hat{p} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a} \times \vec{b}|$.

$$|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + (-1)^2 + 3^2}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{1 + 1 + 9}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{11}$$

So, we have $\hat{p} = \frac{\vec{a} \times \vec{b}}{\sqrt{11}}$

$$\Rightarrow \hat{p} = \frac{1}{\sqrt{11}}(-\hat{i} - \hat{j} + 3\hat{k})$$

Thus, the required unit vector that is perpendicular to both $\vec{a}$ and $\vec{b}$ is $\frac{1}{\sqrt{11}}(-\hat{i} - \hat{j} + 3\hat{k})$.

4. Question

Find the magnitude of vector $\vec{a} = (3\hat{k} + 4\hat{j}) \times (\hat{i} + \hat{j} - \hat{k})$.

Answer

Given $\vec{a} = (3\hat{k} + 4\hat{j}) \times (\hat{i} + \hat{j} - \hat{k})$

$$\Rightarrow \vec{a} = (4\hat{j} + 3\hat{k}) \times (\hat{i} + \hat{j} - \hat{k})$$

We need to find the magnitude of the vector $\vec{a}$.

Recall the cross product of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (0, 4, 3)$ and $(b_1, b_2, b_3) = (1, 1, -1)$

$$\Rightarrow \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 3 \\ 1 & 1 & -1 \end{vmatrix}$$

$$\Rightarrow \vec{a} = \hat{i}[(4)(-1) - (1)(3)] - \hat{j}[(0)(-1) - (1)(3)] + \hat{k}[(0)(1) - (1)(4)]$$

$$\Rightarrow \vec{a} = \hat{i}[-4 - 3] - \hat{j}[0 - 3] + \hat{k}[0 - 4]$$

$$\therefore \vec{a} = -7\hat{i} + 3\hat{j} - 4\hat{k}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a}|$.

$$|\vec{a}| = \sqrt{(-7)^2 + 3^2 + (-4)^2}$$

$$\Rightarrow |\vec{a}| = \sqrt{49 + 9 + 16}$$

$$\therefore |\vec{a}| = \sqrt{74}$$

Thus, magnitude of vector $\vec{a} = \sqrt{74}$

5. Question

If $\vec{a} = 4\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{k}$, then find $|2\hat{b} \times \vec{a}|$.

Answer

Given $\vec{a} = 4\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{k}$

We need to find the magnitude of vector $2\hat{b} \times \vec{a}$.

We know unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

$$\Rightarrow \hat{b} = \frac{\vec{b}}{|\vec{b}|}$$

$$\Rightarrow \hat{b} = \frac{(\hat{i} - 2\hat{k})}{\sqrt{1^2 + (-2)^2}}$$

$$\Rightarrow \hat{b} = \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{k})$$

$$\therefore 2\hat{b} = \frac{2}{\sqrt{5}}(\hat{i} - 2\hat{k}) = \frac{2}{\sqrt{5}}\hat{i} - \frac{4}{\sqrt{5}}\hat{k}$$

Recall the cross product of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = \left(\frac{2}{\sqrt{5}}, 0, -\frac{4}{\sqrt{5}}\right)$ and $(b_1, b_2, b_3) = (4, 3, 1)$

$$\Rightarrow 2\hat{b}\times\vec{a}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{2}{\sqrt{5}} & 0 & -\frac{4}{\sqrt{5}}\\ 4 & 3 & 1\end{vmatrix}$$

$$\Rightarrow 2\hat{b}\times\vec{a}=\hat{i}\left[(0)(1)-(3)\left(-\frac{4}{\sqrt{5}}\right)\right]-\hat{j}\left[\left(\frac{2}{\sqrt{5}}\right)(1)-(4)\left(-\frac{4}{\sqrt{5}}\right)\right]$$
$$+\hat{k}\left[\left(\frac{2}{\sqrt{5}}\right)(3)-(4)(0)\right]$$

$$\Rightarrow 2\hat{b}\times\vec{a}=\hat{i}\left[0+\frac{12}{\sqrt{5}}\right]-\hat{j}\left[\frac{2}{\sqrt{5}}+\frac{16}{\sqrt{5}}\right]+\hat{k}\left[\frac{6}{\sqrt{5}}-0\right]$$

$$\therefore 2\hat{b}\times\vec{a}=\frac{12}{\sqrt{5}}\hat{i}-\frac{18}{\sqrt{5}}\hat{j}+\frac{6}{\sqrt{5}}\hat{k}$$

Recall the magnitude of the vector $x\hat{i}+y\hat{j}+z\hat{k}$ is

$$\left|x\hat{i}+y\hat{j}+z\hat{k}\right|=\sqrt{x^2+y^2+z^2}$$

Now, we find $\left|2\hat{b}\times\vec{a}\right|$.

$$\left|2\hat{b}\times\vec{a}\right|=\sqrt{\left(\frac{12}{\sqrt{5}}\right)^2+\left(-\frac{18}{\sqrt{5}}\right)^2+\left(\frac{6}{\sqrt{5}}\right)^2}$$

$$\Rightarrow \left|2\hat{b}\times\vec{a}\right|=\sqrt{\frac{144}{5}+\frac{324}{5}+\frac{36}{5}}$$

$$\therefore \left|2\hat{b}\times\vec{a}\right|=\sqrt{\frac{504}{5}}$$

Thus, $\left|2\hat{b}\times\vec{a}\right|=\sqrt{\frac{504}{5}}$

6. Question

If $\vec{a}=3\hat{i}-\hat{j}-2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$, find $\left(\vec{a}+2\vec{b}\right)\times\left(2\vec{a}-\vec{b}\right)$.

Answer

Given $\vec{a}=3\hat{i}-\hat{j}-2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$

We need to find the vector $(\vec{a}+2\vec{b})\times(2\vec{a}-\vec{b})$.

$$\vec{a}+2\vec{b}=(3\hat{i}-\hat{j}-2\hat{k})+2(2\hat{i}+3\hat{j}+\hat{k})$$

$$\Rightarrow \vec{a}+2\vec{b}=(3+4)\hat{i}+(-1+6)\hat{j}+(-2+2)\hat{k}$$

$$\therefore \vec{a}+2\vec{b}=7\hat{i}+5\hat{j}$$

$$2\vec{a}-\vec{b}=2(3\hat{i}-\hat{j}-2\hat{k})-(2\hat{i}+3\hat{j}+\hat{k})$$

$$\Rightarrow 2\vec{a}-\vec{b}=(6-2)\hat{i}+(-2-3)\hat{j}+(-4-1)\hat{k}$$

$\therefore 2\vec{a}-\vec{b}=4\hat{i}-5\hat{j}-5\hat{k}$

Recall the cross product of two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (7, 5, 0)$ and $(b_1, b_2, b_3) = (4, -5, -5)$

$$\Rightarrow \left(\vec{a}+2\vec{b}\right)\times\left(2\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 7 & 5 & 0\\ 4 & -5 & -5\end{vmatrix}$$

$$\Rightarrow \left(\vec{a}+2\vec{b}\right)\times\left(2\vec{a}-\vec{b}\right) = \hat{i}[(5)(-5)-(-5)(0)]-\hat{j}[(7)(-5)-(4)(0)] + \hat{k}[(7)(-5)-(4)(5)]$$

$$\Rightarrow \left(\vec{a}+2\vec{b}\right)\times\left(2\vec{a}-\vec{b}\right)=\hat{i}[-25-0]-\hat{j}[-35-0]+\hat{k}[-35-20]$$

$$\therefore \left(\vec{a}+2\vec{b}\right)\times\left(2\vec{a}-\vec{b}\right)=-25\hat{i}+35\hat{j}-55\hat{k}$$

Thus, $\left(\vec{a}+2\vec{b}\right)\times\left(2\vec{a}-\vec{b}\right)=-25\hat{i}+35\hat{j}-55\hat{k}$

7 A. Question

Find a vector of magnitude 49, which is perpendicular to both the vectors $2\hat{i}+3\hat{j}+6\hat{k}$ and $3\hat{i}-6\hat{j}+2\hat{k}$.

Answer

Given two vectors $2\hat{i}+3\hat{j}+6\hat{k}$ and $3\hat{i}-6\hat{j}+2\hat{k}$

Let $\vec{a}=2\hat{i}+3\hat{j}+6\hat{k}$ and $\vec{b}=3\hat{i}-6\hat{j}+2\hat{k}$

We need to find a vector of magnitude 49 that is perpendicular to $\vec{a}$ and $\vec{b}$.

Recall a vector that is perpendicular to two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (2, 3, 6)$ and $(b_1, b_2, b_3) = (3, -6, 2)$

$$\Rightarrow \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 6\\ 3 & -6 & 2\end{vmatrix}$$

$$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[(3)(2)-(-6)(6)]-\hat{j}[(2)(2)-(3)(6)]+\hat{k}[(2)(-6)-(3)(3)]$$

$$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[6+36]-\hat{j}[4-18]+\hat{k}[-12-9]$$

$$\therefore \vec{a}\times\vec{b}=42\hat{i}+14\hat{j}-21\hat{k}$$

Recall the magnitude of the vector $x\hat{i}+y\hat{j}+z\hat{k}$ is

$$\left|x\hat{i}+y\hat{j}+z\hat{k}\right|=\sqrt{x^2+y^2+z^2}$$

Now, we find $\left|\vec{a}\times\vec{b}\right|$.

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{42^2+14^2+(-21)^2}$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\sqrt{1764+196+441}$$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{2401} = 49$

Thus, the vector of magnitude 49 that is perpendicular to both $\vec{a}$ and $\vec{b}$ is $42\hat{i} + 14\hat{j} - 21\hat{k}$.

7 B. Question

Find the vector whose length is 3 and which is perpendicular to the vector $\vec{a} = 3\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 6\hat{i} + 5\hat{j} - 2\hat{k}$.

Answer

Given two vectors $\vec{a} = 3\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 6\hat{i} + 5\hat{j} - 2\hat{k}$

We need to find vector of magnitude 3 that is perpendicular to $\vec{a}$ and $\vec{b}$.

Recall a vector that is perpendicular to two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (3, 1, -4)$ and $(b_1, b_2, b_3) = (6, 5, -2)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -4 \\ 6 & 5 & -2 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(1)(-2) - (5)(-4)] - \hat{j}[(3)(-2) - (6)(-4)] + \hat{k}[(3)(5) - (6)(1)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[-2 + 20] - \hat{j}[-6 + 24] + \hat{k}[15 - 6]$$

$$\therefore \vec{a} \times \vec{b} = 18\hat{i} - 18\hat{j} + 9\hat{k}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a} \times \vec{b}|$.

$$|\vec{a} \times \vec{b}| = \sqrt{18^2 + (-18)^2 + 9^2}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{324 + 324 + 81}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{729} = 27$$

Let the unit vector in the direction of $\vec{a} \times \vec{b}$ be $\hat{p}$.

We know unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

$$\Rightarrow \hat{p} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$\Rightarrow \hat{p} = \frac{\vec{a} \times \vec{b}}{27}$$

$$\therefore \hat{p} = \frac{1}{27}(18\hat{i} - 18\hat{j} + 9\hat{k})$$

So, a vector of magnitude 3 in the direction of $\vec{a} \times \vec{b}$ is

$$3\hat{p} = 3 \times \frac{1}{27}(18\hat{i} - 18\hat{j} + 9\hat{k})$$

$\Rightarrow 3\hat{p} = \frac{1}{9}(18\hat{i} - 18\hat{j} + 9\hat{k})$

$\therefore 3\hat{p} = 2\hat{i} - 2\hat{j} + \hat{k}$

Thus, the vector of magnitude 3 that is perpendicular to both $\vec{a}$ and $\vec{b}$ is $2\hat{i} - 2\hat{j} + \hat{k}$.

8 A. Question

Find the area of the parallelogram determined by the vectors :

$2\hat{i}$ and $3\hat{j}$

Answer

Given two vectors $2\hat{i}$ and $3\hat{j}$ are sides of a parallelogram

Let $\vec{a} = 2\hat{i}$ and $\vec{b} = 3\hat{j}$

Recall the area of the parallelogram whose adjacent sides are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $|\vec{a} \times \vec{b}|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (2, 0, 0)$ and $(b_1, b_2, b_3) = (0, 3, 0)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix}$$

$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(0)(0) - (3)(0)] - \hat{j}[(2)(0) - (0)(0)] + \hat{k}[(2)(3) - (0)(0)]$

$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[0 - 0] - \hat{j}[0 - 0] + \hat{k}[6 - 0]$

$\therefore \vec{a} \times \vec{b} = 6\hat{k}$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$

Now, we find $|\vec{a} \times \vec{b}|$.

$|\vec{a} \times \vec{b}| = \sqrt{0^2 + 0^2 + 6^2}$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{6^2}$

$\therefore |\vec{a} \times \vec{b}| = 6$

Thus, area of the parallelogram is 6 square units.

8 B. Question

Find the area of the parallelogram determined by the vectors :

$2\hat{i} + \hat{j} + 3\hat{k}$ and $\hat{i} - \hat{j}$

Answer

Given two vectors $2\hat{i} + \hat{j} + 3\hat{k}$ and $\hat{i} - \hat{j}$ are sides of a parallelogram

Let $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} - \hat{j}$

Recall the area of the parallelogram whose adjacent sides are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $|\vec{a} \times \vec{b}|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (2, 1, 3)$ and $(b_1, b_2, b_3) = (1, -1, 0)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(1)(0) - (-1)(3)] - \hat{j}[(2)(0) - (1)(3)] + \hat{k}[(2)(-1) - (1)(1)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[0 + 3] - \hat{j}[0 - 3] + \hat{k}[-2 - 1]$$

$$\therefore \vec{a} \times \vec{b} = 3\hat{i} + 3\hat{j} - 3\hat{k}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a} \times \vec{b}|$.

$$|\vec{a} \times \vec{b}| = \sqrt{3^2 + 3^2 + (-3)^2}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{9 + 9 + 9}$$

$$\therefore |\vec{a} \times \vec{b}| = 3\sqrt{3}$$

Thus, area of the parallelogram is $3\sqrt{3}$ square units.

8 C. Question

Find the area of the parallelogram determined by the vectors :

$3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$

Answer

Given two vectors $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$ are sides of a parallelogram

Let $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$

Recall the area of the parallelogram whose adjacent sides are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $|\vec{a} \times \vec{b}|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (3, 1, -2)$ and $(b_1, b_2, b_3) = (1, -3, 4)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(1)(4) - (-3)(-2)] - \hat{j}[(3)(4) - (1)(-2)] + \hat{k}[(3)(-3) - (1)(1)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[4 - 6] - \hat{j}[12 + 2] + \hat{k}[-9 - 1]$$

$$\therefore \vec{a} \times \vec{b} = -2\hat{i} - 14\hat{j} - 10\hat{k}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$|x\hat{i}+y\hat{j}+z\hat{k}|=\sqrt{x^2+y^2+z^2}$

Now, we find $|\vec{a}\times\vec{b}|$.

$|\vec{a}\times\vec{b}|=\sqrt{(-2)^2+(-14)^2+(-10)^2}$

$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{4+196+100}$

$\therefore |\vec{a}\times\vec{b}|=10\sqrt{3}$

Thus, area of the parallelogram is $10\sqrt{3}$ square units.

8 D. Question

Find the area of the parallelogram determined by the vectors :

$\hat{i}-3\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$

Answer

Given two vectors $\hat{i}-3\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$ are sides of a parallelogram

Let $\vec{a}=\hat{i}-3\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}+\hat{k}$

Recall the area of the parallelogram whose adjacent sides are given by the two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ is $|\vec{a}\times\vec{b}|$ where

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, -3, 1)$ and $(b_1, b_2, b_3) = (1, 1, 1)$

$$\Rightarrow \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 1 & 1 & 1\end{vmatrix}$$

$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[(-3)(1)-(1)(1)]-\hat{j}[(1)(1)-(1)(1)]+\hat{k}[(1)(1)-(1)(-3)]$

$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[-3-1]-\hat{j}[1-1]+\hat{k}[1+3]$

$\therefore \vec{a}\times\vec{b}=-4\hat{i}+4\hat{k}$

Recall the magnitude of the vector $x\hat{i}+y\hat{j}+z\hat{k}$ is

$|x\hat{i}+y\hat{j}+z\hat{k}|=\sqrt{x^2+y^2+z^2}$

Now, we find $|\vec{a}\times\vec{b}|$.

$|\vec{a}\times\vec{b}|=\sqrt{(-4)^2+0^2+4^2}$

$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{16+16}$

$\therefore |\vec{a}\times\vec{b}|=4\sqrt{2}$

Thus, the area of the parallelogram is $4\sqrt{2}$ square units.

9 A. Question

Find the area of the parallelogram whose diagonals are :

$4\hat{i}-\hat{j}-3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$

Answer

Given two diagonals of a parallelogram are $4\hat{i}-\hat{j}-3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$

Let $\vec{a}=4\hat{i}-\hat{j}-3\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$

Recall the area of the parallelogram whose diagonals are given by the two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$ where

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (4, -1, -3)$ and $(b_1, b_2, b_3) = (-2, 1, -2)$

$$\Rightarrow \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & -3\\ -2 & 1 & -2\end{vmatrix}$$

$$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[(-1)(-2)-(1)(-3)]-\hat{j}[(4)(-2)-(-2)(-3)] + \hat{k}[(4)(1)-(-2)(-1)]$$

$$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[2+3]-\hat{j}[-8-6]+\hat{k}[4-2]$$

$$\therefore \vec{a}\times\vec{b}=5\hat{i}+14\hat{j}+2\hat{k}$$

Recall the magnitude of the vector $x\hat{i}+y\hat{j}+z\hat{k}$ is

$$|x\hat{i}+y\hat{j}+z\hat{k}|=\sqrt{x^2+y^2+z^2}$$

Now, we find $|\vec{a}\times\vec{b}|$.

$$|\vec{a}\times\vec{b}|=\sqrt{5^2+14^2+2^2}$$

$$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{25+196+4}$$

$$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{225}=15$$

$$\therefore \frac{|\vec{a}\times\vec{b}|}{2}=\frac{15}{2}=7.5$$

Thus, the area of the parallelogram is 7.5 square units.

9 B. Question

Find the area of the parallelogram whose diagonals are :

$2\hat{i}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$

Answer

Given two diagonals of a parallelogram are $2\hat{i}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$

Let $\vec{a}=2\hat{i}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}+\hat{k}$

Recall the area of the parallelogram whose diagonals are given by the two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$ where

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (2, 0, 1)$ and $(b_1, b_2, b_3) = (1, 1, 1)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(0)(1) - (1)(1)] - \hat{j}[(2)(1) - (1)(1)] + \hat{k}[(2)(1) - (1)(0)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[0 - 1] - \hat{j}[2 - 1] + \hat{k}[2 - 0]$$

$$\therefore \vec{a} \times \vec{b} = -\hat{i} - \hat{j} + 2\hat{k}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a} \times \vec{b}|$.

$$|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + (-1)^2 + 2^2}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{1 + 1 + 4}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{6}$$

$$\therefore \frac{|\vec{a} \times \vec{b}|}{2} = \frac{\sqrt{6}}{2}$$

Thus, the area of the parallelogram is $\frac{\sqrt{6}}{2}$ square units.

9 C. Question

Find the area of the parallelogram whose diagonals are :

$3\hat{i} + 4\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$

Answer

Given two diagonals of a parallelogram are $3\hat{i} + 4\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$

Let $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

Recall the area of the parallelogram whose diagonals are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (3, 4, 0)$ and $(b_1, b_2, b_3) = (1, 1, 1)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(4)(1) - (1)(0)] - \hat{j}[(3)(1) - (1)(0)] + \hat{k}[(3)(1) - (1)(4)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[4 - 0] - \hat{j}[3 - 0] + \hat{k}[3 - 4]$$

$$\therefore \vec{a} \times \vec{b} = 4\hat{i} - 3\hat{j} - \hat{k}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a} \times \vec{b}|$.

$|\vec{a}\times\vec{b}| = \sqrt{4^2+(-3)^2+(-1)^2}$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{16+9+1}$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{26}$

$\therefore \frac{|\vec{a}\times\vec{b}|}{2} = \frac{\sqrt{26}}{2}$

Thus, the area of the parallelogram is $\frac{\sqrt{26}}{2}$ square units.

9 D. Question

Find the area of the parallelogram whose diagonals are :

$2\hat{i}+3\hat{j}+6\hat{k}$ and $3\hat{i}-6\hat{j}+2\hat{k}$

Answer

Given two diagonals of a parallelogram are $2\hat{i}+3\hat{j}+6\hat{k}$ and $3\hat{i}-6\hat{j}+2\hat{k}$

Let $\vec{a} = 2\hat{i}+3\hat{j}+6\hat{k}$ and $\vec{b} = 3\hat{i}-6\hat{j}+2\hat{k}$

Recall the area of the parallelogram whose diagonals are given by the two vectors $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$ where

$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

Here, we have $(a_1, a_2, a_3) = (2, 3, 6)$ and $(b_1, b_2, b_3) = (3, -6, 2)$

$\Rightarrow \vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 6\\ 3 & -6 & 2\end{vmatrix}$

$\Rightarrow \vec{a}\times\vec{b} = \hat{i}[(3)(2)-(-6)(6)] - \hat{j}[(2)(2)-(3)(6)] + \hat{k}[(2)(-6)-(3)(3)]$

$\Rightarrow \vec{a}\times\vec{b} = \hat{i}[6+36] - \hat{j}[4-18] + \hat{k}[-12-9]$

$\therefore \vec{a}\times\vec{b} = 42\hat{i}+14\hat{j}-21\hat{k}$

Recall the magnitude of the vector $x\hat{i}+y\hat{j}+z\hat{k}$ is

$|x\hat{i}+y\hat{j}+z\hat{k}| = \sqrt{x^2+y^2+z^2}$

Now, we find $|\vec{a}\times\vec{b}|$.

$|\vec{a}\times\vec{b}| = \sqrt{42^2+14^2+(-21)^2}$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{1764+196+441}$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{2401} = 49$

$\therefore \frac{|\vec{a}\times\vec{b}|}{2} = \frac{49}{2} = 24.5$

Thus, area of the parallelogram is 24.5 square units.

10. Question

If $\vec{a} = 2\hat{i}+5\hat{j}-7\hat{k},\ \vec{b} = -3\hat{i}+4\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-2\hat{j}-3\hat{k}$, compute $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$ and verify that these are not equal.

Answer

Given $\vec{a} = 2\hat{i} + 5\hat{j} - 7\hat{k}$, $\vec{b} = -3\hat{i} + 4\hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} - 3\hat{k}$

We need to find $(\vec{a} \times \vec{b}) \times \vec{c}$.

First, we will find $\vec{a} \times \vec{b}$.

Recall the cross product of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (2, 5, -7)$ and $(b_1, b_2, b_3) = (-3, 4, 1)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 5 & -7 \\ -3 & 4 & 1 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(5)(1) - (4)(-7)] - \hat{j}[(2)(1) - (-3)(-7)] + \hat{k}[(2)(4) - (-3)(5)]$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[5 + 28] - \hat{j}[2 - 21] + \hat{k}[8 + 15]$$

$$\therefore \vec{a} \times \vec{b} = 33\hat{i} + 19\hat{j} + 23\hat{k}$$

Now, we will find $(\vec{a} \times \vec{b}) \times \vec{c}$.

Using the formula for cross product as above, we have

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 33 & 19 & 23 \\ 1 & -2 & -3 \end{vmatrix}$$

$$\Rightarrow (\vec{a} \times \vec{b}) \times \vec{c} = \hat{i}[(19)(-3) - (-2)(23)] - \hat{j}[(33)(-3) - (1)(23)] + \hat{k}[(33)(-2) - (1)(19)]$$

$$\Rightarrow (\vec{a} \times \vec{b}) \times \vec{c} = \hat{i}[-57 + 46] - \hat{j}[-99 - 23] + \hat{k}[-66 - 19]$$

$$\therefore (\vec{a} \times \vec{b}) \times \vec{c} = -11\hat{i} + 122\hat{j} - 85\hat{k}$$

Now, we need to find $\vec{a} \times (\vec{b} \times \vec{c})$.

First, we will find $\vec{b} \times \vec{c}$.

Using the formula for cross product, we have

$$\Rightarrow \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 4 & 1 \\ 1 & -2 & -3 \end{vmatrix}$$

$$\Rightarrow \vec{b} \times \vec{c} = \hat{i}[(4)(-3) - (-2)(1)] - \hat{j}[(-3)(-3) - (1)(1)] + \hat{k}[(-3)(-2) - (1)(4)]$$

$$\Rightarrow \vec{b} \times \vec{c} = \hat{i}[-12 + 2] - \hat{j}[9 - 1] + \hat{k}[6 - 4]$$

$$\therefore \vec{b} \times \vec{c} = -10\hat{i} - 8\hat{j} + 2\hat{k}$$

Now, we will find $\vec{a} \times (\vec{b} \times \vec{c})$.

Using the formula for the cross product as above, we have

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 5 & -7 \\ -10 & -8 & 2 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times (\vec{b} \times \vec{c}) = \hat{i}[(5)(2) - (-8)(-7)] - \hat{j}[(2)(2) - (-10)(-7)] + \hat{k}[(2)(-8) - (-10)(5)]$$

$$\Rightarrow \vec{a} \times (\vec{b} \times \vec{c}) = \hat{i}[10 - 56] - \hat{j}[4 - 70] + \hat{k}[-16 + 50]$$

$$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = -46\hat{i} + 66\hat{j} + 34\hat{k}$$

So, we found $(\vec{a} \times \vec{b}) \times \vec{c} = -11\hat{i} + 122\hat{j} - 85\hat{k}$ and

$\vec{a} \times (\vec{b} \times \vec{c}) = -46\hat{i} + 66\hat{j} + 34\hat{k}$

Therefore, we have $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$.

11. Question

If $|\vec{a}| = 2, |\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, find $\vec{a}.\vec{b}$.

Answer

Given $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$

We know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\, \hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta||\hat{n}|$$

$\hat{n}$ is a unit vector $\Rightarrow |\hat{n}| = 1$

$\Rightarrow 8 = 2 \times 5 \times \sin\theta \times 1$

$\Rightarrow 10 \sin\theta = 8$

$$\therefore \sin\theta = \frac{4}{5}$$

We also have the dot product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

But, we have $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\sqrt{1 - \sin^2\theta}$$

$$\Rightarrow \vec{a}.\vec{b} = 2 \times 5 \times \sqrt{1 - \left(\frac{4}{5}\right)^2}$$

$$\Rightarrow \vec{a}.\vec{b} = 10 \times \sqrt{1 - \frac{16}{25}}$$

$\Rightarrow \vec{a}.\vec{b} = 10 \times \sqrt{\frac{9}{25}}$

$\therefore \vec{a}.\vec{b} = 10 \times \frac{3}{5} = 6$

Thus, $\vec{a}.\vec{b} = 6$

12. Question

Given $\vec{a} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right), \vec{b} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right),\ \vec{c} = \frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right), \hat{i}, \hat{j}, \hat{k}$ being a right handed orthogonal system of unit vectors in space, show that $\vec{a}, \vec{b}, \vec{c}$ is also another system.

Answer

To show that $\vec{a}, \vec{b}, \vec{c}$ is a right handed orthogonal system of unit vectors, we need to prove the following –

(a) $|\vec{a}| = |b| = |\vec{c}| = 1$

(b) $\vec{a} \times \vec{b} = \vec{c}$

(c) $\vec{b} \times \vec{c} = \vec{a}$

(d) $\vec{c} \times \vec{a} = \vec{b}$

Let us consider each of these one at a time.

(a) Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$

First, we will find $|\vec{a}|$.

$|\vec{a}| = \frac{1}{7}\sqrt{2^2 + 3^2 + 6^2}$

$\Rightarrow |\vec{a}| = \frac{1}{7}\sqrt{4 + 9 + 36}$

$\Rightarrow |\vec{a}| = \frac{1}{7}\sqrt{49} = \frac{1}{7} \times 7$

$\therefore |\vec{a}| = 1$

Now, we will find $|\vec{a}|$.

$|\vec{b}| = \frac{1}{7}\sqrt{3^2 + (-6)^2 + 2^2}$

$\Rightarrow |\vec{b}| = \frac{1}{7}\sqrt{9 + 36 + 4}$

$\Rightarrow |\vec{b}| = \frac{1}{7}\sqrt{49} = \frac{1}{7} \times 7$

$\therefore |\vec{b}| = 1$

Finally, we will find $|\vec{c}|$.

$|\vec{c}| = \frac{1}{7}\sqrt{6^2 + 2^2 + (-3)^2}$

$\Rightarrow |\vec{c}| = \frac{1}{7}\sqrt{36 + 4 + 9}$

$\Rightarrow |\vec{c}| = \frac{1}{7}\sqrt{49} = \frac{1}{7} \times 7$

$\therefore |\vec{c}| = 1$

Hence, we have $|\vec{a}| = |b| = |\vec{c}| = 1$

(b) Now, we will evaluate the vector $\vec{a} \times \vec{b}$

Recall the cross product of two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Taking the scalar $\frac{1}{7}$ common, here, we have $(a_1, a_2, a_3) = (2, 3, 6)$ and $(b_1, b_2, b_3) = (3, -6, 2)$

$$\Rightarrow \vec{a} \times \vec{b} = \frac{1}{7} \times \frac{1}{7}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \frac{1}{49}\left(\hat{\imath}[(3)(2) - (-6)(6)] - \hat{\jmath}[(2)(2) - (3)(6)] + \hat{k}[(2)(-6) - (3)(3)]\right)$$

$$\Rightarrow \vec{a} \times \vec{b} = \frac{1}{49}\left(\hat{\imath}[6 + 36] - \hat{\jmath}[4 - 18] + \hat{k}[-12 - 9]\right)$$

$$\Rightarrow \vec{a} \times \vec{b} = \frac{1}{49}\left(42\hat{\imath} + 14\hat{\jmath} - 21\hat{k}\right)$$

$$\therefore \vec{a} \times \vec{b} = \frac{1}{7}\left(6\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) = \vec{c}$$

Hence, we have $\vec{a} \times \vec{b} = \vec{c}$.

(c) Now, we will evaluate the vector $\vec{b} \times \vec{c}$

Taking the scalar $\frac{1}{7}$ common, here, we have $(a_1, a_2, a_3) = (3, -6, 2)$ and $(b_1, b_2, b_3) = (6, 2, -3)$

$$\Rightarrow \vec{b} \times \vec{c} = \frac{1}{7} \times \frac{1}{7}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -6 & 2 \\ 6 & 2 & -3 \end{vmatrix}$$

$$\Rightarrow \vec{b} \times \vec{c} = \frac{1}{49}\left(\hat{\imath}[(-6)(-3) - (2)(2)] - \hat{\jmath}[(3)(-3) - (6)(2)] + \hat{k}[(3)(2) - (6)(-6)]\right)$$

$$\Rightarrow \vec{b} \times \vec{c} = \frac{1}{49}\left(\hat{\imath}[18 - 4] - \hat{\jmath}[-9 - 12] + \hat{k}[6 + 36]\right)$$

$$\Rightarrow \vec{b} \times \vec{c} = \frac{1}{49}\left(14\hat{\imath} + 21\hat{\jmath} + 42\hat{k}\right)$$

$$\therefore \vec{b} \times \vec{c} = \frac{1}{7}\left(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}\right) = \vec{a}$$

Hence, we have $\vec{b} \times \vec{c} = \vec{a}$.

(d) Now, we will evaluate the vector $\vec{c} \times \vec{a}$

Taking the scalar $\frac{1}{7}$ common, here, we have $(a_1, a_2, a_3) = (6, 2, -3)$ and $(b_1, b_2, b_3) = (2, 3, 6)$

$$\Rightarrow \vec{c} \times \vec{a} = \frac{1}{7} \times \frac{1}{7} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 2 & -3 \\ 2 & 3 & 6 \end{vmatrix}$$

$$\Rightarrow \vec{c} \times \vec{a} = \frac{1}{49}\left(\hat{i}[(2)(6) - (3)(-3)] - \hat{j}[(6)(6) - (2)(-3)] + \hat{k}[(6)(3) - (2)(2)]\right)$$

$$\Rightarrow \vec{c} \times \vec{a} = \frac{1}{49}\left(\hat{i}[12 + 9] - \hat{j}[36 + 6] + \hat{k}[18 - 4]\right)$$

$$\Rightarrow \vec{c} \times \vec{a} = \frac{1}{49}\left(21\hat{i} - 42\hat{j} + 14\hat{k}\right)$$

$$\therefore \vec{c} \times \vec{a} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right) = \vec{b}$$

Hence, we have $\vec{c} \times \vec{a} = \vec{b}$.

Thus, $\vec{a}, \vec{b}, \vec{c}$ is also another right handed orthogonal system of unit vectors.

13. Question

If $|\vec{a}| = 13, |\vec{b}| = 5$ and $\vec{a} \cdot \vec{b} = 60,$ then find $|\vec{a} \times \vec{b}|$.

Answer

Given $|\vec{a}| = 2, |\vec{b}| = 5$ and $\vec{a}.\vec{b} = 60$

We know the dot product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow 60 = 13 \times 5 \times \cos\theta$$

⇒ 65 cos θ = 60

$$\therefore \cos\theta = \frac{12}{13}$$

We also know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta||\hat{n}|$$

But, we have $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sqrt{1 - \cos^2\theta}\,|\hat{n}|$$

$\hat{n}$ is a unit vector $\Rightarrow |\hat{n}| = 1$

$$\Rightarrow |\vec{a} \times \vec{b}| = 13 \times 5 \times \sqrt{1 - \left(\frac{12}{13}\right)^2} \times 1$$

$$\Rightarrow |\vec{a} \times \vec{b}| = 13 \times 5 \times \sqrt{1 - \frac{144}{169}}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = 13 \times 5 \times \sqrt{\frac{25}{169}}$$

$\therefore |\vec{a} \times \vec{b}| = 13 \times 5 \times \frac{5}{13} = 25$

Thus, $|\vec{a} \times \vec{b}| = 25$

14. Question

Find the angle between two vectors $\vec{a}$ and $\vec{b}$, if $|\vec{a} \times \vec{b}| = \vec{a}.\vec{b}.$

Answer

Given $|\vec{a} \times \vec{b}| = \vec{a}.\vec{b}$.

Let the angle between vectors $\vec{a}$ and $\vec{b}$ be θ.

We know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta||\hat{n}|$

$\hat{n}$ is a unit vector $\Rightarrow |\hat{n}| = 1$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta| \times 1$

$\therefore |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

We also have the dot product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

But, it is given that $|\vec{a} \times \vec{b}| = \vec{a}.\vec{b}$

$\Rightarrow |\vec{a}||\vec{b}|\sin\theta = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \sin\theta = \cos\theta$

$\Rightarrow \tan\theta = 1$

$\therefore \theta = \frac{\pi}{4}$

Thus, the angle between two vectors is $\frac{\pi}{4}$.

15. Question

If $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \neq \vec{0}$, then show that $\vec{a} + \vec{c} = m\vec{b}$, where m is any scalar.

Answer

Given $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \neq \vec{0}$.

$\Rightarrow \vec{a} \times \vec{b} - \vec{b} \times \vec{c} = \vec{0}$

We have $\vec{b} \times \vec{c} = -(\vec{c} \times \vec{b})$

$\Rightarrow \vec{a} \times \vec{b} - [-(\vec{c} \times \vec{b})] = \vec{0}$

$\Rightarrow \vec{a} \times \vec{b} + \vec{c} \times \vec{b} = \vec{0}$

Using distributive property of vectors, we have

$(\vec{a}+\vec{c})\times\vec{b}=\vec{0}$

We know that if the cross product of two vectors is the null vector, then the vectors are parallel.

Here, $(\vec{a}+\vec{c})\times\vec{b}=\vec{0}$

So, vector $(\vec{a}+\vec{c})$ is parallel to $\vec{b}$.

Thus, $\vec{a}+\vec{c}=m\vec{b}$ for some scalar m.

16. Question

If $|\vec{a}|=2, |\vec{b}|=7$ and $\vec{a}\times\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$.

Answer

Given $|\vec{a}|=2$, $|\vec{b}|=7$ and $\vec{a}\times\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$

Let the angle between vectors $\vec{a}$ and $\vec{b}$ be θ.

We know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$\Rightarrow |\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}||\sin\theta||\hat{n}|$

$\hat{n}$ is a unit vector $\Rightarrow |\hat{n}|=1$

$\Rightarrow |\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}||\sin\theta|\times 1$

$\Rightarrow |\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$

Recall the magnitude of the vector $x\hat{i}+y\hat{j}+z\hat{k}$ is

$|x\hat{i}+y\hat{j}+z\hat{k}|=\sqrt{x^2+y^2+z^2}$

$\Rightarrow \sqrt{3^2+2^2+6^2}=2\times 7\times\sin\theta$

$\Rightarrow \sqrt{9+4+36}=14\sin\theta$

$\Rightarrow \sqrt{49}=14\sin\theta$

$\Rightarrow 14\sin\theta=7$

$\Rightarrow \sin\theta=\frac{7}{14}=\frac{1}{2}$

$\therefore \theta=\frac{\pi}{6}$

Thus, the angle between two vectors is $\frac{\pi}{6}$.

17. Question

What inference can you draw if $\vec{a}\times\vec{b}=\vec{0}$ and $\vec{a}.\vec{b}=0$.

Answer

Given $\vec{a}\times\vec{b}=\vec{0}$ and $\vec{a}.\vec{b}=0$.

To draw inferences from this, we shall analyze these two equations one at a time.

First, let us consider $\vec{a} \times \vec{b} = \vec{0}$.

We know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

So, if $\vec{a} \times \vec{b} = \vec{0}$, we have at least one of the following true –

(a) $|\vec{a}| = 0$

(b) $|\vec{b}| = 0$

(c) $|\vec{a}| = 0$ and $|\vec{b}| = 0$

(d) $\vec{a}$ is parallel to $\vec{b}$

Now, let us consider $\vec{a}.\vec{b} = 0$.

We have the dot product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

So, if $\vec{a}.\vec{b} = 0$, we have at least one of the following true –

(a) $|\vec{a}| = 0$

(b) $|\vec{b}| = 0$

(c) $|\vec{a}| = 0$ and $|\vec{b}| = 0$

(d) $\vec{a}$ is perpendicular to $\vec{b}$

Given both these conditions are true.

Hence, the possibility (d) cannot be true as $\vec{a}$ can't be both parallel and perpendicular to $\vec{b}$ at the same time.

Thus, either one or both of $\vec{a}$ and $\vec{b}$ are zero vectors if we have $\vec{a} \times \vec{b} = \vec{0}$ as well as $\vec{a}.\vec{b} = 0$.

18. Question

If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a} \times \vec{b} = \vec{c}, \vec{b} \times \vec{c} = \vec{a}, \vec{c} \times \vec{a} = \vec{b}$. Show that $\vec{a}, \vec{b}, \vec{c}$ form an orthonormal right handed triad of unit vectors.

Answer

Given $\vec{a} \times \vec{b} = \vec{c}$, $\vec{b} \times \vec{c} = \vec{a}$ and $\vec{c} \times \vec{a} = \vec{b}$.

Considering the first equation, $\vec{c}$ is the cross product of the vectors $\vec{a}$ and $\vec{b}$.

By the definition of the cross product of two vectors, we have $\vec{c}$ perpendicular to both $\vec{a}$ and $\vec{b}$.

Similarly, considering the second equation, we have $\vec{a}$ perpendicular to both $\vec{b}$ and $\vec{c}$.

Once again, considering the third equation, we have $\vec{b}$ perpendicular to both $\vec{c}$ and $\vec{a}$.

From the above three statements, we can observe that the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular.

It is also said that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three unit vectors.

Thus, $\vec{a}, \vec{b}, \vec{c}$ form an orthonormal right handed triad of unit vectors.

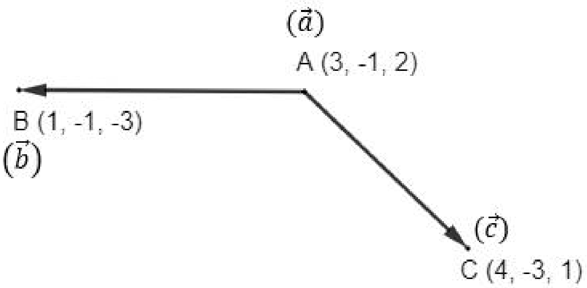
19. Question

Find a unit vector perpendicular to the plane ABC, where the coordinates of A, B and C are A(3, –1, 2), B(1, –1, –3) and C(4, –3, 1).

Answer

Given points A(3, –1, 2), B(1, –1, –3) and C(4, –3, 1)

Let position vectors of the points A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.



We know position vector of a point (x, y, z) is given by $x\hat{i} + y\hat{j} + z\hat{k}$, where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along X, Y and Z directions.

$\Rightarrow \vec{a} = (3)\hat{i} + (-1)\hat{j} + (2)\hat{k}$

$\therefore \vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$

Similarly, we have $\vec{b} = \hat{i} - \hat{j} - 3\hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + \hat{k}$

Plane ABC contains the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

So, a vector perpendicular to this plane is also perpendicular to both of these vectors.

Recall the vector $\overrightarrow{AB}$ is given by

$\overrightarrow{AB} = \text{position vector of B} - \text{position vector of A}$

$\Rightarrow \overrightarrow{AB} = \vec{b} - \vec{a}$

$\Rightarrow \overrightarrow{AB} = (\hat{i} - \hat{j} - 3\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k})$

$\Rightarrow \overrightarrow{AB} = (1 - 3)\hat{i} + (-1 + 1)\hat{j} + (-3 - 2)\hat{k}$

$\therefore \overrightarrow{AB} = -2\hat{i} - 5\hat{k}$

Similarly, the vector $\overrightarrow{AC}$ is given by

$\overrightarrow{AC} = \text{position vector of C} - \text{position vector of A}$

$\Rightarrow \overrightarrow{AC} = \vec{c} - \vec{a}$

$\Rightarrow \overrightarrow{AC} = (4\hat{i} - 3\hat{j} + 1\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k})$

$\Rightarrow \overrightarrow{AC} = (4 - 3)\hat{i} + (-3 + 1)\hat{j} + (1 - 2)\hat{k}$

$\therefore \overrightarrow{AC} = \hat{i} - 2\hat{j} - \hat{k}$

We need to find a unit vector perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Recall a vector that is perpendicular to two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (-2, 0, -5)$ and $(b_1, b_2, b_3) = (1, -2, -1)$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix}$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \hat{i}[(0)(-1) - (-2)(-5)] - \hat{j}[(-2)(-1) - (1)(-5)] + \hat{k}[(-2)(-2) - (1)(0)]$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \hat{i}[0 - 10] - \hat{j}[2 + 5] + \hat{k}[4 - 0]$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = -10\hat{i} - 7\hat{j} + 4\hat{k}$$

Let the unit vector in the direction of $\overrightarrow{AB} \times \overrightarrow{AC}$ be $\hat{p}$.

We know unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

$$\Rightarrow \hat{p} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\overrightarrow{AB} \times \overrightarrow{AC}|$.

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-10)^2 + (-7)^2 + 4^2}$$

$$\Rightarrow |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{100 + 49 + 16}$$

$$\therefore |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{165}$$

So, we have $\hat{p} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{\sqrt{165}}$

$$\Rightarrow \hat{p} = \frac{1}{\sqrt{165}}(-10\hat{i} - 7\hat{j} + 4\hat{k})$$

Thus, the required unit vector that is perpendicular to plane ABC is $\frac{1}{\sqrt{165}}(-10\hat{i} - 7\hat{j} + 4\hat{k})$.

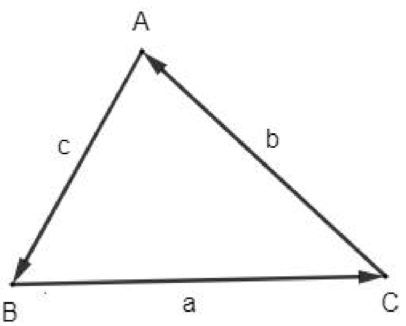
20. Question

If a, b, c are the lengths of sides, BC, CA and AB of a triangle ABC, prove that $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$ and deduce that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Answer

Given ABC is a triangle with BC = a, CA = b and AB = c.

$$\Rightarrow |\overrightarrow{BC}| = a, |\overrightarrow{CA}| = b \text{ and } |\overrightarrow{AB}| = c$$



Firstly, we need to prove $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$.

From the triangle law of vector addition, we have

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

But, we know $\overrightarrow{AC} = -\overrightarrow{CA}$

$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA}$

$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

$\therefore \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$

Let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$

$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0} ----(\mathrm{I})$

By taking cross product with $\vec{a}$, we get

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

$\Rightarrow \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}\ [\because \vec{a} \times \vec{0} = \vec{0}]$

$\Rightarrow \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}\ [\because \vec{a} \times \vec{a} = \vec{0}]$

$\Rightarrow \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$

$\Rightarrow \vec{a} \times \vec{b} = -(\vec{a} \times \vec{c})$

$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$

We know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

Here, all the vectors are coplanar. So, the unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is same as that of $\vec{b}$ and $\vec{c}$.

$\Rightarrow |\vec{a}||\vec{b}| \sin C = |\vec{c}||\vec{a}| \sin B$

$\Rightarrow |\vec{b}| \sin C = |\vec{c}| \sin B$

$\Rightarrow b \sin C = c \sin B\ [\because |\overrightarrow{CA}| = b \text{ and } |\overrightarrow{AB}| = c]$

$\therefore \frac{b}{\sin B} = \frac{c}{\sin C} ----(\mathrm{II})$

Consider equation (I) again.

We have $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

By taking cross product with $\vec{a}$, we get

$\vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times \vec{0}$

$\Rightarrow \vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}\ [\because \vec{b} \times \vec{0} = \vec{0}]$

$\Rightarrow \vec{b} \times \vec{a} + \vec{0} + \vec{b} \times \vec{c} = \vec{0}\ [\because \vec{b} \times \vec{b} = \vec{0}]$

$\Rightarrow \vec{b} \times \vec{a} + \vec{b} \times \vec{c} = \vec{0}$

$\Rightarrow \vec{b} \times \vec{c} = -(\vec{b} \times \vec{a})$

$\Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b}$

$\Rightarrow |\vec{b}||\vec{c}| \sin A = |\vec{a}||\vec{b}| \sin C$

$\Rightarrow |\vec{c}| \sin A = |\vec{a}| \sin C$

$\Rightarrow c \sin A = a \sin C \left[\because |\overrightarrow{AB}| = c \text{ and } |\overrightarrow{BC}| = a\right]$

$\therefore \frac{c}{\sin C} = \frac{a}{\sin A} - - - - -(III)$

From (II) and (III), we get $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Thus, $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$ and $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ in ΔABC.

21. Question

If $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, and $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$, then find $\vec{a} \times \vec{b}$. Verify that $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular to each other.

Answer

Given $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$

Recall the cross product of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, -2, 3)$ and $(b_1, b_2, b_3) = (2, 3, -5)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 3 & -5 \end{vmatrix}$$

$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(-2)(-5) - (3)(3)] - \hat{j}[(1)(-5) - (2)(3)] + \hat{k}[(1)(3) - (2)(-2)]$

$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[10 - 9] - \hat{j}[-5 - 6] + \hat{k}[3 + 4]$

$\therefore \vec{a} \times \vec{b} = \hat{i} + 11\hat{j} + 7\hat{k}$

We need to prove $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular to each other.

We know that two vectors are perpendicular if their dot product is zero.

So, we will evaluate $\vec{a}.(\vec{a} \times \vec{b})$.

$\vec{a}.(\vec{a} \times \vec{b}) = (\hat{i} - 2\hat{j} + 3\hat{k}).(\hat{i} + 11\hat{j} + 7\hat{k})$

$\Rightarrow \vec{a}.(\vec{a} \times \vec{b}) = \hat{i}.(\hat{i} + 11\hat{j} + 7\hat{k}) - 2\hat{j}.(\hat{i} + 11\hat{j} + 7\hat{k}) + 3\hat{k}.(\hat{i} + 11\hat{j} + 7\hat{k})$

But, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are mutually perpendicular.

$\Rightarrow \vec{a}.(\vec{a} \times \vec{b}) = \hat{i}.\hat{i} - 2\hat{j}.11\hat{j} + 3\hat{k}.7\hat{k}$

$\Rightarrow \vec{a}.(\vec{a} \times \vec{b}) = \hat{i}.\hat{i} - 22(\hat{j}.\hat{j}) + 21(\hat{k}.\hat{k})$

$\Rightarrow \vec{a}.(\vec{a} \times \vec{b}) = 1 - 22 + 21$

$\therefore \vec{a}.(\vec{a} \times \vec{b}) = 0$

Thus $\vec{a} \times \vec{b} = \hat{i} + 11\hat{j} + 7\hat{k}$ and it is perpendicular to $\vec{a}$.

22. Question

If $\bar{p}$ and $\bar{q}$ are unit vectors forming an angle of 30^0, find the area of the parallelogram having $\bar{a} = \bar{p} + 2\bar{q}$ and $\bar{b} = 2\bar{p} + \bar{q}$ as its diagonals.

Answer

Given two unit vectors $\vec{p}$ and $\vec{q}$ forming an angle of 30°.

We know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

$$\Rightarrow \vec{p} \times \vec{q} = |\vec{p}||\vec{q}|\sin 30°\,\hat{n}$$

$$\Rightarrow \vec{p} \times \vec{q} = 1 \times 1 \times \frac{1}{2} \times \hat{n}$$

$$\therefore \vec{p} \times \vec{q} = \frac{1}{2}\hat{n}$$

Given two diagonals of parallelogram $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$

Recall the area of the parallelogram whose diagonals are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$.

$$\Rightarrow \text{Area} = \frac{1}{2}|(\vec{p} + 2\vec{q}) \times (2\vec{p} + \vec{q})|$$

$$\Rightarrow \text{Area} = \frac{1}{2}|\vec{p} \times (2\vec{p} + \vec{q}) + 2\vec{q} \times (2\vec{p} + \vec{q})|$$

$$\Rightarrow \text{Area} = \frac{1}{2}|\vec{p} \times 2\vec{p} + \vec{p} \times \vec{q} + 2\vec{q} \times 2\vec{p} + 2\vec{q} \times \vec{q}|$$

We have $\vec{p} \times \vec{p} = \vec{q} \times \vec{q} = \vec{0}$

$$\Rightarrow \text{Area} = \frac{1}{2}|\vec{p} \times \vec{q} + 4(\vec{q} \times \vec{p})|$$

We have $\vec{q} \times \vec{p} = -(\vec{p} \times \vec{q})$

$$\Rightarrow \text{Area} = \frac{1}{2}|\vec{p} \times \vec{q} + 4[-(\vec{p} \times \vec{q})]|$$

$$\Rightarrow \text{Area} = \frac{1}{2}|\vec{p} \times \vec{q} - 4(\vec{p} \times \vec{q})|$$

$$\Rightarrow \text{Area} = \frac{1}{2}|-3(\vec{p} \times \vec{q})|$$

$$\Rightarrow \text{Area} = \frac{3}{2}|\vec{p} \times \vec{q}|$$

But, we found $\vec{p} \times \vec{q} = \frac{1}{2}\hat{n}$.

$$\Rightarrow \text{Area} = \frac{3}{2}\left|\frac{1}{2}\hat{n}\right|$$

$$\Rightarrow \text{Area} = \frac{3}{2} \times \frac{1}{2}|\hat{n}|$$

$\hat{n}$ is a unit vector $\Rightarrow |\hat{n}| = 1$

$\therefore$ Area $= \frac{3}{2} \times \frac{1}{2} \times 1 = \frac{3}{4}$

Thus, area of the parallelogram is $\frac{3}{4}$ square units.

23. Question

For any two vectors $\vec{a}$ and $\vec{b}$, prove that $\left|\vec{a} \times \vec{b}\right|^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix}$.

Answer

Let the angle between vectors $\vec{a}$ and $\vec{b}$ be θ.

We know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta||\hat{n}|$

$\hat{n}$ is a unit vector $\Rightarrow |\hat{n}| = 1$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta| \times 1$

$\therefore |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

Now, consider the LHS of the given expression.

$|\vec{a} \times \vec{b}|^2 = \left(|\vec{a}||\vec{b}| \sin\theta\right)^2$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 \sin^2\theta$

But, we have $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 (1 - \cos^2\theta)$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2 \cos^2\theta$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - \left(|\vec{a}||\vec{b}| \cos\theta\right)^2$

We know $\vec{a}.\vec{a} = |\vec{a}|^2$, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$ and $\vec{b}.\vec{b} = |\vec{b}|^2$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = (\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = (\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})(\vec{a}.\vec{b})$

But $\vec{a}.\vec{b} = b.\vec{a}$ as dot product is commutative

$\Rightarrow |\vec{a} \times \vec{b}|^2 = (\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{b}.\vec{a})(\vec{a}.\vec{b})$

$\therefore |\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix}$

Thus, $|\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix}$

24. Question

Define $\vec{a} \times \vec{b}$ and prove that $\left|\vec{a} \times \vec{b}\right| = \left(\vec{a} . \vec{b}\right) \tan \theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Answer

Cross Product: The vector or cross product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is defined as

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin \theta\, \hat{n}$

where θ is the angle between $\vec{a}$ and $\vec{b}$, 0≤θ≤π and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

We have $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin \theta\, \hat{n}$

$\Rightarrow \left|\vec{a} \times \vec{b}\right| = |\vec{a}||\vec{b}||\sin\theta||\hat{n}|$

$\hat{n}$ is a unit vector $\Rightarrow |\hat{n}| = 1$

$\Rightarrow \left|\vec{a} \times \vec{b}\right| = |\vec{a}||\vec{b}||\sin\theta| \times 1$

$\therefore \left|\vec{a} \times \vec{b}\right| = |\vec{a}||\vec{b}| \sin\theta$

But, we have the dot product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Now, we divide these two equations.

$$\Rightarrow \frac{\left|\vec{a} \times \vec{b}\right|}{\vec{a}.\vec{b}} = \frac{|\vec{a}||\vec{b}| \sin\theta}{|\vec{a}||\vec{b}|\cos\theta}$$

$$\Rightarrow \frac{\left|\vec{a} \times \vec{b}\right|}{\vec{a}.\vec{b}} = \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \frac{\left|\vec{a} \times \vec{b}\right|}{\vec{a}.\vec{b}} = \tan\theta$$

$\therefore \left|\vec{a} \times \vec{b}\right| = \left(\vec{a}.\vec{b}\right) \tan\theta$

Thus, $\left|\vec{a} \times \vec{b}\right| = \left(\vec{a}.\vec{b}\right) \tan\theta$

25. Question

If $\left|\vec{a}\right| = \sqrt{26}, \left|\vec{b}\right| = 7$ and $\left|\vec{a} \times \vec{b}\right| = 35$, find $\vec{a} . \vec{b}$.

Answer

Given $|\vec{a}| = \sqrt{26}$, $\left|\vec{b}\right| = 7$ and $\left|\vec{a} \times \vec{b}\right| = 35$

We know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin \theta\, \hat{n}$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$\Rightarrow \left|\vec{a} \times \vec{b}\right| = |\vec{a}||\vec{b}||\sin\theta||\hat{n}|\sqrt{26}$

$\hat{n}$ is a unit vector $\Rightarrow |\hat{n}| = 1$

$\Rightarrow 35 = \sqrt{26} \times 7 \times \sin\theta \times 1$

$\Rightarrow 35 = 7\sqrt{26} \sin\theta$

$\Rightarrow \sqrt{26} \sin\theta = 5$

$\therefore \sin\theta = \frac{5}{\sqrt{26}}$

We also have the dot product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

But, we have $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\sqrt{1-\sin^2\theta}$

$\Rightarrow \vec{a}.\vec{b} = \sqrt{26} \times 7 \times \sqrt{1-\left(\frac{5}{\sqrt{26}}\right)^2}$

$\Rightarrow \vec{a}.\vec{b} = 7\sqrt{26} \times \sqrt{1-\frac{25}{26}}$

$\Rightarrow \vec{a}.\vec{b} = 7\sqrt{26} \times \sqrt{\frac{1}{26}}$

$\therefore \vec{a}.\vec{b} = 7\sqrt{26} \times \frac{1}{\sqrt{26}} = 7$

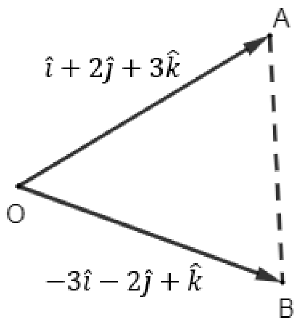
Thus, $\vec{a}.\vec{b} = 7$

26. Question

Find the area of the triangle formed by O, A, B when $\overline{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\overline{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$.

Answer

Given $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\overrightarrow{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$ are two adjacent sides of a triangle.



Recall the area of the triangle whose adjacent sides are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, 2, 3)$ and $(b_1, b_2, b_3) = (-3, -2, 1)$

$$\Rightarrow \overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix}$$

$\Rightarrow \overrightarrow{OA} \times \overrightarrow{OB} = \hat{i}[(2)(1) - (-2)(3)] - \hat{j}[(1)(1) - (-3)(3)] + \hat{k}[(1)(-2) - (-3)(2)]$

$\Rightarrow \overrightarrow{OA} \times \overrightarrow{OB} = \hat{i}[2 + 6] - \hat{j}[1 + 9] + \hat{k}[-2 + 6]$

$\therefore \overrightarrow{OA} \times \overrightarrow{OB} = 8\hat{i} - 10\hat{j} + 4\hat{k}$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$

Now, we find $|\overrightarrow{OA} \times \overrightarrow{OB}|$.

$|\overrightarrow{OA} \times \overrightarrow{OB}| = \sqrt{8^2 + (-10)^2 + 4^2}$

$\Rightarrow |\overrightarrow{OA} \times \overrightarrow{OB}| = \sqrt{64 + 100 + 16}$

$\Rightarrow |\overrightarrow{OA} \times \overrightarrow{OB}| = \sqrt{180} = 6\sqrt{5}$

$\therefore \frac{|\overrightarrow{OA} \times \overrightarrow{OB}|}{2} = \frac{6\sqrt{5}}{2} = 3\sqrt{5}$

Thus, area of the triangle is $3\sqrt{5}$ square units.

27. Question

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d} = 15$.

Answer

Given $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

We need to find a vector $\vec{d}$ perpendicular to $\vec{a}$ and $\vec{b}$ such that $\vec{c}.\vec{d} = 15$.

Recall a vector that is perpendicular to two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, 4, 2)$ and $(b_1, b_2, b_3) = (3, -2, 7)$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[(4)(7) - (-2)(2)] - \hat{j}[(1)(7) - (3)(2)] + \hat{k}[(1)(-2) - (3)(4)]$

$\Rightarrow \vec{a} \times \vec{b} = \hat{i}[28 + 4] - \hat{j}[7 - 6] + \hat{k}[-2 - 12]$

$\therefore \vec{a} \times \vec{b} = 32\hat{i} - \hat{j} - 14\hat{k}$

So, $\vec{d}$ is a vector parallel to $\vec{a} \times \vec{b}$.

Let $\vec{d} = \lambda(\vec{a} \times \vec{b})$ for some scalar λ.

$\Rightarrow \vec{d} = \lambda(32\hat{i} - \hat{j} - 14\hat{k})$

We have $\vec{c}.\vec{d} = 15$.

$\Rightarrow (2\hat{i} - \hat{j} + 4\hat{k}).[\lambda(32\hat{i} - \hat{j} - 14\hat{k})] = 15$

$\Rightarrow \lambda\left[\left(2\hat{i}-\hat{j}+4\hat{k}\right).\left(32\hat{i}-\hat{j}-14\hat{k}\right)\right]=15$

$\Rightarrow \lambda[(2)(32) + (-1)(-1) + (4)(-14)] = 15$

$\Rightarrow \lambda(64 + 1 - 56) = 15$

$\Rightarrow 9\lambda = 15$

$\therefore \lambda = \frac{15}{9} = \frac{5}{3}$

So, we have $\vec{d} = \frac{5}{3}\left(32\hat{i}-\hat{j}-14\hat{k}\right)$.

Thus, $\vec{d} = \frac{5}{3}\left(32\hat{i}-\hat{j}-14\hat{k}\right)$

28. Question

Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$.

Answer

Given $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$

We need to find the vector perpendicular to both the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

$\vec{a}+\vec{b} = \left(3\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-2\hat{k}\right)$

$\Rightarrow \vec{a}+\vec{b} = (3+1)\hat{i}+(2+2)\hat{j}+(2-2)\hat{k}$

$\therefore \vec{a}+\vec{b} = 4\hat{i}+4\hat{j}$

$\vec{a}-\vec{b} = \left(3\hat{i}+2\hat{j}+2\hat{k}\right)-\left(\hat{i}+2\hat{j}-2\hat{k}\right)$

$\Rightarrow \vec{a}-\vec{b} = (3-1)\hat{i}+(2-2)\hat{j}+(2+2)\hat{k}$

$\therefore \vec{a}-\vec{b} = 2\hat{i}+4\hat{k}$

Recall a vector that is perpendicular to two vectors $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ is

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (4, 4, 0)$ and $(b_1, b_2, b_3) = (2, 0, 4)$

$$\Rightarrow \left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$\Rightarrow \left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right) = \hat{i}[(4)(4)-(0)(0)]-\hat{j}[(4)(4)-(2)(0)]+\hat{k}[(4)(0)-(2)(4)]$$

$\Rightarrow \left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right) = \hat{i}[16-0]-\hat{j}[16-0]+\hat{k}[0-8]$

$\therefore \left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right) = 16\hat{i}-16\hat{j}-8\hat{k}$

Let the unit vector in the direction of $\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)$ be $\hat{p}$.

We know unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

$$\Rightarrow \hat{p} = \frac{(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})}{|(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})|}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})|$.

$$|(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})| = \sqrt{16^2 + (-16)^2 + (-8)^2}$$

$$\Rightarrow |(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})| = \sqrt{256 + 256 + 64}$$

$$\therefore |(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})| = \sqrt{576} = 24$$

So, we have $\hat{p} = \frac{(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})}{24}$

$$\Rightarrow \hat{p} = \frac{1}{24}(16\hat{i} - 16\hat{j} - 8\hat{k})$$

$$\therefore \hat{p} = \frac{1}{3}(2\hat{i} - 2\hat{j} - \hat{k})$$

Thus, the required unit vector that is perpendicular to both $\vec{a}$ and $\vec{b}$ is $\frac{1}{3}(2\hat{i} - 2\hat{j} - \hat{k})$.

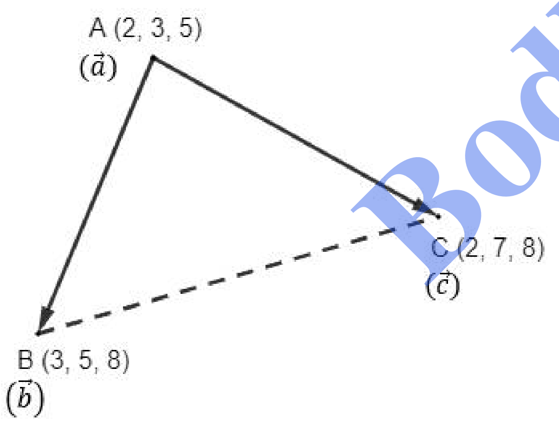
29. Question

Using vectors, find the area of the triangle with vertices A(2, 3, 5), B(3, 5, 8) and C(2, 7, 8).

Answer

Given three points A(2, 3, 5), B(3, 5, 8) and C(2, 7, 8) forming a triangle.

Let position vectors of the vertices A, B and C of ΔABC be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.



We know position vector of a point (x, y, z) is given by $x\hat{i} + y\hat{j} + z\hat{k}$, where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along X, Y and Z directions.

$$\Rightarrow \vec{a} = (2)\hat{i} + (3)\hat{j} + (5)\hat{k}$$

$$\therefore \vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$$

Similarly, we have $\vec{b} = 3\hat{i} + 5\hat{j} + 8\hat{k}$ and $\vec{c} = 2\hat{i} + 7\hat{j} + 8\hat{k}$

To find area of ΔABC, we need to find at least two sides of the triangle. So, we will find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Recall the vector $\overrightarrow{AB}$ is given by

$$\overrightarrow{AB} = \text{position vector of B} - \text{position vector of A}$$

$\Rightarrow \overrightarrow{AB} = \vec{b} - \vec{a}$

$\Rightarrow \overrightarrow{AB} = (3\hat{i} + 5\hat{j} + 8\hat{k}) - (2\hat{i} + 3\hat{j} + 5\hat{k})$

$\Rightarrow \overrightarrow{AB} = (3 - 2)\hat{i} + (5 - 3)\hat{j} + (8 - 5)\hat{k}$

$\therefore \overrightarrow{AB} = \hat{i} + 2\hat{j} + 3\hat{k}$

Similarly, the vector $\overrightarrow{AC}$ is given by

$\overrightarrow{AC} = \text{position vector of C} - \text{position vector of A}$

$\Rightarrow \overrightarrow{AC} = \vec{c} - \vec{a}$

$\Rightarrow \overrightarrow{AC} = (2\hat{i} + 7\hat{j} + 8\hat{k}) - (2\hat{i} + 3\hat{j} + 5\hat{k})$

$\Rightarrow \overrightarrow{AC} = (2 - 2)\hat{i} + (7 - 3)\hat{j} + (8 - 5)\hat{k}$

$\therefore \overrightarrow{AC} = 4\hat{j} + 3\hat{k}$

Recall the area of the triangle whose adjacent sides are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, 2, 3)$ and $(b_1, b_2, b_3) = (0, 4, 3)$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \hat{i}[(2)(3) - (4)(3)] - \hat{j}[(1)(3) - (0)(3)] + \hat{k}[(1)(4) - (0)(2)]$

$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \hat{i}[6 - 12] - \hat{j}[3 - 0] + \hat{k}[4 - 0]$

$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = -6\hat{i} - 3\hat{j} + 4\hat{k}$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$

Now, we find $|\overrightarrow{AB} \times \overrightarrow{AC}|$.

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-6)^2 + (-3)^2 + 4^2}$

$\Rightarrow |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{36 + 9 + 16}$

$\Rightarrow |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{61}$

$$\therefore \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{\sqrt{61}}{2}$$

Thus, area of the triangle is $\frac{\sqrt{61}}{2}$ square units.

30. Question

If $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}, \vec{b} = -\hat{i} + \hat{k}, \vec{c} = 2\hat{j} - \hat{k}$ are three vectors, find the area of the parallelogram having diagonals $(\vec{a} + \vec{b})$ and $(\vec{b} + \vec{c})$.

Answer

Given $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$, $\vec{b} = -\hat{\imath} + \hat{k}$ and $\vec{c} = 2\hat{\jmath} - \hat{k}$

We need to find area of the parallelogram with vectors $\vec{a} + \vec{b}$ and $\vec{b} + \vec{c}$ as diagonals.

$$\vec{a} + \vec{b} = (2\hat{\imath} - 3\hat{\jmath} + \hat{k}) + (-\hat{\imath} + \hat{k})$$

$$\Rightarrow \vec{a} + \vec{b} = (2-1)\hat{\imath} + (-3)\hat{\jmath} + (1+1)\hat{k}$$

$$\therefore \vec{a} + \vec{b} = \hat{\imath} - 3\hat{\jmath} + 2\hat{k}$$

$$\vec{b} + \vec{c} = (-\hat{\imath} + \hat{k}) + (2\hat{\jmath} - \hat{k})$$

$$\Rightarrow \vec{b} + \vec{c} = (-1)\hat{\imath} + (2)\hat{\jmath} + (1-1)\hat{k}$$

$$\therefore \vec{b} + \vec{c} = -\hat{\imath} + 2\hat{\jmath}$$

Recall the area of the parallelogram whose diagonals are given by the two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, -3, 2)$ and $(b_1, b_2, b_3) = (-1, 2, 0)$

$$\Rightarrow (\vec{a} + \vec{b}) \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix}$$

$$\Rightarrow (\vec{a} + \vec{b}) \times (\vec{b} + \vec{c}) = \hat{\imath}[(-3)(0) - (2)(2)] - \hat{\jmath}[(1)(0) - (-1)(2)] + \hat{k}[(1)(2) - (-1)(-3)]$$

$$\Rightarrow (\vec{a} + \vec{b}) \times (\vec{b} + \vec{c}) = \hat{\imath}[0 - 4] - \hat{\jmath}[0 + 2] + \hat{k}[2 - 3]$$

$$\therefore (\vec{a} + \vec{b}) \times (\vec{b} + \vec{c}) = -4\hat{\imath} - 2\hat{\jmath} - \hat{k}$$

Recall the magnitude of the vector $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is

$$|x\hat{\imath} + y\hat{\jmath} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})|$.

$$|(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})| = \sqrt{(-4)^2 + (-2)^2 + (-1)^2}$$

$$\Rightarrow |(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})| = \sqrt{16 + 4 + 1}$$

$$\Rightarrow |(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})| = \sqrt{21}$$

$$\therefore \frac{|(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})|}{2} = \frac{\sqrt{21}}{2}$$

Thus, area of the parallelogram is $\frac{\sqrt{21}}{2}$ square units.

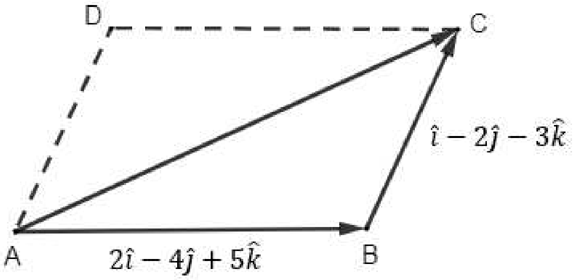
31. Question

The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to one of its diagonals. Also, find its area.

Answer

Let ABCD be a parallelogram with sides AB and AC given.

We have $\overrightarrow{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\overrightarrow{BC} = \hat{i} - 2\hat{j} - 3\hat{k}$



We need to find unit vector parallel to diagonal $\overrightarrow{AC}$.

From the triangle law of vector addition, we have

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\Rightarrow \overrightarrow{AC} = \left(2\hat{i} - 4\hat{j} + 5\hat{k}\right) + \left(\hat{i} - 2\hat{j} - 3\hat{k}\right)$

$\Rightarrow \overrightarrow{AC} = (2 + 1)\hat{i} + (-4 - 2)\hat{j} + (5 - 3)\hat{k}$

$\therefore \overrightarrow{AC} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

Let the unit vector in the direction of $\overrightarrow{AC}$ be $\hat{p}$.

We know unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

$$\Rightarrow \hat{p} = \frac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$\left|x\hat{i} + y\hat{j} + z\hat{k}\right| = \sqrt{x^2 + y^2 + z^2}$

Now, we find $\left|\overrightarrow{AC}\right|$.

$\left|\overrightarrow{AC}\right| = \sqrt{3^2 + (-6)^2 + 2^2}$

$\Rightarrow \left|\overrightarrow{AC}\right| = \sqrt{9 + 36 + 4}$

$\therefore \left|\overrightarrow{AC}\right| = \sqrt{49} = 7$

So, we have $\hat{p} = \frac{\overrightarrow{AC}}{7}$

$$\Rightarrow \hat{p} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$$

Thus, the required unit vector that is parallel to diaonal $\overrightarrow{AC}$ is $\frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$.

Now, we have to find the area of parallelogram ABCD.

Recall the area of the parallelogram whose adjacent sides are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $\left|\vec{a} \times \vec{b}\right|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (2, -4, 5)$ and $(b_1, b_2, b_3) = (1, -2, -3)$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{BC} = \hat{i}[(-4)(-3) - (-2)(5)] - \hat{j}[(2)(-3) - (1)(5)] + \hat{k}[(2)(-2) - (1)(-4)]$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{BC} = \hat{i}[12 + 10] - \hat{j}[-6 - 5] + \hat{k}[-4 + 4]$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{BC} = 22\hat{i} + 11\hat{j}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$\left|x\hat{i} + y\hat{j} + z\hat{k}\right| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$.

$$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{22^2 + 11^2 + 0^2}$$

$$\Rightarrow \left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{484 + 121}$$

$$\therefore \left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{605} = 11\sqrt{5}$$

Thus, area of the parallelogram is $11\sqrt{5}$ square units.

32. Question

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

Answer

We know $\vec{a} \times \vec{b} = \vec{0}$ if either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

To verify if the converse is true, we suppose $\vec{a} \times \vec{b} = \vec{0}$

We know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

So, if $\vec{a} \times \vec{b} = \vec{0}$, we have at least one of the following true –

(a) $|\vec{a}| = 0$

(b) $|\vec{b}| = 0$

(c) $|\vec{a}| = 0$ and $|\vec{b}| = 0$

(d) $\vec{a}$ is parallel to $\vec{b}$

The first three possibilities mean that either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or both of them are true.

However, there is another possibility that $\vec{a} \times \vec{b} = \vec{0}$ when the two vectors are parallel. Thus, the converse is not true.

We will justify this using an example.

Given $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = 2\vec{a} = 2\hat{i} + 6\hat{j} - 4\hat{k}$

Recall the cross product of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, 3, -2)$ and $(b_1, b_2, b_3) = (2, 6, -4)$

$$\Rightarrow \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 3 & -2\\ 2 & 6 & -4\end{vmatrix}$$

$$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[(3)(-4)-(6)(-2)]-\hat{j}[(1)(-4)-(2)(-2)]+\hat{k}[(1)(6)-(2)(3)]$$

$$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[-12+12]-\hat{j}[-4+4]+\hat{k}[6-6]$$

$$\therefore \vec{a}\times\vec{b}=0\hat{i}-0\hat{j}+0\hat{k}=\vec{0}$$

Hence, we have $\vec{a}\times\vec{b}=\vec{0}$ even when $\vec{a}\neq\vec{0}$ and $\vec{b}\neq\vec{0}$.

Thus, the converse of the given statement is not true.

33. Question

If $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \bar{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\bar{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then verify that $\bar{a}=\left(\bar{b}+\bar{c}\right)=\bar{a}\times\bar{b}+\bar{a}\times\bar{c}$.

Answer

Given $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

We need to verify that $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

$$\vec{b}+\vec{c}=\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)+\left(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right)$$

$$\therefore \vec{b}+\vec{c}=(b_1+c_1)\hat{i}+(b_2+c_2)\hat{j}+(b_3+c_3)\hat{k}$$

First, we will find $\vec{a}\times\left(\vec{b}+\vec{c}\right)$.

Recall the cross product of two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$\Rightarrow \vec{a}\times\left(\vec{b}+\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$$

$$\Rightarrow \vec{a}\times\left(\vec{b}+\vec{c}\right)$$
$$=\hat{i}[(a_2)(b_3+c_3)-(b_2+c_2)(a_3)]$$
$$-\hat{j}[(a_1)(b_3+c_3)-(b_1+c_1)(a_3)]$$
$$+\hat{k}[(a_1)(b_2+c_2)-(b_1+c_1)(a_2)]$$

$$\therefore \vec{a}\times\left(\vec{b}+\vec{c}\right)$$
$$=\hat{i}(a_2b_3+a_2c_3-b_2a_3-c_2a_3)-\hat{j}(a_1b_3+a_1c_3-b_1a_3-c_1a_3)$$
$$+\hat{k}(a_1b_2+a_1c_2-b_1a_2-c_1a_2)$$

Now, we will find $\vec{a}\times\vec{b}$.

We have $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

$$\Rightarrow \vec{a}\times\vec{b} = \hat{\imath}[(a_2)(b_3)-(b_2)(a_3)] - \hat{\jmath}[(a_1)(b_3)-(b_1)(a_3)] + \hat{k}[(a_1)(b_2)-(b_1)(a_2)]$$

$$\therefore \vec{a}\times\vec{b} = \hat{\imath}(a_2b_3-b_2a_3) - \hat{\jmath}(a_1b_3-b_1a_3) + \hat{k}(a_1b_2-b_1a_2)$$

Finally, we will find $\vec{a}\times\vec{c}$.

We have $\vec{a}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\Rightarrow \vec{a}\times\vec{c} = \hat{\imath}[(a_2)(c_3)-(c_2)(a_3)] - \hat{\jmath}[(a_1)(c_3)-(c_1)(a_3)] + \hat{k}[(a_1)(c_2)-(c_1)(a_2)]$$

$$\therefore \vec{a}\times\vec{c} = \hat{\imath}(a_2c_3-c_2a_3) - \hat{\jmath}(a_1c_3-c_1a_3) + \hat{k}(a_1c_2-c_1a_2)$$

So, $\vec{a}\times\vec{b}+\vec{a}\times\vec{c} = \left[\hat{\imath}(a_2b_3-b_2a_3) - \hat{\jmath}(a_1b_3-b_1a_3) + \hat{k}(a_1b_2-b_1a_2)\right] + \left[\hat{\imath}(a_2c_3-c_2a_3) - \hat{\jmath}(a_1c_3-c_1a_3) + \hat{k}(a_1c_2-c_1a_2)\right]$

$$\Rightarrow \vec{a}\times\vec{b}+\vec{a}\times\vec{c} = \hat{\imath}(a_2b_3-b_2a_3+a_2c_3-c_2a_3) - \hat{\jmath}(a_1b_3-b_1a_3+a_1c_3-c_1a_3) + \hat{k}(a_1b_2-b_1a_2+a_1c_2-c_1a_2)$$

$$\therefore \vec{a}\times\vec{b}+\vec{a}\times\vec{c} = \hat{\imath}(a_2b_3+a_2c_3-b_2a_3-c_2a_3) - \hat{\jmath}(a_1b_3+a_1c_3-b_1a_3-c_1a_3) + \hat{k}(a_1b_2+a_1c_2-b_1a_2-c_1a_2)$$

Observe that that RHS of both $\vec{a}\times(\vec{b}+\vec{c})$ and $\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ are the same.

Thus, $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

34 A. Question

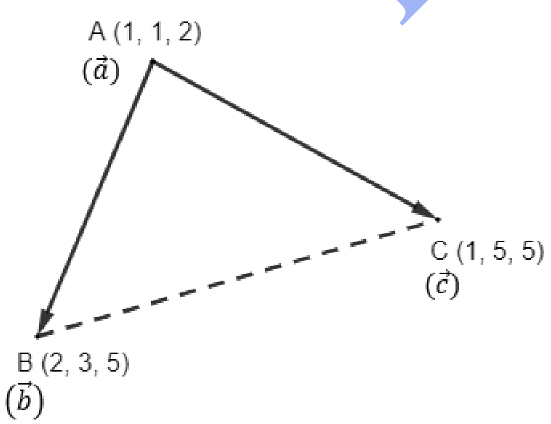
Using vectors, find the area of the triangle with vertices

A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5)

Answer

Given three points A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5) forming a triangle.

Let position vectors of the vertices A, B and C of ΔABC be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.



We know position vector of a point (x, y, z) is given by $x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vectors along X, Y and Z directions.

$$\Rightarrow \vec{a} = (1)\hat{\imath}+(1)\hat{\jmath}+(2)\hat{k}$$

$$\therefore \vec{a} = \hat{\imath}+\hat{\jmath}+2\hat{k}$$

Similarly, we have $\vec{b} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} + 5\hat{j} + 5\hat{k}$

To find area of ΔABC, we need to find at least two sides of the triangle. So, we will find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Recall the vector $\overrightarrow{AB}$ is given by

$\overrightarrow{AB} = \text{position vector of B} - \text{position vector of A}$

$\Rightarrow \overrightarrow{AB} = \vec{b} - \vec{a}$

$\Rightarrow \overrightarrow{AB} = \left(2\hat{i} + 3\hat{j} + 5\hat{k}\right) - \left(\hat{i} + \hat{j} + 2\hat{k}\right)$

$\Rightarrow \overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k}$

$\therefore \overrightarrow{AB} = \hat{i} + 2\hat{j} + 3\hat{k}$

Similarly, the vector $\overrightarrow{AC}$ is given by

$\overrightarrow{AC} = \text{position vector of C} - \text{position vector of A}$

$\Rightarrow \overrightarrow{AC} = \vec{c} - \vec{a}$

$\Rightarrow \overrightarrow{AC} = \left(\hat{i} + 5\hat{j} + 5\hat{k}\right) - \left(\hat{i} + \hat{j} + 2\hat{k}\right)$

$\Rightarrow \overrightarrow{AC} = (1-1)\hat{i} + (5-1)\hat{j} + (5-2)\hat{k}$

$\therefore \overrightarrow{AC} = 4\hat{j} + 3\hat{k}$

Recall the area of the triangle whose adjacent sides are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $\frac{1}{2}\left|\vec{a} \times \vec{b}\right|$ where

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, 2, 3)$ and $(b_1, b_2, b_3) = (0, 4, 3)$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \hat{i}[(2)(3) - (4)(3)] - \hat{j}[(1)(3) - (0)(3)] + \hat{k}[(1)(4) - (0)(2)]$

$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \hat{i}[6-12] - \hat{j}[3-0] + \hat{k}[4-0]$

$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = -6\hat{i} - 3\hat{j} + 4\hat{k}$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$\left|x\hat{i} + y\hat{j} + z\hat{k}\right| = \sqrt{x^2 + y^2 + z^2}$

Now, we find $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$.

$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2}$

$\Rightarrow \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{36 + 9 + 16}$

$\Rightarrow \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{61}$

$$\therefore \frac{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}{2} = \frac{\sqrt{61}}{2}$$

Thus, area of the triangle is $\frac{\sqrt{61}}{2}$ square units.

34 B. Question

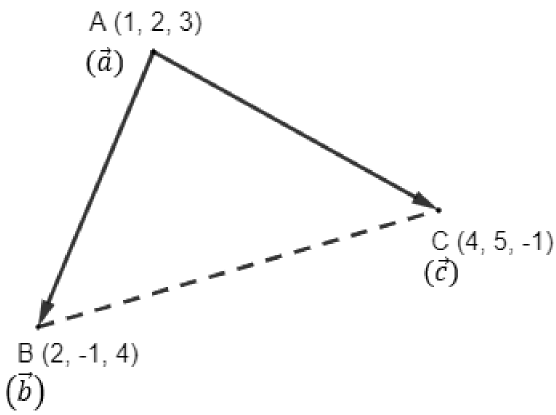
Using vectors, find the area of the triangle with vertices

A(1, 2, 3), B(2, –1, 4) and C(4, 5, –1)

Answer

Given three points A(1, 2, 3), B(2, –1, 4) and C(4, 5, –1) forming a triangle.

Let position vectors of the vertices A, B and C of ΔABC be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.



We know position vector of a point (x, y, z) is given by $x\hat{i} + y\hat{j} + z\hat{k}$, where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along X, Y and Z directions.

$$\Rightarrow \vec{a} = (1)\hat{i} + (2)\hat{j} + (3)\hat{k}$$

$$\therefore \vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Similarly, we have $\vec{b} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{c} = 4\hat{i} + 5\hat{j} - \hat{k}$

To find area of ΔABC, we need to find at least two sides of the triangle. So, we will find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Recall the vector $\overrightarrow{AB}$ is given by

$$\overrightarrow{AB} = \text{position vector of B} - \text{position vector of A}$$

$$\Rightarrow \overrightarrow{AB} = \vec{b} - \vec{a}$$

$$\Rightarrow \overrightarrow{AB} = (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\Rightarrow \overrightarrow{AB} = (2 - 1)\hat{i} + (-1 - 2)\hat{j} + (4 - 3)\hat{k}$$

$$\therefore \overrightarrow{AB} = \hat{i} - 3\hat{j} + \hat{k}$$

Similarly, the vector $\overrightarrow{AC}$ is given by

$$\overrightarrow{AC} = \text{position vector of C} - \text{position vector of A}$$

$$\Rightarrow \overrightarrow{AC} = \vec{c} - \vec{a}$$

$$\Rightarrow \overrightarrow{AC} = (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\Rightarrow \overrightarrow{AC} = (4 - 1)\hat{i} + (5 - 2)\hat{j} + (-1 - 3)\hat{k}$$

$$\therefore \overrightarrow{AC} = 3\hat{j} + 3\hat{j} - 4\hat{k}$$

Recall the area of the triangle whose adjacent sides are given by the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$ where

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, –3, 1)$ and $(b_1, b_2, b_3) = (3, 3, –4)$

$$\Rightarrow \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}$$

$$\Rightarrow \overrightarrow{AB}\times\overrightarrow{AC}=\hat{i}[(-3)(-4)-(3)(1)]-\hat{j}[(1)(-4)-(3)(1)] + \hat{k}[(1)(3)-(3)(-3)]$$

$$\Rightarrow \overrightarrow{AB}\times\overrightarrow{AC}=\hat{i}[12-3]-\hat{j}[-4-3]+\hat{k}[3+9]$$

$$\therefore \overrightarrow{AB}\times\overrightarrow{AC}=9\hat{i}+7\hat{j}+12\hat{k}$$

Recall the magnitude of the vector $x\hat{i}+y\hat{j}+z\hat{k}$ is

$$\left|x\hat{i}+y\hat{j}+z\hat{k}\right|=\sqrt{x^2+y^2+z^2}$$

Now, we find $\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$.

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{9^2+7^2+12^2}$$

$$\Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{81+49+144}$$

$$\Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{274}$$

$$\therefore \frac{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}{2}=\frac{\sqrt{274}}{2}$$

Thus, area of the triangle is $\frac{\sqrt{274}}{2}$ square units.

35. Question

Find all vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane of $\hat{i}+2\hat{j}+\hat{k}$ and $-\hat{i}+3\hat{j}+4\hat{k}$.

Answer

Given two vectors $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=-\hat{i}+3\hat{j}+4\hat{k}$

We need to find vectors of magnitude $10\sqrt{3}$ perpendicular to $\vec{a}$ and $\vec{b}$.

Recall a vector that is perpendicular to two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Here, we have $(a_1, a_2, a_3) = (1, 2, 1)$ and $(b_1, b_2, b_3) = (–1, 3, 4)$

$$\Rightarrow \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 1\\ -1 & 3 & 4\end{vmatrix}$$

$$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[(2)(4)-(3)(1)]-\hat{j}[(1)(4)-(-1)(1)]+\hat{k}[(1)(3)-(-1)(2)]$$

$$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[8-3]-\hat{j}[4+1]+\hat{k}[3+2]$$

$$\therefore \vec{a}\times\vec{b}=5\hat{i}-5\hat{j}+5\hat{k}$$

Let the unit vector in the direction of $\vec{a}\times\vec{b}$ be $\hat{p}$.

We know unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

$$\Rightarrow \hat{p} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Recall the magnitude of the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

$$|x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Now, we find $|\vec{a} \times \vec{b}|$.

$$|\vec{a} \times \vec{b}| = \sqrt{5^2 + (-5)^2 + 5^2}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{25 + 25 + 25}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{75} = 5\sqrt{3}$$

So, we have $\hat{p} = \frac{\vec{a} \times \vec{b}}{5\sqrt{3}}$

$$\Rightarrow \hat{p} = \frac{1}{5\sqrt{3}}\left(5\hat{i} - 5\hat{j} + 5\hat{k}\right)$$

$$\therefore \hat{p} = \frac{1}{\sqrt{3}}\left(\hat{i} - \hat{j} + \hat{k}\right)$$

So, a vector of magnitude $10\sqrt{3}$ in the direction of $\vec{a} \times \vec{b}$ is

$$10\sqrt{3}\hat{p} = 10\sqrt{3} \times \frac{1}{\sqrt{3}}\left(\hat{i} - \hat{j} + \hat{k}\right)$$

$$\Rightarrow 10\sqrt{3}\hat{p} = 10\left(\hat{i} - \hat{j} + \hat{k}\right)$$

$$\therefore 10\sqrt{3}\hat{p} = 10\hat{i} - 10\hat{j} + 10\hat{k}$$

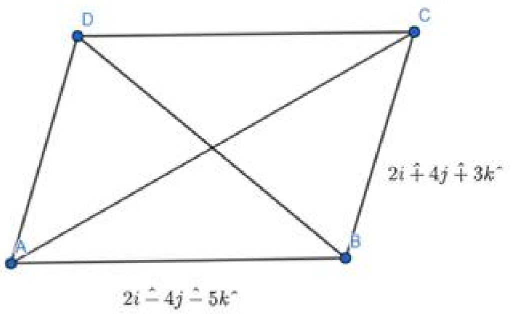
Observe that $-10\sqrt{3}\hat{p}$ is also a unit vector perpendicular to the same plane. This vector is along the direction opposite to the direction of vector $10\sqrt{3}\hat{p}$.

Thus, the vectors of magnitude $10\sqrt{3}$ that are perpendicular to plane of both $\vec{a}$ and $\vec{b}$ are $\pm\left(10\hat{i} - 10\hat{j} + 10\hat{k}\right)$.

36. Question

The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} - 5\hat{k}$ and $2\hat{i} + 2\hat{j} + 3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram.

Answer



We need to find a unit vector parallel to $\overline{AC}$.

Now from the Parallel law of vector Addition, we know that,

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Therefore,

$\overrightarrow{AC} = 2\,\hat{\imath} - 4\,\hat{\jmath} - 5\,\hat{k} + (2\,\hat{\imath} + 3\,\hat{\jmath} + 3\,\hat{k})$

$\overrightarrow{AC} = 4\,\hat{\imath} - \hat{\jmath} - 2\,\hat{k}$

Now we need to find the unit vector parallel to $\overrightarrow{AC}$

Any unit vector is given by,

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|}$$

Therefore, $\widehat{AC} = \frac{\overrightarrow{AC}}{|\overrightarrow{AC}|}$

$|\overrightarrow{AC}| = \sqrt{(4)^2 + (1)^2 + (2)^2}$

$|\overrightarrow{AC}| = \sqrt{21}$

$$\widehat{AC} = \frac{4\,\hat{\imath} - \hat{\jmath} - 2\,\hat{k}}{\sqrt{21}}$$

Now, we need to find Area of parallelogram. From the figure above it can be easily found by the cross product of adjacent sides.

Therefore, Area of Parallelogram = $|\overrightarrow{AB} \times \overrightarrow{BC}|$

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here, we have,

(a_1, a_2, a_3) = (2, -4, -5) and (b_1, b_2, b_3) = (2, 3, 3)

$$\Rightarrow \overrightarrow{(AB)} \times \overrightarrow{(BC)} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -4 & -5 \\ 2 & 3 & 2 \end{vmatrix}$$

$\overrightarrow{AB} \times \overrightarrow{BC} = \hat{\imath}\,(-8 + 15) - \hat{\jmath}\,(4 + 10) + \hat{k}\,(6 + 8)$

$\overrightarrow{AB} \times \overrightarrow{BC} = 7\,\hat{\imath} - 14\,\hat{\jmath} + 14\,\hat{k}$

$|\overrightarrow{AB} \times \overrightarrow{BC}| = \sqrt{(7)^2 + (14)^2 + (14)^2}$

$|\overrightarrow{AB} \times \overrightarrow{BC}| = 21$

Area of Parallelogram = 21 sq units.

37. Question

If $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a}.\vec{b}\right|^2 = 400$ and $\left|\vec{a}\right| = 5$, then write the value of $\left|\vec{b}\right|$.

Answer

Given $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a}.\vec{b}\right|^2 = 400$ and $|\vec{a}| = 5$

We know the dot product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow |\vec{a}.\vec{b}| = |\vec{a}||\vec{b}||\cos\theta|$

$\therefore |\vec{a}.\vec{b}| = 5|\vec{b}||\cos\theta|$

We also know the cross product of two vectors $\vec{a}$ and $\vec{b}$ forming an angle θ is

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

where $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta||\hat{n}|$

$\hat{n}$ is a unit vector $\Rightarrow |\hat{n}| = 1$

$\therefore |\vec{a} \times \vec{b}| = 5|\vec{b}||\sin\theta|$

We have $|\vec{a} \times \vec{b}|^2 + |\vec{a}.\vec{b}|^2 = 400$

$\Rightarrow (5|\vec{b}||\sin\theta|)^2 + (5|\vec{b}||\cos\theta|)^2 = 400$

$\Rightarrow 25|\vec{b}|^2|\sin\theta|^2 + 25|\vec{b}|^2|\cos\theta|^2 = 400$

$\Rightarrow 25|\vec{b}|^2(|\sin\theta|^2 + |\cos\theta|^2) = 400$

$\Rightarrow 25|\vec{b}|^2(\sin^2\theta + \cos^2\theta) = 400$

But, we know $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow 25|\vec{b}|^2 = 400$

$\Rightarrow |\vec{b}|^2 = 16$

$\therefore |\vec{b}| = \sqrt{16} = 4$

Thus, $|\vec{b}| = 4$

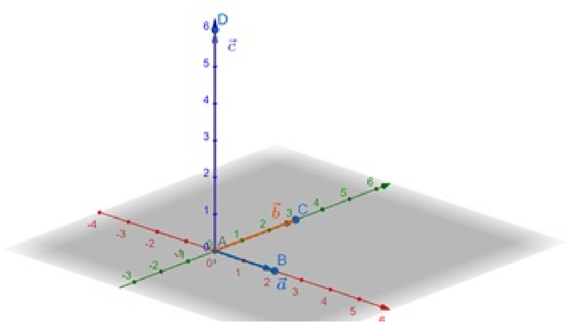
Very short answer

1. Question

Define vector product of two vectors.

Answer

Definition: VECTOR PRODUCT: When multiplication of two vectors yields another vector then it is called vector product of two vectors.



Example:

Figure 1: Vector Product

$\vec{c} = \vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \hat{n}$

[where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$(referred to the figure provided)]

2. Question

Write the value $\left(\hat{i} \times \hat{j}\right) \cdot \hat{k} + \hat{i} \cdot \hat{j}$.

Answer

$(\hat{\imath} \times \hat{\jmath}) \cdot \hat{k} + \hat{\imath} \cdot \hat{\jmath} = 1$.

We know, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are 3 unit vectors along x, y and z axis whose magnitudes are unity.

$\hat{\imath} \times \hat{\jmath} = |\hat{\imath}||\hat{\jmath}|\sin 90^{o}\hat{n}$

[where $\hat{n}$ is a unit vector perpendicular to the plane containing $\hat{\imath}$ and $\hat{\jmath}$]

$= 1 \times 1 \times 1 \times \hat{k}$

$= \hat{k}$ [here $\hat{n}$ is $\hat{k}$, as $\hat{k}$ is perpendicular to both $\hat{\imath}$ and $\hat{\jmath}$]

And, $\hat{\imath} \cdot \hat{\jmath} = |\hat{\imath}||\hat{\jmath}|\cos 90^{o} = 0$.

So, $(\hat{\imath} \times \hat{\jmath}) \cdot \hat{k} + \hat{\imath} \cdot \hat{\jmath}$

$= \hat{k} \cdot \hat{k} + 0$

$= |\hat{k}||\hat{k}|\cos 0^{o}$

$= 1$ [$\because \hat{k}$ is an unit vector].

3. Question

Write the value of $\hat{i} \cdot \left(\hat{j} \times \hat{k}\right) + \hat{j} \cdot \left(\hat{k} \times \hat{i}\right) + \hat{k} \cdot \left(\hat{j} \times \hat{i}\right)$.

Answer

$\hat{\imath} \cdot (\hat{\jmath} \times \hat{k}) + \hat{\jmath} \cdot (\hat{k} \times \hat{\imath}) + \hat{k}.(\hat{\jmath} \times \hat{\imath}) = 1$.

We know, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are 3 unit vectors along x, y and z axis whose magnitudes are unity.

We have,

$\hat{\jmath} \times \hat{k} = |\hat{\jmath}||\hat{k}|\sin 90^{o}\hat{\imath} = \hat{\imath}$,

$\hat{k} \times \hat{\imath} = |\hat{k}||\hat{\imath}|\sin 90^{o}\hat{\jmath}$ and

$\hat{\jmath} \times \hat{\imath} = |\hat{\jmath}||\hat{\imath}|\sin 90^{o}(-\hat{k}) = -\hat{k}$

And, $\hat{\imath} \cdot \hat{\imath} = |\hat{\imath}||\hat{\imath}|\cos 0^{o} = 1$,

$\hat{\jmath} \cdot \hat{\jmath} = |\hat{\jmath}||\hat{\jmath}|\cos 90^{o} = 1$ and

$\hat{k}.(-\hat{k}) = |\hat{k}||\hat{k}|\cos 180^{o} = -1$.

$\therefore \hat{\imath} \cdot (\hat{\jmath} \times \hat{k}) + \hat{\jmath} \cdot (\hat{k} \times \hat{\imath}) + \hat{k}.(\hat{\jmath} \times \hat{\imath})$

$= \hat{\imath} \cdot \hat{\imath} + \hat{\jmath} \cdot \hat{\jmath} + \hat{k}.(-\hat{k})$

=1+1+(-1)

=1.

4. Question

Write the value of $\hat{i}\cdot\left(\hat{j}\times\hat{k}\right)+\hat{j}\cdot\left(\hat{k}\times\hat{i}\right)+\hat{k}\cdot\left(\hat{i}\times\hat{j}\right)$.

Answer

$\hat{\imath}\cdot(\hat{\jmath}\times\hat{k})+\hat{\jmath}\cdot(\hat{k}\times\hat{\imath})+\hat{k}\cdot(\hat{\imath}\times\hat{\jmath})=3$.

We know, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are 3 unit vectors along x, y and z axis whose magnitudes are unity.

We have,

$\because \hat{\jmath}\times\hat{k}=|\hat{\jmath}||\hat{k}|\sin 90^{o}\hat{\imath}=\hat{\imath}$,

$\hat{k}\times\hat{\imath}=|\hat{k}||\hat{\imath}|\sin 90^{o}\hat{\jmath}$ and

$\hat{\imath}\times\hat{\jmath}=|\hat{\imath}||\hat{\jmath}|\sin 90^{o}\hat{k}=\hat{k}$

And, $\hat{\imath}.\hat{\imath}=|\hat{\imath}||\hat{\imath}|\cos 0^{o}=1$,

$\hat{\jmath}\cdot\hat{\jmath}=|\hat{\imath}||\hat{\jmath}|\cos 0^{o}=1$ and

$\hat{k}\cdot\hat{k}=|\hat{k}||\hat{k}|\cos 0^{o}=1$.

$\therefore \hat{\imath}\cdot(\hat{\jmath}\times\hat{k})+\hat{\jmath}\cdot(\hat{k}\times\hat{\imath})+\hat{k}\cdot(\hat{\imath}\times\hat{\jmath})$

$=\hat{\imath}\cdot\hat{\imath}+\hat{\jmath}\cdot\hat{\jmath}+\hat{k}\cdot\hat{k}$

=1+1+1

=3

5. Question

Write the value of $\hat{i}\times\left(\hat{j}+\hat{k}\right)+\hat{j}\times\left(\hat{k}+\hat{i}\right)+\hat{k}\times\left(\hat{i}+\hat{j}\right)$.

Answer

We know, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are 3 unit vectors along x, y and z axis whose magnitudes are unity.

We have, $\hat{\imath}\times(\hat{\jmath}+\hat{k})=|\hat{\imath}||\hat{\jmath}|\sin 90^{o}\hat{k}+|\hat{\imath}||\hat{k}|\sin 90^{o}(-\hat{\jmath})=\hat{k}-\hat{\jmath}$,

$\hat{\jmath}\times(\hat{k}+\hat{\imath})=|\hat{\jmath}||\hat{k}|\sin 90^{o}\hat{\imath}+|\hat{\jmath}||\hat{\imath}|\sin 90^{o}(-\hat{k})=\hat{\imath}-\hat{k}$ and

$\hat{k}\times(\hat{\imath}+\hat{\jmath})=|\hat{k}||\hat{\imath}|\sin 90^{o}\hat{\jmath}+|\hat{k}||\hat{\jmath}|\sin 90^{o}(-\hat{\imath})+=\hat{\jmath}-\hat{\imath}$.

$\therefore \hat{\imath}\times(\hat{\jmath}+\hat{k})+\hat{\jmath}\times(\hat{k}+\hat{\imath})+\hat{k}\times(\hat{\imath}+\hat{\jmath})$

$=\hat{k}-\hat{\jmath}+\hat{\imath}-\hat{k}+\hat{\jmath}-\hat{\imath}$

=0

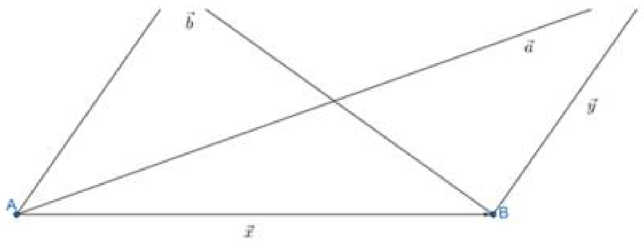
6. Question

Write the expression for the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its diagonals.

Answer

Area of parallelogram $=\frac{1}{2}|\vec{a}\times\vec{b}|$

Figure 2: Parallelogram



From the figure, it is clear that, $\vec{x} + \vec{y} = \vec{a}$ and

$\vec{y} + (-\vec{x}) = \vec{b}$ i.e. $\vec{y} - \vec{x} = \vec{b}$.

Now, $\vec{a} \times \vec{b} = (\vec{x} + \vec{y}) \times (\vec{y} - \vec{x})$

$= \vec{x} \times (\vec{y} - \vec{x}) + \vec{y} \times (\vec{y} - \vec{x})$

$= \{(\vec{x} \times \vec{y}) - (\vec{x} \times \vec{x})\} + \{(\vec{y} \times \vec{y}) - (\vec{y} \times \vec{x})\}$

$= 2(\vec{x} \times \vec{y})$.

$[\because (\vec{x} \times \vec{x}) = 0, (\vec{y} \times \vec{y}) = 0 \text{ and } (\vec{y} \times \vec{x}) = -(\vec{x} \times \vec{y})]$

Now, we know, area of parallelogram $= |\vec{x} \times \vec{y}|$.

So, Area of parallelogram $= \frac{1}{2}|\vec{a} \times \vec{b}|$. $[\because \vec{a} \times \vec{b} = 2(\vec{x} \times \vec{y})]$

7. Question

For any two vectors $\vec{a}$ and $\vec{b}$ write the value of $\left(\vec{a} \cdot \vec{b}\right)^2 + |\vec{a} \times \vec{b}|^2$ in terms of their magnitudes.

Answer

$(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = (|\vec{a}||\vec{b}|)^2$.

We know, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

and $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta$.

So, $(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2$

$= (|\vec{a}||\vec{b}|\cos\theta)^2 + \left||\vec{a}||\vec{b}|\sin\theta\right|^2$

$= (|\vec{a}||\vec{b}|)^2(\cos^2\theta + \sin^2\theta)$

$= (|\vec{a}||\vec{b}|)^2$. $[\because (\cos^2\theta + \sin^2\theta) = 1]$

8. Question

If $\vec{a}$ and $\vec{b}$ are two vectors of magnitudes 3 and $\frac{\sqrt{2}}{3}$ respectively such that $\vec{a} \times \vec{b}$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.

Answer

Angle between $\vec{a}$ and $\vec{b} = 45^\circ$.

Given, $|\vec{a}| = 3, |\vec{b}| = \frac{\sqrt{2}}{3}$

Also given, $\vec{a} \times \vec{b}$ is a unit vector

i.e. $\vec{a} \times \vec{b} = 1$.

$\therefore \vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta$

$= 3 \times \frac{\sqrt{2}}{3} \times \sin\theta$

$= \sqrt{2} \times \sin\theta = 1$

$\Rightarrow \sqrt{2} \times \sin\theta = 1$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$

$\therefore$Angle between $\vec{a}$ and $\vec{b} = 45^\circ$

9. Question

If $|\vec{a}| = 10, |\vec{b}| = 2$ and $|\vec{a} \times \vec{b}| = 16$, and $\vec{a} \cdot \vec{b}$.

Answer

$\vec{a} \cdot \vec{b} = 12$.

Given, $|\vec{a}| = 10, |\vec{b}| = 2$ and $\vec{a} \times \vec{b} = 16$

$\therefore \vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta = 10 \times 2 \times \sin\theta = 20 \times \sin\theta = 16$

$\Rightarrow 20 \times \sin\theta = 16$

$\Rightarrow \sin\theta = \frac{16}{20} = \frac{4}{5}$

$\therefore \cos\theta = \sqrt{1 - \left(\frac{4}{5}\right)^2}$

$= \sqrt{1 - \frac{16}{25}}$

$= \sqrt{\frac{25 - 16}{25}}$

$= \sqrt{\frac{9}{25}}$

$= \frac{3}{5}$

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$= 10 \times 2 \times \frac{3}{5}$

=12

10. Question

For any two vectors $\vec{a}$ and $\vec{b}$, find $\vec{a}\cdot(\vec{b}\times\vec{a})$.

Answer

$\vec{a}\cdot(\vec{b}\times\vec{a}) = 0$.

We know,

$(\vec{b}\times\vec{a})$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

So, $\vec{a}\cdot(\vec{b}\times\vec{a}) = 0$ [$\because \vec{a}$ and $(\vec{b}\times\vec{a})$ are perpendicular to each other]

11. Question

If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{b}\times\vec{a}| = \sqrt{3}$ and $\vec{a}\cdot\vec{b} = 1$, find the angle between.

Answer

The angle between $\vec{a}$ and $\vec{b}$ is 60°.

We have, $|\vec{b}\times\vec{a}| = \sqrt{3}$ and $\vec{a}\cdot\vec{b} = 1$.

$\therefore |\vec{b}\times\vec{a}| = |\vec{b}||\vec{a}|\sin\theta = \sqrt{3}$ .... (1)

and $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 1$ .... (2)

Dividing equation (1) by equation (2),

$$\frac{|\vec{b}||\vec{a}|\sin\theta}{|\vec{a}||\vec{b}|\cos\theta} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\Rightarrow \tan\theta = \sqrt{3}$$

$$\Rightarrow \theta = \tan^{-1}\sqrt{3} = 60^\circ$$

$\therefore$ The angle between $\vec{a}$ and $\vec{b}$ is 60°.

12. Question

For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ write the value of $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b})$.

Answer

$\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b}) = 0$.

$\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b})$

$= (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c}) + (\vec{b}\times\vec{a}) + (\vec{c}\times\vec{a}) + (\vec{c}\times\vec{b})$

$= (\vec{a}\times\vec{b}) - (\vec{c}\times\vec{a}) + (\vec{b}\times\vec{c}) - (\vec{a}\times\vec{b}) + (\vec{c}\times\vec{a}) - (\vec{b}\times\vec{c})$

= 0

13. Question

For any two vectors $\vec{a}$ and $\vec{b}$, find $(\vec{a}\times\vec{b})\cdot\vec{b}$.

Answer

$(\vec{a}\times\vec{b})\cdot\vec{b} = 0$

We know, $(\vec{a} \times \vec{b})$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

So, $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$ [$\because \vec{b}$ and $(\vec{a} \times \vec{b})$ are perpendicular to each other]

14. Question

Write the value of $\hat{i} \times (\hat{j} \times \hat{k})$.

Answer

$\hat{i} \times (\hat{j} \times \hat{k}) = 0$.

We know, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are 3 unit vectors along x, y and z axis whose magnitudes are unity.

$\hat{i} \times (\hat{j} \times \hat{k}) = \hat{i} \times \hat{i} = |\hat{i}||\hat{i}| \sin 0° = 0$.

15. Question

If $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$, then find $(\vec{a} \times \vec{b})\vec{a}$.

Answer

NOTE: The product of $(\vec{a} \times \vec{b})$ and $\vec{a}$ is not mentioned here.

$(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$ and $(\vec{a} \times \vec{b}) \times \vec{a} = 19\hat{i} + 17\hat{j} - 20\hat{k}$.

We know, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are 3 unit vectors along x, y and z axis whose magnitudes are unity.

Given, $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$.

$\therefore (\vec{a} \times \vec{b}) \cdot \vec{a} = [(3\hat{i} - \hat{j} + 2\hat{k}) \times (2\hat{i} + \hat{j} - \hat{k})] \cdot (3\hat{i} - \hat{j} + 2\hat{k})$

$= (-\hat{i} + 7\hat{j} + 5\hat{k}) \cdot (3\hat{i} - \hat{j} + 2\hat{k})$

$= -3 - 7 + 10$

$= 0$.

"FOR CROSS PRODUCT"

$\therefore (\vec{a} \times \vec{b}) \times \vec{a} = [(3\hat{i} - \hat{j} + 2\hat{k}) \times (2\hat{i} + \hat{j} - \hat{k})] \times (3\hat{i} - \hat{j} + 2\hat{k})$

$= (-\hat{i} + 7\hat{j} + 5\hat{k}) \times (3\hat{i} - \hat{j} + 2\hat{k})$

$= 19\hat{i} + 17\hat{j} - 20\hat{k}$.

16. Question

Write a unit vector perpendicular to $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$.

Answer

We know that cross product of two vectors gives us a vector which is perpendicular to both the vectors.

Let $\vec{M} = \hat{\imath} + \hat{\jmath}$ *and* $\vec{N} = \hat{\jmath} + \hat{k}$ and $\vec{O}$ be the vector perpendicular to vectors $\vec{M}$ and $\vec{N}$.

$$\therefore \vec{O} = \vec{M} \times \vec{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{vmatrix}$$

$= (M_2N_3 - M_3N_2)\hat{\imath} - (M_1N_3 - M_3N_1)\hat{\jmath} + (M_1N_2 - M_2N_1)\hat{k}$

Inserting the given values we get,

$$\vec{O} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$= (1 \times 1 - 0 \times 1)\hat{\imath} - (1 \times 1 - 0 \times 0)\hat{\jmath} + (1 \times 1 - 1 \times 0)\hat{k}$

$= (1 - 0)\hat{\imath} + (1 - 0)\hat{\jmath} + (1 - 0)\hat{k}$

$= \hat{\imath} - \hat{\jmath} + \hat{k}$

Now, as we know unit vector can be obtained by dividing the given vector by its magnitude.

$\vec{O} = \hat{\imath} - \hat{\jmath} + \hat{k}$ *and* $|\vec{C}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

Unit vector in the direction of $\vec{O} = \frac{\vec{c}}{|\vec{c}|}$

∴Desired unit vector is $\frac{1}{\sqrt{3}}(\hat{\imath} - \hat{\jmath} + \hat{k})$

17. Question

If $|\vec{a} \times \vec{b}|^2 + \left(\vec{a} \cdot \vec{b}\right)^{-2} = 144$ and $|\vec{a}| = 4$, find $|\vec{b}|$.

[Correction in the Question – $(\vec{a}.\vec{b})^{-2}$should be $(\vec{a}.\vec{b})^2$ or else it's not possible to find the value$|\vec{b}|$.]

Answer

We know that,

$(\vec{a} \times \vec{b}) = |\vec{a}||\vec{b}| \sin\theta \rightarrow (1)$

$(\vec{a}.\vec{b}) = |\vec{a}||\vec{b}| \cos\theta \rightarrow (2)$

Now,

$|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2 = 144$

$|\vec{a}|^2|\vec{b}|^2 sin^2\theta + |\vec{a}|^2|\vec{b}|^2 cos^2\theta = 144 \rightarrow From\ (1)\ and\ (2)$

$|\vec{a}|^2|\vec{b}|^2(sin^2\theta + cos^2\theta) = 144$

$|\vec{a}|^2|\vec{b}|^2 = 144 \rightarrow \sin^2\theta + \cos^2\theta = 1$

$4^2 \times |\vec{b}|^2 = 144$

$16 \times |\vec{b}|^2 = 144$

$$|\vec{b}|^2 = \frac{144}{16} = 9$$

$|\vec{b}| = 3$

18. Question

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then write the value of $|\vec{r} \times \hat{i}|^2$.

Answer

So we have $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$and $\hat{\imath}$, in order to find $|\vec{r} \times \hat{\imath}|^2$ we need to work out the problem by finding cross product through determinant.

$$\therefore \vec{r}\times\hat{\imath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_1 & r_2 & r_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (r_2\times 0 - r_3\times 0)\hat{\imath} - (r_1\times 0 - r_3\times 1)\hat{\jmath} + (r_1\times 0 - r_2\times 1)\hat{k}$$

$$\vec{r}\times\hat{\imath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix} = (y\times 0 - z\times 0)\hat{\imath} - (x\times 0 - z\times 1)\hat{\jmath} + (x\times 0 - y\times 1)\hat{k}$$

$$= 0\hat{\imath} + z\hat{\jmath} - y\hat{k} = z\hat{\jmath} - y\hat{k} \rightarrow (1)$$

Now then,

$|\vec{r}\times\hat{\imath}| = \sqrt{z^2+(-y)^2} = \sqrt{z^2+y^2} \rightarrow$ From (1)

$|\vec{r}\times\hat{\imath}|^2 = z^2 + y^2$

19. Question

If $\vec{a}$ and $\vec{b}$ are unit vectors such that $\vec{a}\times\vec{b}$ is also a unit vector, find the angle between $\vec{a}$ and $\vec{b}$.

Answer

Let's see what all things we know from the given question.

$|\vec{a}| = 1, |\vec{b}| = 1$ and $|\vec{a}\times\vec{b}| = 1 \rightarrow$ Unit Vectors

$Also, |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|sin\theta$

1 = (1)(1) sinθ

sinθ=1

$\theta = \frac{\pi}{2}$

20. Question

If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$, write the angle between $\vec{a}$ and $\vec{b}$.

Answer

Equations we already have –

$|\vec{a}.\vec{b}| = |\vec{a}||\vec{b}||cos\theta| \rightarrow (1)$

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}||sin\theta| \rightarrow (2)$

Now,

$|\vec{a}\times\vec{b}| = |\vec{a}.\vec{b}| \rightarrow (Given)$

$|\vec{a}||\vec{b}||sin\theta| = |\vec{a}||\vec{b}||cos\theta| \rightarrow (1\ and\ 2)$

sin θ = cos θ

$\theta = \frac{\pi}{4}$

21. Question

If $\vec{a}$ and $\vec{b}$ are unit vectors, then write the value of $|\vec{a}\times\vec{b}|^2 + \left(\vec{a}\times\vec{b}\right)^2$.

Answer

Let's have a look at everything we have before proceeding to solve the question.

$|\vec{a}| = 1 \; and \; |\vec{b}| = 1$ → Given (Unit Vectors)

$(\vec{a} \times \vec{b}) = |\vec{a}||\vec{b}| \sin\theta$

$(\vec{a}.\vec{b}) = |\vec{a}||\vec{b}| \cos\theta$

Now then,

$|\vec{a} \times \vec{b}|^2 + (\vec{a} \times \vec{b})^2$

$= (|\vec{a}||\vec{b}|sin\theta)^2 + (|\vec{a}||\vec{b}|sin\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2 sin^2\theta + |\vec{a}|^2|\vec{b}|^2 sin^2\theta$

$= 2|\vec{a}|^2|\vec{b}|^2 sin^2\theta$

$= 2(1)(1)\sin^2\theta$

$= 2 \sin^2\theta$

In case, the question asks for $|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2$

$= (|\vec{a}||\vec{b}|sin\theta)^2 + (|\vec{a}||\vec{b}|cos\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2 sin^2\theta + |\vec{a}|^2|\vec{b}|^2 cos^2\theta$

$= |\vec{a}|^2|\vec{b}|^2(sin^2\theta + cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2$

$=(1)(1)$

$= 1$

22. Question

If $\vec{a}$ is a unit vector such that $\vec{a} \times \hat{i} = \hat{j}$, find $\vec{a} \cdot \hat{i}$.

Answer

We know that →

$\hat{\imath} \times \hat{\jmath} = \hat{k} \rightarrow (1)$

$\hat{\jmath} \times \hat{k} = \hat{\imath} \rightarrow (2)$

$\hat{k} \times \hat{\imath} = \hat{\jmath} \rightarrow (3)$

$\hat{\imath}.\hat{\jmath} = \hat{\imath}.\hat{k} = \hat{\jmath}.\hat{k} = 0 \rightarrow (4)$

Now,

$\vec{a} \times \hat{\imath} = \hat{k} \times \hat{\imath}$ → Given and (3)

On comparing LHS and RHS we get :

$\vec{a} = \hat{k} \rightarrow (5)$

$\vec{a}.\hat{\imath} = \hat{k}.\hat{\imath}$ → From (5)

$\vec{a}.\hat{\imath} = 0$ → From (4)

23. Question

If $\vec{c}$ is a unit vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$, write another unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

Answer

We know that cross product of two vectors gives us a vector which is perpendicular to both the vectors. And keeping in mind that $\vec{c}$ is a Unit vector we get the equation –

$\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \vec{c} \rightarrow$ (Vector divided its magnitude gives unit vector)

$\frac{\vec{b}\times\vec{a}}{|\vec{a}\times\vec{b}|} = -\vec{c} \;\therefore\; -\vec{c}$ is perpendicular to $\vec{a}$ and $\vec{b}$

Thus, $-\vec{c}$ is another unit vector perpendicular to $\vec{a}\ and\ \vec{b}.$

Alternative Solution –

Since $\vec{c}$ is perpendicular to $\vec{a}\ and\ \vec{b}$, any unit vector parallel/anti-parallel to $\vec{c}$ will be perpendicular to $\vec{a}\ and\ \vec{b}$.

24. Question

Find the angle between two vectors $\vec{a}$ and $\vec{b}$, with magnitudes 1 and 2 respectively and when $|\vec{a}\times\vec{b}|=\sqrt{3}$.

Answer

$|\vec{a}| = 1, |\vec{b}| = 2, |\vec{a}\times\vec{b}| = \sqrt{3} \rightarrow$ Given

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}||sin\theta|$

$\sqrt{3} = 1 \times 2 \times \sin\theta$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\sin\theta = \frac{\pi}{3}$

25. Question

Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=\sqrt{3}, |\vec{b}|=\frac{2}{3}$ and $(\vec{a}\times\vec{b})$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.

Answer

Let's have a look at everything given in the problem.

$|\vec{a}| = \sqrt{3}$

$|\vec{b}| = \frac{2}{3}$

$|\vec{a}\times\vec{b}| = 1$

We can use the basic cross product formula to solve the question –

$|\vec{a}\times\vec{b}| = |a||b|sin\theta$

$1 = \sqrt{3}\times\frac{2}{3}\times sin\theta$

$\sin\theta = \frac{3}{2} \times \frac{1}{\sqrt{3}} = \frac{3}{2} \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}$

26. Question

Find λ, if $(2\hat{i}+6\hat{j}+14\hat{k})\times(\hat{i}-\lambda\hat{j}+7\hat{k})=\vec{0}$.

Answer

We need to solve the problem by finding cross product through determinant.

Let $\vec{M} = 2\hat{\imath} + 6\hat{\jmath} + 14\hat{k}$ *and* $\vec{N} = \hat{\imath} - \lambda\hat{\jmath} + 7\hat{k}$, also $\vec{M} \times \vec{N} = 0$ (Given)

$$\vec{M} \times \vec{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{vmatrix}$$

$= (M_2N_3 - M_3N_2)\hat{\imath} - (M_1N_3 - M_3N_1)\hat{\jmath} + (M_1N_2 - M_2N_1)\hat{k}$

Inserting the given values we get,

$$\vec{0} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 6 & 14 \\ 1 & -\lambda & 7 \end{vmatrix}$$

$= (6 \times 7 - (14 \times -\lambda))\hat{\imath} - (2 \times 7 - 14 \times 1)\hat{\jmath} + ((2 \times -\lambda) - 6 \times 1)\hat{k}$

$(42 + 14\lambda)\hat{\imath} - 0\hat{\jmath} + (-2\lambda - 6)\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

On comparing LHS and RHS we get,

42+14λ=0 and-2λ-6=0

14λ= -42and-2λ=6

λ= -3andλ=-3

27. Question

Write the value of the area of the parallelogram determined by the vectors $2\hat{i}$ and $3\hat{j}$.

Answer

Area of the parallelogram is give by $|\vec{a} \times \vec{b}|$

Let, $\vec{a} = 2\hat{\imath}$ *and* $\vec{b} = 3\hat{\jmath}$

$Area = |\vec{a} \times \vec{b}|$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix}$$

$= (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (6-0)\hat{k}$

$= 6\hat{k} = 6|\hat{k}| = 6(1) \rightarrow$ ($\hat{k}$ is an unit vector)

$= 6$ sq. units.

28. Question

Write the value of $\left(\hat{i}\times\hat{j}\right)\cdot\hat{k}+\left(\hat{j}+\hat{k}\right)\cdot\hat{j}$.

Answer

We know that,

$\hat{\imath}\times\hat{\jmath}=\hat{k}\rightarrow(1)$

$\hat{\jmath}\times\hat{k}=\hat{\imath}\rightarrow(2)$

$\hat{k}\times\hat{\imath}=\hat{\jmath}\rightarrow(3)$

$\hat{\imath}.\hat{\imath}=\hat{\jmath}.\hat{\jmath}=\hat{k}.\hat{k}=1\ \rightarrow(4)$

$\hat{\imath}.\hat{\jmath}=\hat{\imath}.\hat{k}=\hat{\jmath}.\hat{k}=0\rightarrow(5)$

Now,

$=\ (\hat{\imath}\times\hat{\jmath}).\hat{k}+\ (\hat{\jmath}+\hat{k}).\hat{\jmath}$

$=\hat{k}.\hat{k}+\ \hat{\jmath}.\hat{\jmath}+\ \hat{k}.\hat{\jmath}$ → (From 1)

= 1+1+0 → (From 4 and 5)

= 2

29. Question

Find a vector of magnitude $\sqrt{171}$ which is perpendicular to both of the vectors $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$.

Answer

We know that cross product of two vectors gives us a vector which is perpendicular to both the vectors. If we can find an unit vector

perpendicular to the given vectors, we can easily get the answer by multiplying $\sqrt{171}\ to$ the unit vector.

Unit vectors perpendicular to the given vectors $=\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Now,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=(a_2b_3-a_3b_2)\hat{\imath}-\ (a_1b_3-a_3b_1)\hat{\jmath}+(a_1b_2-a_2b_1)\hat{k}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 2 & -3\\ 3 & -1 & 2\end{vmatrix}$$
$$=(2\times2-(-3\times-1))\hat{\imath}-\ (1\times2-(-3\times3))\hat{\jmath}$$
$$+((1\times-1)-2\times3)\hat{k}$$

$\vec{a}\times\vec{b}=\ \hat{\imath}-\ 11\hat{\jmath}-\ 7\hat{k}$

$|\vec{a}\times\vec{b}|=\ \sqrt{1^2+(-11)^2+(-7)^2}=\ \sqrt{171}$

∵ Unit vectors perpendicular to $\vec{a}$ and $\vec{b}\ =\ \pm\frac{\hat{\imath}-11\hat{\jmath}-7\hat{k}}{\sqrt{171}}$

Vectors of magnitude $\sqrt{171}$ which are perpendicular to $\vec{a}$ and $\vec{b}$ →

$\sqrt{171} \times \pm\frac{\hat{\imath}-11\hat{\jmath}-7\hat{k}}{\sqrt{171}} = \pm(\hat{\imath}-\ 11\hat{\jmath}-\ 7\hat{k})$

30. Question

Write the number of vectors of unit length perpendicular to both the vectors $\vec{a} = 2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = \hat{j}+\hat{k}$.

Answer

As we know, for vectors $\vec{a}$ and $\vec{b}$ unit vectors perpendicular to them is give by $\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Unit vector can be $\perp$ either in positive or negative direction.

Hence, the number of vectors of unit length perpendicular to both the vectors $\vec{a} = 2\hat{\imath}+\hat{\jmath}+2\hat{k}\ and\ \vec{b} = \hat{\jmath}+\hat{k}$ is 2.

31. Question

Write the angle between the vectors $\vec{a}\times\vec{b}$ and $\vec{a}\times\vec{b}$.

Answer

Given question gives us two same vectors so the angle is $0°$.

In case, it asks write the angle between the vectors $\vec{a}\times\vec{b}\ and\ \vec{b}\times\vec{a}$ –

The angle between the vectors will be 180° as they are equal in magnitude and opposite in direction.

MCQ

1. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ is any vector, then $(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2 =$

A. $\vec{a}^2$

B. $2\vec{a}^2$

C. $3\vec{a}^2$

D. $4\vec{a}^2$

Answer

Let $\vec{a} = a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$

$\vec{a}\times\hat{\imath} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ 1 & 0 & 0\end{vmatrix}$

$= a_3\hat{\jmath}-a_2\hat{k}$

$(\vec{a}\times\hat{\imath})^2 = {a_3}^2+{a_2}^2\ \because \hat{\jmath}.\hat{k} = 0$

$\vec{a}\times\hat{\jmath} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ 0 & 1 & 0\end{vmatrix}$

$= -a_3\hat{\imath}+a_1\hat{k}$

$(\vec{a} \times j)^2 = {a_3}^2 + {a_1}^2 \because \hat{\imath}.\hat{k} = 0$

$$\vec{a} \times \hat{k} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 1 \end{vmatrix}$$

$a_2\hat{\imath} - a_1\hat{\jmath}$

$(\vec{a} \times k)^2 = {a_1}^2 + {a_2}^2 \because \hat{\jmath}.\hat{\imath} = 0$

$$\left(\vec{a} \times \hat{i}\right)^2 + \left(\vec{a} \times \hat{j}\right)^2 + \left(\vec{a} \times \hat{k}\right)^2 = {a_3}^2 + {a_2}^2 + {a_3}^2 + {a_1}^2 + {a_1}^2 + {a_2}^2$$

$= 2({a_1}^2 + {a_2}^2 + {a_3}^2)$

$= 2\vec{a}^2$

(B)

2. Question

Mark the correct alternative in each of the following:

If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}, \vec{a} \neq 0$, then

A. $\vec{b} = \vec{c}$

B. $\vec{b} = \vec{0}$

C. $\vec{b} + \vec{c} = \vec{0}$

D. None of these

Answer

$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$

$\vec{a}.\vec{b} - \vec{a}.\vec{c}$

$\vec{a}(\vec{b} - \vec{c}) = 0$ ...(1)

$\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

$\vec{a} \times \vec{b} - \vec{a} \times \vec{c} = 0$

$\vec{a} \times (\vec{b} - \vec{c}) = 0$

Let Q be the angle between $\vec{a}$ and $\vec{b} - \vec{c}$

$|\vec{a}||\vec{b} - \vec{c}| \sin Q = 0$ ...(2)

Out of the four options the only option that satisfies both (1) and (2) is

$\vec{b} - \vec{c} = 0$

$\vec{b} = \vec{c}$ (A)

3. Question

Mark the correct alternative in each of the following:

The vector $\vec{b} = 3\hat{i} + 4\hat{k}$ is to be written as sum of a vector $\vec{\alpha}$ parallel to $\vec{a} = \hat{i} + \hat{j}$ and a vector $\vec{\beta}$ perpendicular to $\vec{a}$. Then $\vec{\alpha} =$

A. $\frac{3}{2}(\hat{i}+\hat{j})$

B. $\frac{2}{3}(\hat{i}+\hat{j})$

C. $\frac{1}{2}(\hat{i}+\hat{j})$

D. $\frac{1}{3}(\hat{i}+\hat{j})$

Answer

$\vec{b} = 3\hat{i} + 4\hat{k}$

$\vec{a} = \hat{i} + \hat{j}$

$\vec{b} = \vec{\alpha} + \vec{\beta}$

Let $\vec{\alpha} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

Since $\vec{\alpha}||\vec{a}$

$\vec{\alpha} = \gamma\vec{a}$

$a\hat{\imath} + b\hat{\jmath} + c\hat{k} = \gamma(\hat{\imath} + \hat{\jmath})$

$\alpha = \gamma\hat{\imath} + \gamma\hat{\jmath}$

$\beta = \vec{b} - \alpha$

$= (3-\gamma)\hat{\imath} - \gamma\hat{\jmath} + 4\hat{k}$

Since β is perpendicular to a

$\vec{a}.\beta = 0$

3-γ-γ=0

$\gamma = \frac{3}{2}$

$\therefore\ \alpha = \frac{3}{2}(\hat{\imath} + \hat{\jmath})$ (A)

4. Question

Mark the correct alternative in each of the following:

The unit vector perpendicular to the plane passing through points $P(\hat{i}-\hat{j}+2\hat{k})$, $Q(2\hat{i}-\hat{k})$ and $R(2\hat{j}+\hat{k})$ is

A. $2\hat{i}+\hat{j}+\hat{k}$

B. $\sqrt{6}(2\hat{i}+\hat{j}+\hat{k})$

C. $\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})$

D. $\frac{1}{6}\left(2\hat{i}+\hat{j}+\hat{k}\right)$

Answer

The equations of the plane is given by

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

Where A,B and C are the drs of the normal to the plane.

Putting the first point,

=A(x-1)+B(y+1)+C(z-2)=0 ...(1)

Putting the second point in Eqn (1)

=A(2-1)+B(0+1)+C(-1-2)=0

A+B-3C=0 ...(a)

Putting the third point in Eqn (1)

=A(0-1)+B(2+1)+C(1-2)=0

= -A+3B-C=0 ...(b)

Solving (a) and (b) using cross multiplication method

A+B-3C=0

-A+3B-C=0

$$\frac{A}{-1-(-9)}=\frac{-B}{-1-3}=\frac{C}{3-(-1)}=\alpha$$

$A=8\alpha; B=4\alpha; C=4\alpha$

Put these in Eqn(1)

=8α(x-1)+4α(y+1)+4α(z-2)=0

=2(x-1)+(y+1)+(z-2)=0

=2x+2+y+1+z-2=0

2x+y+z+1=0

Now the vector perpendicular to this plane is

$\vec{c}=2\hat{\imath}+\hat{\jmath}+\hat{k}$

Now the unit vector of $\vec{c}$ is given by

$$\hat{c}=\frac{\vec{c}}{|\vec{c}|}$$

$|\vec{c}|=\sqrt{2^2+1^2+1^2}=\sqrt{6}$

$\hat{c}=\frac{1}{\sqrt{6}}(2\hat{\imath}+\hat{\jmath}+\hat{k})$(C)

5. Question

Mark the correct alternative in each of the following:

If $\vec{a},\vec{b}$ represent the diagonals of a rhombus, then

A. $\vec{a}\times\vec{b}=\vec{0}$

B. $\vec{a} \cdot \vec{b} = 0$

C. $\vec{a} \cdot \vec{b} = 1$

D. $\vec{a} \times \vec{b} = \vec{a}$

Answer

The diagnols of a rhombus are always perpendicular

It means $\vec{a}$is perpendicular to$\vec{b}$

Q=90°

$\cos Q = 0$

$\vec{a} \cdot \vec{b} = 0$ (B)

6. Question

Mark the correct alternative in each of the following:

Vectors $\vec{a}$ and $\vec{b}$ are inclined at angle θ = 120°. If $|\vec{a}| = 1, |\vec{b}| = 2$, then $\left[\left(\vec{a} + 3\vec{b}\right) \times \left(3\vec{a} - \vec{b}\right)\right]^2$ is equal to

A. 300

B. 325

C. 275

D. 225

Answer

$[(\vec{a} + 3\vec{b}) \times (3\vec{a} - \vec{b})]^2$

$= [3(\vec{a} \times \vec{a}) - \overrightarrow{(a \times b)} - 9(\vec{b} \times \vec{a}) - (3\vec{b} \times \vec{b})]^2$

$[3(0$ $\because$ Angle between the same vector is 0° and $\sin 0 = 0) - \overrightarrow{(a \times b)}$
$- 3(\vec{a} \times \vec{b}) - 3(\vec{b} \times \vec{b} = 0)]^2$ $\because$ $(\vec{b} \times \vec{a}) = -\overrightarrow{(a \times b)}$

$= \left[-10\left(\overline{|\vec{a}||\vec{b}| \sin \frac{2\pi}{3}}\right)\right]^2$

$= 100 \times 1 \times 4 \times \frac{3}{4}$

$\because \sin \frac{2\pi}{3} = \sin \pi - \frac{\pi}{3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$= 300$ (A)

7. Question

Mark the correct alternative in each of the following:

If $\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$, then a unit vector normal to the vectors $\vec{a} + \vec{b}$ and $\vec{b} - \vec{c}$ is

A. $\hat{i}$

B. $\hat{j}$

C. $\hat{k}$

D. None of these

Answer

$\vec{a}+\vec{b}=3\hat{j}+\hat{k}$

$\vec{b}-\vec{c}=3\hat{k}$

Let $\vec{c}$ be perpendicular to both of these vectors

$\vec{c}=(\vec{a}+\vec{b})\times(\vec{b}-\vec{c})$

$=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 0 & 3 & 1\\ 0 & 0 & 3\end{vmatrix}$

$=\hat{\imath}(9-0)-\hat{\jmath}(0-0)+\hat{k}(0-0)$

$=9\hat{\imath}$

Now the unit vector of $\vec{c}$ is given by

$\hat{c}=\frac{\vec{c}}{|\vec{c}|}$

$|\vec{c}|=\sqrt{9^2}=9$

$\hat{c}=\frac{1}{9}(9\hat{\imath})=\hat{\imath}$ (A)

8. Question

Mark the correct alternative in each of the following:

A unit vector perpendicular to both $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$ is

A. $\hat{i}-\hat{j}+\hat{k}$

B. $\hat{i}+\hat{j}+\hat{k}$

C. $\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right)$

D. $\frac{1}{\sqrt{3}}\left(\hat{i}-\hat{j}+\hat{k}\right)$

Answer

Let $\vec{a}=\hat{\imath}+\hat{\jmath}$ and $\vec{b}=\hat{\jmath}+\vec{k}$

A vector perpendicular to both of them is given by $\vec{a}\times\vec{b}$

$\vec{a}\times\vec{b}=\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}$

$=\hat{\imath}(1-0)-\hat{\jmath}(1-0)+\hat{k}(1-0)$

$=\hat{\imath}-\hat{\jmath}+\hat{k}$

Now the unit vector of $\vec{c}$ is given by

$\hat{c} = \frac{\vec{c}}{|\vec{c}|}$

$|\vec{c}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

$\hat{c} = \frac{1}{\sqrt{3}}(\hat{\imath} - \hat{\jmath} + \hat{k})$ (D)

9. Question

Mark the correct alternative in each of the following:

If $\vec{a} = 2\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 4\hat{j} - 2\hat{k}$, then $\vec{a} \times \vec{b}$ is

A. $10\hat{i} + 2\hat{j} + 11\hat{k}$

B. $10\hat{i} + 3\hat{j} + 11\hat{k}$

C. $10\hat{i} - 3\hat{j} + 11\hat{k}$

D. $10\hat{i} - 2\hat{j} - 10\hat{k}$

Answer

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -3 & -1 \\ -2 & 4 & -2 \end{vmatrix}$

$= \hat{\imath}(6 - (-4)) - \hat{\jmath}(-4 - (-1)) + \hat{k}(8 - (-3))$

$= \hat{\imath}(10) - \hat{\jmath}(-3) + \hat{k}(11)$

$= 10\hat{\imath} + 3\hat{\jmath} + 11\hat{k}$ (B)

10. Question

Mark the correct alternative in each of the following:

If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors, then

A. $\hat{i} \cdot \hat{j} = 1$

B. $\hat{i} \cdot \hat{i} = 1$

C. $\hat{i} \times \hat{j} = 1$

D. $\hat{i} \times (\hat{j} \times \hat{k}) = 1$

Answer

$\hat{i}, \hat{j}, \hat{k}$ are unit vectors and angle between each of them is 90°

So, $\cos Q = \cos\frac{\pi}{2} = 0$

So (A) is false $\because \hat{\imath}.\hat{\jmath} = 0$

Option (B) is true because angle between them is 0°

So, cosQ=cos0=1

$\hat{i} \cdot \hat{i} = 1 \because |\hat{\imath}| = |\hat{\jmath}| = 1$

(C) False as $\hat{\imath} \times \hat{\jmath} = \hat{k}$

(D) is False as $\hat{\jmath} \times \hat{k} = \hat{\imath}$

And then $\hat{\imath} \times \hat{\imath} = 0$ as $\sin Q = 0$

(B)

11. Question

Mark the correct alternative in each of the following:

If θ is the angle between the vectors $2\hat{i} - 2\hat{j} + 4\hat{k}$ and $3\hat{i} + \hat{j} + 2\hat{k}$, then sin θ =

A. $\frac{2}{3}$

B. $\frac{2}{\sqrt{7}}$

C. $\frac{\sqrt{2}}{7}$

D. $\sqrt{\frac{2}{7}}$

Answer

Let $\vec{a} = 2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$ and $\vec{b} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 4 \\ 3 & 1 & 2 \end{vmatrix}$$

$= \hat{\imath}(-4 - 4) - \hat{\jmath}(4 - 12) + \hat{k}(2 - (-6))$

$= -8\hat{\imath} + 8\hat{\jmath} + 8\hat{k}$

We know

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin Q||\hat{n}|$

$\Rightarrow \sqrt{(-8)^2 + 8^2 + 8^2} = \sqrt{2^2 + (-2)^2 + 4^2}\sqrt{3^2 + 1^2 + 2^2} \sin Q$

$\Rightarrow 8\sqrt{3} = 2\sqrt{6}.\sqrt{14} \sin Q$

$\Rightarrow \frac{2}{\sqrt{7}} = \sin Q$ (B)

12. Question

Mark the correct alternative in each of the following:

If $|\vec{a} \times \vec{b}| = 4, |\vec{a}.\vec{b}| = 2$, then $|\vec{a}|^2 |\vec{b}|^2 =$

A. 6

B. 2

C. 20

D. 8

Answer

We know,

$$\left(\vec{a}.\vec{b}\right)^2 + \left|\vec{a}\times\vec{b}\right| = |\vec{a}|^2\left|\vec{b}\right|^2$$

$$\Rightarrow 2^2 + 4^2 = |\vec{a}|^2\left|\vec{b}\right|^2$$

$$\Rightarrow 4 + 16 = |\vec{a}|^2\left|\vec{b}\right|^2$$

$$\Rightarrow 20 = |\vec{a}|^2\left|\vec{b}\right|^2$$

13. Question

Mark the correct alternative in each of the following:

The value of $\left(\vec{a}\times\vec{b}\right)^2$ is

A. $|\vec{a}|^2 + |\vec{b}|^2 - \left(\vec{a}\cdot\vec{b}\right)^2$

B. $|\vec{a}|^2|\vec{b}|^2 - \left(\vec{a}\cdot\vec{b}\right)^2$

C. $|\vec{a}|^2 + |\vec{b}|^2 - 2\left(\vec{a}\cdot\vec{b}\right)$

D. $|\vec{a}|^2 + |\vec{b}|^2 - \vec{a}\cdot\vec{b}$

Answer

Let Q be the angle between vectors a and b

$$=\left(\vec{a}\times\vec{b}\right)^2$$

$$= (|a||b||\sin Q||\hat{n}|)^2$$

$$= |a|^2|b|^2 sin^2 Q$$

$$= |a|^2|b|^2(1 - cos^2 Q)$$

$\because$ $\sin^2 Q = 1 - \cos^2 Q$

$$= |a|^2|b|^2 - |a|^2|b|^2 cos^2 Q$$

$= |a|^2|b|^2 - (a.b)^2$ (B) $\because (a.b) = |a||b|\cos Q$

14. Question

Mark the correct alternative in each of the following:

The value of $\hat{i}\cdot\left(\hat{j}\times\hat{k}\right) + \hat{j}\cdot\left(\hat{i}\times\hat{k}\right) + \hat{k}\cdot\left(\hat{i}\times\hat{j}\right)$, is

A. 0

B. -1

C. 1

D. 3

Answer

We know,

$(\hat{\imath} \times \hat{\imath}) = 0;$

$(\hat{\jmath} \times \hat{\jmath}) = 0;$

$(\hat{k} \times \hat{k}) = 0;$

$(\hat{\imath} \times \hat{\jmath}) = \hat{k};$

$(\hat{\jmath} \times \hat{k}) = \hat{\imath};$

$(\hat{k} \times \hat{\imath}) = \hat{\jmath};$

$(\hat{\jmath} \times \hat{\imath}) = -\hat{k};$

$(\hat{k} \times \hat{\jmath}) = -\hat{\imath};$

$(\hat{\imath} \times \hat{k}) = -\hat{\jmath};$

Using them,

$\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$

$= \hat{\imath}.\hat{\imath} - \hat{\jmath}.\hat{\jmath} + \hat{k}.\hat{k}$

We know,

$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1 = 1 - 1 + 1$

$= 1$ (C)

15. Question

Mark the correct alternative in each of the following:

If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to

A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. π

Answer

$|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$

$|\vec{a}||\vec{b}| \cos Q = |\vec{a}||\vec{b}| \sin Q$

$\tan Q = 1$

$Q = \frac{\pi}{4}$