13. Derivative as a Rate Measurer

Exercise 13.1

1. Question

Find the rate of change of the total surface area of a cylinder of radius r and height h, when the radius varies.

Answer

Total Surface Area of Cylinder = $2\pi r^2 + 2\pi r h$

Given: Radius of the Cylinder varies.

Therefore, We need to find $\frac{dS}{dr}$ where S = Surface Area of Cylinder and r = radius of Cylinder.

$$\frac{dS}{dr} = 4\pi r + 2\pi h$$

Hence, Rate of change of total surface area of the cylinder when the radius is varying is given by $(4\pi r + 2\pi h)$.

2. Question

Find the rate of change of the volume of a sphere with respect to its diameter.

Answer

The volume of a Sphere = $\frac{1}{6}\pi D^3$

Where D = diameter of the Sphere

We need to find, $\frac{dV}{dD}$ where V = Volume of the sphere and D = Diameter of the Sphere.

$$\frac{dV}{dD} = \frac{\pi D^2}{2}$$

Hence, Rate of change of Volume of Sphere with respect to the diameter of the Sphere is given by $\frac{\pi D^2}{2}$.

3. Question

Find the rate of change of the volume of a sphere with respect to its surface area when the radius is 2 cm.

Answer

Volume of Sphere = $\frac{4}{3}\pi r^3$

Surface Area of Sphere = $4\pi r^2$

We need to find, $\frac{dV}{dS}$ where V = Volume of the Sphere and S = Surface Area of the Sphere.

$$\frac{dV}{dS} = \frac{dV}{dr} \times \frac{dr}{dS}$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dS}{dr} = 8\pi r$$

$$\frac{dV}{dS} = \frac{4\pi r^2}{8\pi r}$$

$$\frac{dV}{dS} = \frac{r}{2}$$

$$\left(\frac{dV}{dS}\right)_{at\ r=2} = \frac{2}{2} = 1\ cm$$

4. Question

Find the rate of change of the area of a circular disc with respect to its circumference when the radius is 3 cm.

Answer

Area of a Circular disc = π r^2

Circumference of a Circular disc = 2 π r

Where r = radius of Circular Disc.

Now we need to find $\frac{dA}{dC}$ where A = Area of Circular disc and C = Circumference of the Circular disk.

$$\frac{dA}{dC} = \frac{dA}{dr} \times \frac{dr}{dC}$$

$$\frac{dA}{dr} = 2\,\pi\,r$$

$$\frac{dC}{dr} = 2\,\pi$$

$$\frac{dA}{dC} = \frac{2\pi r}{2\pi} = r$$

$$\left(\frac{dA}{dr}\right)_{at\ x=3} = 3\ cm$$

5. Question

Find the rate of change of the volume of a cone with respect to the radius of its base.

Answer

The volume of Cone = $\frac{1}{3}\pi r^2 h$

Where r = radius of the cone

h = height of the cone

We need to find, $\frac{dV}{dr}$ where V = Volume of cone and r = radius of the cone.

$$\frac{dV}{dr} = \frac{2}{3}\pi r h$$

6. Question

Find the rate of change of the area of a circle with respect to its radius r when r = 5 cm.

Answer

Area of a Circle = π r^2

Where, r = radius of a circle.

We need to find, $\frac{dA}{dr}$, where A = Area of the Circle and r = radius of Circle.

$$\frac{dA}{dr} = 2\pi r$$

At r = 5 cm we have,

$$\frac{dA}{dr} = 2\pi 5$$

$$\frac{dA}{dr} = 10\pi \text{ cm}$$

7. Question

Find the rate of change of the volume of a ball with respect to its radius r. How fast is the volume changing with respect to the radius when the radius is 2 cm?

Answer

Volume of a Ball = Volume of Sphere = $\frac{4}{3}\pi r^3$

Where r = radius of the ball.

We need to find $\frac{dV}{dr}$, where V = Volume of Ball and r = radius of the ball.

$$\frac{dV}{dr} = 4\pi r^2$$

At r = 2 cm,

$$\frac{dV}{dr} = 4\pi (2)^2$$

$$\frac{dV}{dr} = 16\pi \text{ cm}^2$$

8. Question

The total cost C (x) associated with the production of x units of an item is given by C (x) = 0.007 x^3 – 0.003x^2 + 15x + 4000. Find the marginal cost when 17 units are produced.

Answer

Given: Total Cost, C (x) = 0.007 x^3 – 0.003x^2 + 15x + 4000

Where x = Number of units

Marginal Cost is given by,

Marginal Cost M(x) = $\frac{dC}{dx}$

Therefore, M(x) = 0.007 × 3 x^2 – 0.003 × 2 x + 15

At Number of units = 17, x = 17.

So, M(17) = 0.021 $(17)^2$ – 0.006 (17) + 15

M(17) = 20.97

Hence, the Marginal Cost of 17 units is 20.97.

9. Question

The total revenue received from the sale of x units of a product is given by R(x) = 13 x^2 + 26 x + 15. Find the marginal revenue when x = 7?

Answer

Given: Total Revenue, R(x) = 13 x^2 + 26 x + 15

Where x = number of units.

Marginal Revenue is given by,

$M(x) = \frac{d}{dx} R(x)$

Therefore, $\frac{d}{dx} R(x) = 26x + 26$

M(x) = 26 (x + 1)

For x = 7,

M(x) = 26(7 + 1)

M(x) = 26 × 8 = Rs. 207

10. Question

The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (Marginal Revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by

$R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when x = 5, and write which value does the question indicate.

Answer

Given: Marginal Revenue, $M(x) = \frac{d}{dx} R(x)$

M(x) proportional to Money spent on the welfare of employees.

Thus, M(x) = k (Money spent on the welfare of employees)

M(x) = 6 x + 36

At x = 5, M(x) = 66

Now at x = 6, M(x) = 72

Thus Marginal Revenue for x = 5 is 66

And when x increased from x = 5 to x = 6,

Marginal revenue also increases and thus the money spent on the welfare of employees increases.

Exercise 13.2

1. Question

The side of a square sheet is increasing at the rate of 4 cm per minute. At what rate is the area increasing when the side is 8 cm long?

Answer

Given: the side of a square sheet is increasing at the rate of 4 cm per minute.

To find rate of area increasing when the side is 8 cm long

let the side of the given square sheet be x cm at any instant time.

Then according to the given criteria,

Rate of side of the sheet increasing is, $\frac{dx}{dt} = 4 \text{ cm/min} \ldots (i)$

Then the area of the square sheet at any time t will be

$A = x^2 \text{ cm}^2$.

Applying derivative with respect to time on both sides we get,

$$\frac{dA}{dt} = \frac{d(x^2)}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2x\frac{dx}{dt}$$

$\Rightarrow \frac{dA}{dt} = 2x \times 4 = 8x \ldots..(ii)$ [from from equation(i)]

So when the side is 8cm long, the rate of area increasing will become

$\Rightarrow \frac{dA}{dt} = 8 \times 8$ [from equation(ii)]

$$\Rightarrow \frac{dA}{dt} = 64cm^2/min$$

Hence the area is increasing at the rate of $64cm^2/min$ when the side is 8 cm long

2. Question

An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge is 1 cm long?

Answer

Given: the edge of a variable cube is increasing at the rate of 3 cm per second.

To find rate of volume of the cube increasing when the edge is 1 cm long

Let the edge of the given cube be x cm at any instant time.

Then according to the given criteria,

Rate of edge of the cube increasing is, $\frac{dx}{dt} = 3 \text{ cm/sec} \ldots (i)$

Then the volume of the cube at any time t will be

$V = x^3 \text{ cm}^3$.

Applying derivative with respect to time on both sides we get,

$$\frac{dV}{dt} = \frac{d(x^3)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$\Rightarrow \frac{dV}{dt} = 3x^2 \times 3 = 9x^2 \ldots..(ii)$ [from equation(i)]

When the edge of the cube is 1cm long the rate of volume increasing becomes

$$\Rightarrow \frac{dV}{dt} = 9 \times (1)^2 = 9cm^3/sec$$

Hence the volume of the cube increasing at the rate of $9cm^3/sec$ when the edge of the cube is 1 cm long

3. Question

The side of a square is increasing at the rate of 0.2 cm/sec. Find the rate of increase of the perimeter of the square.

Answer

Given: the side of a square is increasing at the rate of 0.2 cm/sec.

To find rate of increase of the perimeter of the square

Let the edge of the given cube be x cm at any instant time.

Then according to the given criteria,

Rate of side of the square increasing is, $\frac{dx}{dt} = 0.2 \text{ cm/sec} \ldots \text{(i)}$

Then the perimeter of the square at any time t will be

P = 4x cm.

Applying derivative with respect to time on both sides we get,

$$\frac{dP}{dt} = \frac{d(4x)}{dt}$$

$$\Rightarrow \frac{dP}{dt} = 4\frac{dx}{dt}$$

$\Rightarrow \frac{dP}{dt} = 4 \times 0.2 = 0.8 \text{cm/sec}$ [from equation(i)]

Hence the rate of increase of the perimeter of the square will be 0.8cm/sec

4. Question

The radius of a circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?

Answer

Given: the radius of a circle is increasing at the rate of 0.7 cm/sec.

To find rate of increase of its circumference

Let the radius of the given circle be r cm at any instant time.

Then according to the given criteria,

Rate of radius of a circle is increasing is, $\frac{dr}{dt} = 0.7 \text{ cm/sec} \ldots \text{(i)}$

Then the circumference of the circle at any time t will be

C = 2πr cm.

Applying derivative with respect to time on both sides we get,

$$\frac{dC}{dt} = \frac{d(2\pi r)}{dt}$$

$$\Rightarrow \frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

$\Rightarrow \frac{dC}{dt} = 2\pi \times 0.7 = 1.4\pi \text{ cm/sec}$ [from equation(i)]

Hence the rate of increase of the circle’s circumference will be 1.4π cm/sec

5. Question

The radius of a spherical soap bubble is increasing at the rate of 0.2 cm/sec. Find the rate of increase of its surface area, when the radius is 7 cm.

Answer

Given: the radius of a spherical soap bubble is increasing at the rate of 0.2 cm/sec.

To find rate of increase of its surface area, when the radius is 7 cm

Let the radius of the given spherical soap bubble be r cm at any instant time.

Then according to the given criteria,

Rate of radius of the spherical soap bubble is increasing is, $\frac{dr}{dt} = 0.2 \text{ cm/sec} \ldots (i)$

Then the surface area of the spherical soap bubble at any time t will be

$S = 4\pi r^2 \text{ cm}^2$.

Applying derivative with respect to time on both sides we get,

$$\frac{dS}{dt} = \frac{d(4\pi r^2)}{dt}$$

$$\Rightarrow \frac{dS}{dt} = 4\pi \frac{d(r^2)}{dt}$$

$$\Rightarrow \frac{dS}{dt} = 4\pi \times 2r \frac{dr}{dt}$$

$\Rightarrow \frac{dS}{dt} = 4\pi \times 2r \times 0.2 = 1.6\pi r \ldots\ldots (ii)$ [from equation(i)]

So when the radius is 7cm, the rate of surface area will become,

$$\Rightarrow \frac{dS}{dt} = 1.6\pi \times (7)$$

$$\Rightarrow \frac{dS}{dt} = 11.2\pi \text{ cm}^2/\text{sec}$$

Hence the rate of increase of its surface area, when the radius is 7 cm is 11.2π cm^2/sec

6. Question

A balloon which always remains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.

Answer

Given: Spherical balloon inflated by pumping in 900 cubic centimetres of gas per second

To find the rate at which the radius of the balloon is increasing when the radius is 15 cm

Let the radius of the given spherical balloon be r cm and let V be the volume of the spherical balloon at any instant time

Then according to the given criteria,

As the balloon is inflated by pumping 900 cubic centimetres of gas per second hence the rate of volume of the spherical balloon increases by, $\frac{dV}{dt} = 900 \text{ cm}^3/\text{sec} \ldots (i)$

We know volume of the spherical balloon is $V = \frac{4}{3}\pi r^3$.

Applying derivative with respect to time on both sides we get,

$$\frac{dV}{dt} = \frac{d\left(\frac{4}{3}\pi r^3\right)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \frac{d(r^3)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt}$$

$\Rightarrow 900 = 4\pi r^2 \frac{dr}{dt}$ [from equation(i)]

So when the radius is 15cm, the above equation becomes,

$$\Rightarrow 900 = 4\pi \times (15)^2 \frac{dr}{dt}$$

$$\Rightarrow 900 = 900\pi \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{\pi} cm/sec$$

Hence the rate at which the radius of the balloon is increasing when the radius is 15 cm will be $\frac{1}{\pi}$ cm/sec.

7. Question

The radius of an air bubble is increasing at the rate of 0.5 cm/sec. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Answer

Given: radius of an air bubble is increasing at the rate of 0.5 cm/sec

To find the rate at which the volume of the bubble increasing when the radius is 1 cm

Let the radius of the given air bubble be r cm and let V be the volume of the air bubble at any instant time

Then according to the given criteria,

Rate of increase in the radius of the air bubble is, $\frac{dr}{dt} = 0.5\ cm/sec \ldots (i)$

We know volume of the air bubble is $V = \frac{4}{3}\pi r^3$.

Applying derivative with respect to time on both sides we get,

$$\frac{dV}{dt} = \frac{d\left(\frac{4}{3}\pi r^3\right)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \frac{d(r^3)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt}$$

$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \times 0.5$ [from equation(i)]

So when the radius is 1cm, the above equation becomes,

$$\Rightarrow \frac{dV}{dt} = 4\pi \times (1)^2 \times 0.5$$

$$\Rightarrow \frac{dV}{dt} = 2\pi\ cm^3/sec$$

Hence the rate at which the volume of the air bubble is increasing when the radius is 1 cm will be 2π cm^3/sec.

8. Question

A man 2 metres high walks at a uniform speed of 5 km/hr away from a lamp - post 6 metres high. Find the rate at which the length of his shadow increases.

Answer

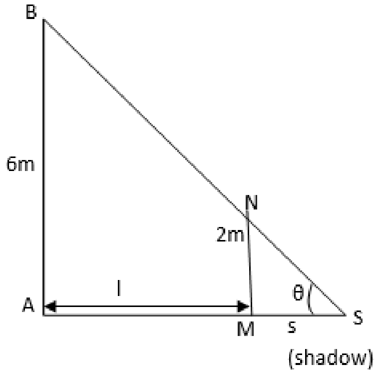
Given: man 2 metres high walks at a uniform speed of 5 km/hr away from a lamp - post 6 metres high

To find the rate at which the length of his shadow increases

Let AB be the lamp post and let MN be the man of height 2m.

Let AL = l meter and MS be the shadow of the man

Let length of the shadow MS = s (as shown in the below figure)



Given man walks at the speed of 5 km/hr

$$\therefore \frac{dl}{dt} = 5\frac{km}{h} \ldots.(i)$$

So the rate at which the length of the man's shadow increases will be $\frac{ds}{dt}$

Consider ΔASB

$$\tan\theta = \frac{AB}{AS}$$

$$\Rightarrow \tan\theta = \frac{6}{l + s} \ldots.(ii)$$

Now consider ΔMSN, we get

$$\tan\theta = \frac{MN}{MS}$$

$$\Rightarrow \tan\theta = \frac{2}{s} \ldots.(iii)$$

So from equation(ii) and (iii),

$$\frac{6}{l + s} = \frac{2}{s}$$

$$\Rightarrow 6s = 2(l + s)$$

$$\Rightarrow 6s - 2s = 2l$$

$$\Rightarrow l = 2s$$

Applying derivative with respect to time on both sides we get,

$$\frac{dl}{dt} = \frac{d(2s)}{dt}$$

$$\Rightarrow \frac{dl}{dt} = 2\frac{ds}{dt}$$

$\Rightarrow 5 = 2\frac{ds}{dt}$ [from equation(i)]

$$\Rightarrow \frac{ds}{dt} = \frac{5}{2} = 2.5km/hr$$

Hence the rate at which the length of his shadow increases by 2.5 km/hr, and it is independent to the current distance of the man from the base of the light.

9. Question

A stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm/sec. At the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Answer

Given: a stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm/sec.

To find the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing

Let r be the radius of the circle and A be the area of the circle

When stone is dropped into the lake waves moves in circle at speed of 4cm/sec.

i.e., radius of the circle increases at a rate of 4cm/sec

i.e., $\frac{dr}{dt} = 4cm/sec \dots\dots(i)$

We know that

Area of the circle is πr^2

Now,

$$\frac{dA}{dt} = \frac{d(\pi r^2)}{dt}$$

$$\Rightarrow \frac{dA}{dt} = \pi \frac{d(r^2)}{dt}$$

$$\Rightarrow \frac{dA}{dt} = \pi \times 2r\frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \times 4 \dots\dots..(ii)$$

So when the radius of the circular wave is 10 cm, the above equation becomes,

$$\Rightarrow \frac{dA}{dt} = 2\pi \times 10 \times 4$$

$$\Rightarrow \frac{dA}{dt} = 80\pi \ cm^2 \backslash sec$$

Hence the enclosed area is increasing at the rate of 80π cm^2/sec

10. Question

A man 160 cm tall walks away from a source of light situated at the top of a pole 6 m high, at the rate of 1.1m/sec. How fast is the length of his shadow increasing when he is 1m away from the pole?

Answer

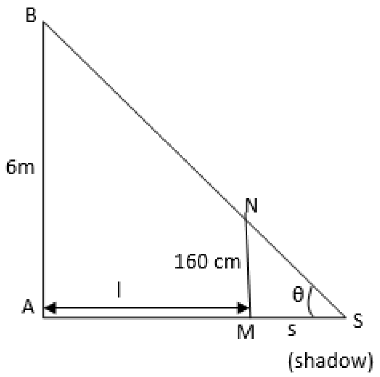
Given: man 160cm tall walks away from a source of light situated at the top of a pole 6 m high, at the rate of 1.1m/sec

To find the rate at which the length of his shadow increases when he is 1m away from the pole

Let AB be the lamp post and let MN be the man of height 160cm or 1.6m.

Let AL = l meter and MS be the shadow of the man

Let length of the shadow MS = s (as shown in the below figure)



Given man walks at the speed of 1.1 m/sec

$$\therefore \frac{dl}{dt} = 1.1 m/sec \ldots. (i)$$

So the rate at which the length of the man’s shadow increases will be $\frac{ds}{dt}$

Consider ΔASB

$$\tan\theta = \frac{AB}{AS}$$

$$\Rightarrow \tan\theta = \frac{6}{l + s} \ldots. (ii)$$

Now consider ΔMSN, we get

$$\tan\theta = \frac{MN}{MS}$$

$$\Rightarrow \tan\theta = \frac{1.6}{s} \ldots. (iii)$$

So from equation(ii) and (iii),

$$\frac{6}{l + s} = \frac{1.6}{s}$$

$$\Rightarrow 6s = 1.6(l + s)$$

$$\Rightarrow 6s - 1.6s = 1.6l$$

$$\Rightarrow l = 2.75s$$

Applying derivative with respect to time on both sides we get,

$$\frac{dl}{dt} = \frac{d(2.75s)}{dt}$$

$$\Rightarrow \frac{dl}{dt} = 2.75\frac{ds}{dt}$$

$\Rightarrow 1.1 = 2.75\frac{ds}{dt}$ [from equation(i)]

$$\Rightarrow \frac{ds}{dt} = 0.4 \text{ m/sec}$$

Hence the rate at which the length of his shadow increases by 0.4 m/sec, and it is independent to the current distance of the man from the base of the light.

11. Question

A man 180 cm tall walks at a rate of 2 m/sec. away, from a source of light that is 9 m above the ground. How fast is the length of his shadow increasing when he is 3 m away from the base of light?

Answer

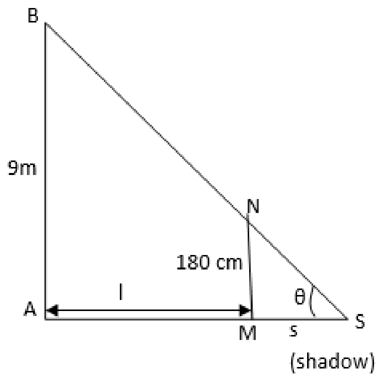
Given: man 180cm tall walks at a rate of 2 m/sec away; from a source of light that is 9 m above the ground

To find the rate at which the length of his shadow increases when he is 3m away from the pole

Let AB be the lamp post and let MN be the man of height 180cm or 1.8m.

Let AL = l meter and MS be the shadow of the man

Let length of the shadow MS = s (as shown in the below figure)



Given man walks at the speed of 2m/sec

$$\therefore \frac{dl}{dt} = 2m/sec \ldots.(i)$$

So the rate at which the length of the man's shadow increases will be $\frac{ds}{dt}$

Consider ΔASB

$$\tan\theta = \frac{AB}{AS}$$

$$\Rightarrow \tan\theta = \frac{9}{l + s} \ldots.(ii)$$

Now consider ΔMSN, we get

$$\tan\theta = \frac{MN}{MS}$$

$$\Rightarrow \tan\theta = \frac{2}{s} \ldots.(iii)$$

So from equation(ii) and (iii),

$$\frac{9}{l + s} = \frac{2}{s}$$

$$\Rightarrow 9s = 2(l + s)$$

$$\Rightarrow 9s - 2s = 2l$$

$$\Rightarrow l = 3.5s$$

Applying derivative with respect to time on both sides we get,

$$\frac{dl}{dt} = \frac{d(3.5s)}{dt}$$

$\Rightarrow \frac{dl}{dt} = 3.5\frac{ds}{dt}$

$\Rightarrow 2 = 3.5\frac{ds}{dt}$ [from equation(i)]

$\Rightarrow \frac{ds}{dt} = 0.6\ m/sec$

Hence the rate at which the length of his shadow increases by 0.6 m/sec, and it is independent to the current distance of the man from the base of the light.

12. Question

A ladder 13 m long leans against a wall. The foot of the ladder is pulled along the ground away from the wall, at the rate of 1.5m/sec. How fast is the angle θ between the ladder and the ground is changing when the foot of the ladder is 12 m away from the wall.

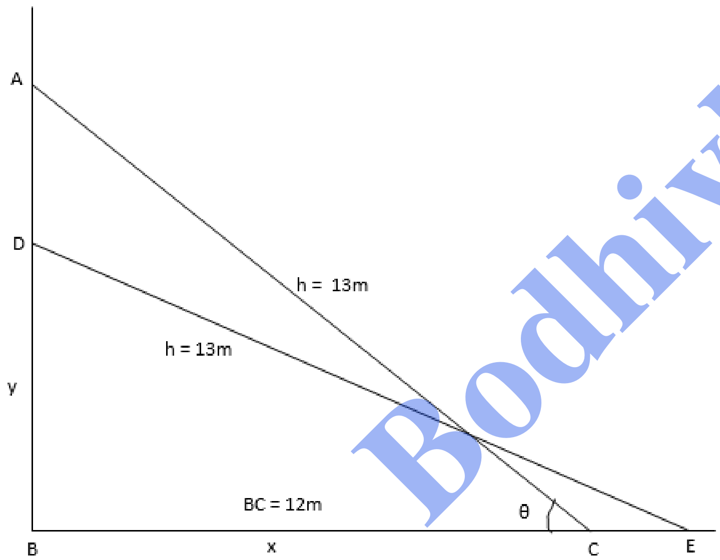
Answer

Given: a ladder 13 m long leans against a wall. The foot of the ladder is pulled along the ground away from the wall, at the rate of 1.5m/sec

To find how fast is the angle θ between the ladder and the ground is changing when the foot of the ladder is 12 m away from the wall

Let AC be the position of the ladder initially, then AC = 13m.

DE be the position of the ladder after being pulled at the rate of 1.5m/sec, then DE = 13m as shown in the below figure.



So it is given that foot of the ladder is pulled along the ground away from the wall, at the rate of 1.5m/sec

$\therefore \frac{dx}{dt} = 1.5m/sec$..............(i)

Consider ΔABC, it is right angled triangle, so by applying the Pythagoras theorem, we get

$AB^2 + BC^2 = AC^2$

$\Rightarrow y^2 + x^2 = h^2$..........(ii)

$\Rightarrow y^2 + (12)^2 = (13)^2$

$\Rightarrow y^2 = 169 - 144 = 25$

$\Rightarrow y = 5$

And in same triangle,

$$\sec\theta = \frac{AC}{BC} = \frac{13}{12} \ldots . (iii)$$

Now differentiate equation(ii) with respect to time, we get

$$\frac{d(y^2 + x^2)}{dt} = \frac{d(h^2)}{dt}$$

$$\Rightarrow \frac{d(y^2)}{dt} + \frac{d(x^2)}{dt} = \frac{d(h^2)}{dt}$$

$$\Rightarrow 2y\frac{dy}{dt} + 2x\frac{dx}{dt} = 2h\frac{dh}{dt}$$

Now substituting the values of x, y, h and $\frac{dx}{dt}$(from equation(i)), we get

$$\Rightarrow 2(5)\frac{dy}{dt} + 2(12)(1.5) = 2(13)\frac{dh}{dt}$$

The value of h is always constant as the ladder is not increasing or decreasing in size, hence the above equation becomes,

$$\Rightarrow 10\frac{dy}{dt} + 24(1.5) = 2(13)(0)$$

$$\Rightarrow 10\frac{dy}{dt} = -36$$

$$\Rightarrow \frac{dy}{dt} = -3.6 \ldots \ldots (iv)$$

And considering the same triangle,

$$\tan\theta = \frac{AB}{BC} = \frac{y}{x}$$

Differentiating the above equation with respect to time we get

$$\frac{d(\tan\theta)}{dt} = \frac{d\left(\frac{y}{x}\right)}{dt}$$

$\Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$ [applying quotient rule $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

Substituting the values of $\sec\theta$ (from equation(iii)), x, y, $\frac{dy}{dt}$ (from equation(iv) and $\frac{dx}{dt}$ (from equation(i)) the above equation becomes,

$$\Rightarrow \left(\frac{13}{12}\right)^2\frac{d\theta}{dt} = \frac{12 \times (-3.6) - 5 \times 1.5}{(12)^2}$$

$$\Rightarrow \left(\frac{169}{144}\right)\frac{d\theta}{dt} = \frac{-43.2 - 7.5}{144}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{-50.7}{144} \times \frac{144}{169}$$

$$\Rightarrow \frac{d\theta}{dt} = -0.3 rad/sec$$

Hence the angle θ between the ladder and the ground is changing at the rate of - 0.3 rad/sec when the foot of the ladder is 12 m away from the wall.

13. Question

A particle moves along the curve $y = x^2 + 2x$. At what point(s) on the curve are the x and y coordinates of the particle changing at the same rate?

Answer

Given: a particle moves along the curve $y = x^2 + 2x$.

To find the points at which the curve are the x and y coordinates of the particle changing at the same rate

Equation of curve is $y = x^2 + 2x$

Differentiating the above equation with respect to x, we get

$$\frac{dy}{dx} = \frac{d(x^2 + 2x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d(x^2)}{dx} + \frac{d(2x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 2x\frac{dx}{dx} + 2\frac{dx}{dx} = 2x + 2 \ldots\ldots\ldots (i)$$

When x and y coordinates of the particle are changing at the same rate, we get

$$\frac{dy}{dt} = \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dt}$$

$$\Rightarrow \frac{dy}{dx} = 1$$

Now substitute the value from eqn(i), we get

$2x + 2 = 1 \Rightarrow 2x = -1$

$$\Rightarrow x = -\frac{1}{2}$$

Substitute this value of x in the given equation of curve, we get

$y = x^2 + 2x$

$$\Rightarrow y = \left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right)$$

$$\Rightarrow y = \frac{1}{4} - 1$$

$$\Rightarrow y = -\frac{3}{4}$$

Hence the points at which the curve are the x and y coordinates of the particle changing at the same rate is $\left(-\frac{1}{2}, -\frac{3}{4}\right)$

14. Question

If $y = 7x - x^3$ and x increases at the rate of 4 units per second, how fast is the slope of the curve changing when x = 2?

Answer

Given: equation of curve $y = 7x - x^3$ and x increases at the rate of 4 units per second.

To find how fast is the slope of the curve changing when x = 2

Equation of curve is y = 7x – x^3

Differentiating the above equation with respect to x, we get slope of the curve

$$\frac{dy}{dx} = \frac{d(7x - x^3)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d(7x)}{dx} - \frac{d(x^3)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 7\frac{dx}{dx} - 3x^2\frac{dx}{dx} = 7 - 3x^2 \ldots\ldots\ldots(i)$$

Let m be the slope of the given curve then the above equation becomes,

m = 7 - $3x^2$.........(ii)

And it is given x increases at the rate of 4 units per second, so

$$\frac{dx}{dt} = 4 \text{ units/sec} \ldots\ldots..(iii)$$

Now differentiating the equation of slope i.e., equation(ii) we get

$$\frac{dm}{dt} = \frac{d(7 - 3x^2)}{dt}$$

$$\Rightarrow \frac{dm}{dt} = \frac{d(7)}{dt} - \frac{d(3x^2)}{dt}$$

$$\Rightarrow \frac{dm}{dt} = 0 - (3 \times 2x)\frac{dx}{dt}$$

$\Rightarrow \frac{dm}{dt} = -6x\frac{dx}{dt} = -6x \times 4 \ldots..(iv)$ [by substituting the value of $\frac{dx}{dt}$ from equation (iii)]

When x = 2, equation(iv) becomes,

$$\Rightarrow \frac{dm}{dt} = -6x \times 4 = -6 \times 2 \times 4 = -48$$

Hence the slope of the curve is changing at the rate of - 48 units/sec when x = 2

15. Question

A particle moves along the curve y = x^3. Find the points on the curve at which the y - coordinate changes three times more rapidly than the x - coordinate.

Answer

Given: a particle moves along the curve y = x^3.

To find the points on the curve at which the y - coordinate changes three times more rapidly than the x - coordinate

Equation of curve is y = x^3

Differentiating the above equation with respect to t, we get

$$\frac{dy}{dt} = \frac{d(x^3)}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{d(x^3)}{dt}$$

$$\Rightarrow \frac{dy}{dt} = 3x^2\frac{dx}{dt} \ldots\ldots\ldots(i)$$

When y - coordinate changes three times more rapidly than the x - coordinate, i.e.,

$$\frac{dy}{dt} = 3\frac{dx}{dt} \ldots\ldots (ii)$$

Equating equation (i) and equation (ii), we get

$$3x^2\frac{dx}{dt} = 3\frac{dx}{dt}$$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

When x = 1, $y = x^3 = (1)^3 \Rightarrow y = 1$

When x = - 1, $y = x^3 = (-1)^3 \Rightarrow y = -1$

Hence the points on the curve at which the y - coordinate changes three times more rapidly than the x - coordinate are (1, 1) and (- 1, - 1).

16 A. Question

Find an angle θ

which increases twice as fast as its cosine.

Answer

which increases twice as fast as its cosine.

As per the given condition,

θ = 2 cos θ

Now differentiating the above equation with respect to time we get

$$\frac{d\theta}{dt} = 2\frac{d(\cos\theta)}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = 2\frac{d(\cos\theta)}{d\theta}.\frac{d\theta}{dt}$$

$$\Rightarrow 1 = 2(-\sin\theta)$$

$$\Rightarrow \sin\theta = -\frac{1}{2}$$

Hence, $\theta = \frac{7\pi}{6}$

So the value of angle θ which increases twice as fast as its cosine is $\frac{7\pi}{6}$

16 B. Question

Find an angle θ

whose rate of increase twice is twice the rate of decrease of its cosine.

Answer

whose rate of increase twice is twice the rate of decrease of its consine

As per the given condition,

$\frac{d\theta}{dt} = -2\frac{d(\cos\theta)}{dt}$ as the rate of increase is twice the rate of decrease, hence the minus sign.

$$\Rightarrow \frac{d\theta}{dt} = -2\frac{d(\cos\theta)}{d\theta}.\frac{d\theta}{dt}$$

$$\Rightarrow 1 = -2(-\sin\theta)$$

$\Rightarrow \sin\theta = \frac{1}{2}$

Hence, $\theta = \frac{\pi}{6}$

So the value of angle θ hose rate of increase twice is twice the rate of decrease of its consine is $\frac{\pi}{6}$

17. Question

The top of a ladder 6 metre long is resting against a vertical wall on a level pavement, when the ladder begins to slide outwards. At the moment when the foot of the ladder is 4 metres from the wall, it is sliding away from the wall at the rate of 0.5 m/sec. How fast is the top - sliding downwards at this instance?

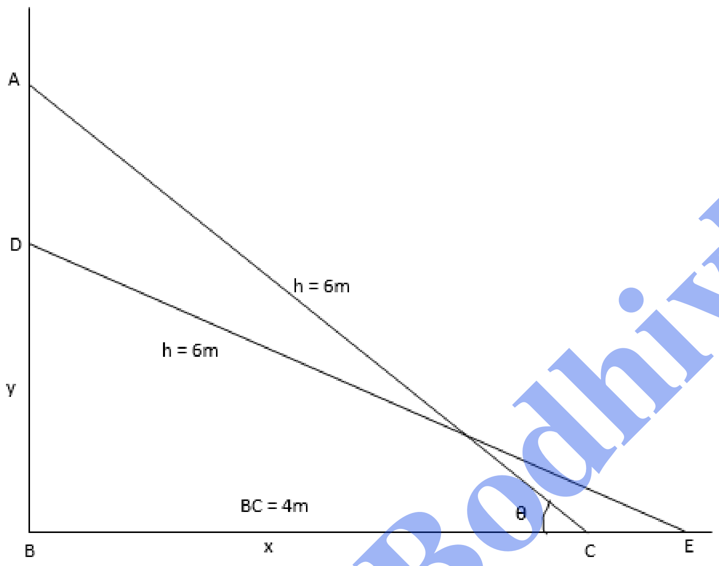
Answer

Given: the top of a ladder 6 metre long is resting against a vertical wall on a level pavement, when the ladder begins to slide outwards, at the moment when the foot of the ladder is 4 metres from the wall, it is sliding away from the wall at the rate of 0.5 m/sec

To find how fast is the top - sliding downwards at this instance

Let AC be the position of the ladder initially, then AC = 6m.

DE be the position of the ladder after being pulled at the rate of 0.5m/sec, then DE = 6m as shown in the below figure.



So it is given that foot of the ladder is pulled along the ground away from the wall, at the rate of 0.5m/sec

$\therefore \frac{dx}{dt} = 0.5\text{m/sec}$..............(i)

Consider ΔABC, it is right angled triangle, so by applying the Pythagoras theorem, we get

$AB^2 + BC^2 = AC^2$

$\Rightarrow y^2 + x^2 = h^2$..........(ii)

$\Rightarrow y^2 + (4)^2 = (6)^2$

$\Rightarrow y^2 = 36 - 16 = 20$

$\Rightarrow y = 4.47$

And in same triangle,

$\sec\theta = \frac{AC}{BC} = \frac{6}{4}\ldots.(iii)$

Now differentiate equation(ii) with respect to time, we get

$$\frac{d(y^2 + x^2)}{dt} = \frac{d(h^2)}{dt}$$

$$\Rightarrow \frac{d(y^2)}{dt} + \frac{d(x^2)}{dt} = \frac{d(h^2)}{dt}$$

$$\Rightarrow 2y\frac{dy}{dt} + 2x\frac{dx}{dt} = 2h\frac{dh}{dt}$$

Now substituting the values of x, y, h and $\frac{dx}{dt}$(from equation(i)), we get

$$\Rightarrow 2(4.47)\frac{dy}{dt} + 2(4)(0.5) = 2(6)\frac{dh}{dt}$$

The value of h is always constant as the ladder is not increasing or decreasing in size, hence the above equation becomes,

$$\Rightarrow 8.94\frac{dy}{dt} + 8(0.5) = 2(6)(0)$$

$$\Rightarrow 8.94\frac{dy}{dt} = -4$$

$$\Rightarrow \frac{dy}{dt} = -0.447 \ldots \ldots (iv)$$

And considering the same triangle,

$$\tan\theta = \frac{AB}{BC} = \frac{y}{x}$$

Differentiating the above equation with respect to time we get

$$\frac{d(\tan\theta)}{dt} = \frac{d\left(\frac{y}{x}\right)}{dt}$$

$\Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$ [applying quotient rule $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

Substituting the values of $\sec\theta$ (from equation(iii)), x, y, $\frac{dy}{dt}$ (from equation(iv) and $\frac{dx}{dt}$ (from equation(i)) the above equation becomes,

$$\Rightarrow \left(\frac{6}{4}\right)^2\frac{d\theta}{dt} = \frac{4 \times (-0.447) - 4.47 \times 0.5}{(4)^2}$$

$$\Rightarrow \left(\frac{36}{16}\right)\frac{d\theta}{dt} = \frac{-1.788 - 2.235}{16}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{-4.023}{16} \times \frac{16}{36}$$

$$\Rightarrow \frac{d\theta}{dt} = -0.11 rad/sec$$

Hence the angle θ between the ladder and the ground is changing at the rate of - 0.11 rad/sec when the foot of the ladder is 4 m away from the wall.

18. Question

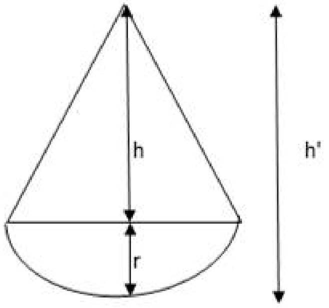
A balloon in the form of a right circular cone surmounted by a hemisphere, having a diameter equal to the height of the cone, is being inflated. How fast is its volume changing with respect to its total height h, when h = 9 cm.

Answer

Given: a balloon in the form of a right circular cone surmounted by a hemisphere, having a diameter equal to the height of the cone, is being inflated

To find: how fast is its volume changing with respect to its total height h, when h = 9 cm

Solution:



Let height of the cone be h'

And the radius of the hemisphere be r

As per the given criteria,

$$h' = 2r \Rightarrow r = \frac{h'}{2}$$

Let the total height of the balloon be h

Then $h = h' + r = h' + \frac{h'}{2} = \frac{3h'}{2}$

$\Rightarrow h' = \frac{2h}{3}$............(i)

So,

Volume of the balloon (V) = Volume of the cone + Volume of the hemisphere

$$V = \frac{1}{3}\pi r^2 h' + \frac{2}{3}\pi r^3$$

$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{h'}{2}\right)^2 h' + \frac{2}{3}\pi\left(\frac{h'}{2}\right)^3$$

$$\Rightarrow V = \frac{1}{12}\pi h'^3 + \frac{1}{12}\pi h'^3 = \frac{2}{12}\pi h'^3$$

$$\Rightarrow V = \frac{1}{6}\pi h'^3$$

This is the volume of the balloon

Now will substitute the value of h' from equation (i), we get

$$V = \frac{1}{6}\pi\left(\frac{2h}{3}\right)^3$$

$$\Rightarrow V = \frac{4}{81}\pi h^3$$

Now differentiate the above equation with respect to h, we get

$$\frac{dV}{dh} = \frac{d\left(\frac{4}{81}\pi h^3\right)}{dh}$$

$$\Rightarrow \frac{dV}{dh} = \frac{4\pi}{81} \times 3h^2 \frac{dh}{dh}$$

$$\Rightarrow \frac{dV}{dh} = \frac{4}{27}\pi h^2$$

When h = 9cm, we get

$$\frac{dV}{dh} = \frac{4}{27}\pi h^2$$

$$\Rightarrow \frac{dV}{dh} = \frac{4}{27}\pi(9)^2$$

$$\Rightarrow \frac{dV}{dh} = 12\pi\ cm^2$$

Hence at the rate of 12 π cm^2 the volume changes with respect to its total height.

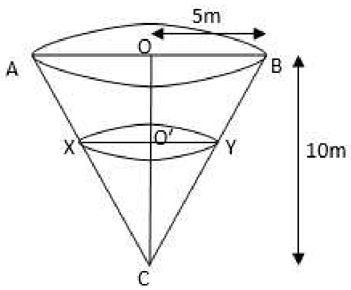
19. Question

Water is running into an inverted cone at the rate of π cubic metres per minute. The height of the cone is 10 metres, and the radius of its base is 5 m. How fast the water level is rising when the water stands 7.5 m below the base.

Answer

Given: The water is running into an inverted cone at the rate of π cubic metres per minute. The height of the cone is 10 metres, and the radius of its base is 5 m.

To find how fast the water level is rising when the water stands 7.5 m below the base.



Let the height of the cone be H = 10m (given)

Let the radius of the base be R = 5m (given)

Let O'Y = r and CO' = h

Now from the above figure,

ΔCOB~ΔCO'Q

So,

$$\frac{O'Y}{OB} = \frac{CO'}{CO} \Rightarrow \frac{r}{5} = \frac{h}{10} \Rightarrow r = \frac{h}{2} \ldots\ldots(i)$$

Let the volume of the water in the vessel at any time t be V

Then,

$$V = \frac{1}{3}\pi r^2 h$$

$\Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ (from equation (i))

$$\Rightarrow V = \frac{1}{12}\pi h^3$$

Now differentiate the above equation with respect to t, we get

$$\frac{dV}{dt} = \frac{d\left(\frac{1}{12}\pi h^3\right)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{1}{12}\pi \times 3h^2 \frac{dh}{dt}$$

But given the water is running at the rate of πm^3/min, i.e., $\frac{dV}{dt} = \pi$

So the above equation becomes

$$\Rightarrow \pi = \frac{1}{12}\pi \times 3h^2 \frac{dh}{dt}$$

$$\Rightarrow 1 = \frac{1}{4}h^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{4}{h^2}$$

So when the water stands 7.5 m below the base

So h = 10 - 7.5 = 2.5m, the rate becomes

$$\Rightarrow \frac{dh}{dt} = \frac{4}{h^2}$$

$$\Rightarrow \frac{dh}{dt} = \frac{4}{(2.5)^2} = 0.64 \text{ m/min}$$

Hence the rate of water level rising when the water stands 7.5m below the base is 0.64 metres per min

20. Question

A man 2 metres high walks at a uniform speed of 6 km/h away from a lamp - post 6 metres high. Find the rate at which the length of his shadow increases.

Answer

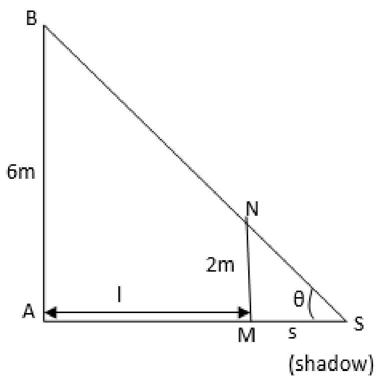
Given: man 2m tall walks at a speed of 6km/h away; from a source of light that is 6 m above the ground

To find the rate at which the length of his shadow increases

Let AB be the lamp post and let MN be the man of height 2m.

Let AL = l meter and MS be the shadow of the man

Let length of the shadow MS = s (as shown in the below figure)



Given man walks at the speed of 6 km/h

$$\therefore \frac{dl}{dt} = 6 \text{ km/h} \ldots (i)$$

So the rate at which the length of the man's shadow increases will be $\frac{ds}{dt}$

Consider ΔASB

$$\tan\theta = \frac{AB}{AS}$$

$$\Rightarrow \tan\theta = \frac{9}{l + s} \ldots. (ii)$$

Now consider ΔMSN, we get

$$\tan\theta = \frac{MN}{MS}$$

$$\Rightarrow \tan\theta = \frac{2}{s} \ldots. (iii)$$

So from equation(ii) and (iii),

$$\frac{9}{l + s} = \frac{2}{s}$$

$$\Rightarrow 9s = 2(l + s)$$

$$\Rightarrow 9s - 2s = 2l$$

$$\Rightarrow l = 3.5s$$

Applying derivative with respect to time on both sides we get,

$$\frac{dl}{dt} = \frac{d(3.5s)}{dt}$$

$$\Rightarrow \frac{dl}{dt} = 3.5\frac{ds}{dt}$$

$\Rightarrow 2 = 3.5\frac{ds}{dt}$ [from equation(i)]

$$\Rightarrow \frac{ds}{dt} = 0.6 \text{ km/hr}$$

Hence the rate at which the length of his shadow increases by 0.6 km/hr, and it is independent to the current distance of the man from the base of the light.

21. Question

The surface area of a spherical bubble is increasing at the rate of 2 cm^2/s. When the radius of the bubble is 6 cm, at what rate is the volume of the bubble increasing?

Answer

Given: the surface area of a spherical bubble is increasing at the rate of 0.2 cm^2/sec

To find rate of increase of its volume, when the radius is 6cm

Let the radius of the given spherical bubble be r cm at any instant time.

It is given that the surface area of a spherical bubble is increasing at the rate of 0.2 cm^2/sec

So the surface area of the bubble will be,

$SA = 4\pi r^2$

Now differentiating the above equation with respect to time we get

$$\frac{d(SA)}{dt} = \frac{d(4\pi r^2)}{dt}$$

$$\Rightarrow \frac{d(SA)}{dt} = 4\pi \times 2r\frac{dr}{dt}$$

This is the rate of surface area increasing = 0.2cm^2/sec, hence the rate at which the radius of the bubble is increasing becomes,

$$\Rightarrow \frac{dr}{dt} = \frac{d(SA)}{dt} \times \frac{1}{8\pi r}$$

$$\Rightarrow \frac{dr}{dt} = 0.2 \times \frac{1}{8\pi r} \ldots.(i)$$

Then the volume of the spherical bubble at any time t will be

V = 4πr^3 cm^3.

Applying derivative with respect to time on both sides we get,

$$\frac{dV}{dt} = \frac{d(4\pi r^3)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi\frac{d(r^3)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi \times 3r^2\frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 12\pi r^2 \times 0.2 \times \frac{1}{8\pi r}$$ [from equation(i)]

$$\Rightarrow \frac{dV}{dt} = 0.3r \ldots\ldots (ii)$$

So when the radius is 6cm, the rate of volume will become,

$$\Rightarrow \frac{dV}{dt} = 0.3(6)$$

$$\Rightarrow \frac{dV}{dt} = 1.8\ cm^3/sec$$

Hence the rate of increase of its volume, when the radius is 6cm is 1.8 cm^3/sec

22. Question

The radius of a cylinder is increasing at the rate 2 cm/sec and its altitude is decreasing at the rate of 3 cm/sec. Find the rate of change of volume when radius is 3 cm and altitude 5cm.

Answer

Given: the radius of a cylinder is increasing at the rate 2 cm/sec and its altitude is decreasing at the rate of 3 cm/sec

To find the rate of change of volume when radius is 3 cm and altitude 5cm

Let V be the volume of the cylinder, r be its radius and h be its altitude at any instant of time ‘t’.

We know volume of the cylinder is

V = πr^{2}h

Differentiating this with respect to time we get

$$\frac{dV}{dt} = \frac{d(\pi r^2 h)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \pi\left[\frac{d(r^2h)}{dt}\right]$$

Now will apply the product rule of differentiation, i.e.,

$\frac{d(uv)}{dx} = v\frac{d(u)}{dx} + u\frac{d(v)}{dx}$, so the above equation becomes,

$$\Rightarrow \frac{dV}{dt} = \pi\left[h\frac{d(r^2)}{dt} + r^2\frac{d(h)}{dt}\right]$$

$$\Rightarrow \frac{dV}{dt} = \pi\left[h \times 2r\frac{dr}{dt} + r^2\frac{dh}{dt}\right] \ldots..(i)$$

But given of a cylinder is increasing at the rate 2 cm/sec, i.e., $\frac{dr}{dt} = 2cm/sec$ and its altitude is decreasing at the rate of 3 cm/sec, i.e., $\frac{dh}{dt} = -3cm/sec$, by subsitituting the above values in equation (i) we get

$$\Rightarrow \frac{dV}{dt} = \pi[h \times 2r(2) + r^2(-3)] = \pi(4hr - 3r^2) \ldots.(ii)$$

When radius of the cylinder, r = 3cm and its altitude, h = 5cm, the equation (ii) becomes,

$$\Rightarrow \frac{dV}{dt} = \pi(4 \times 5 \times 3 - 3(3)^2)$$

$$\Rightarrow \frac{dV}{dt} = \pi(60 - 27)$$

$$\Rightarrow \frac{dV}{dt} = 33\pi\ cm^3/sec$$

Hence the rate of change of volume when radius is 3 cm and altitude 5cm is 33π cm^3/sec

23. Question

The volume of metal in a hollow sphere is constant. If the inner radius is increasing at the rate of 1 cm/sec, find the rate of increase of the outer radius when the radii are 4 cm and 8cm respectively.

Answer

Let the inner radius be r, outer radius be R and volume be V of a hollow sphere at any instant of time

We know the volume of the hollow sphere is

$$V = \frac{4}{3}\pi(R^3 - r^3)$$

Differentiating the volume with respect to time, we get

$$\frac{dV}{dt} = \frac{d\left(\frac{4}{3}\pi(R^3 - r^3)\right)}{dt}$$

This is the rate of the volume of the hollow sphere and it is given this is constant hence

$$\Rightarrow 0 = \frac{4}{3}\pi\frac{d\left((R^3 - r^3)\right)}{dt}$$

$$\Rightarrow \frac{d(R^3)}{dt} - \frac{d(r^3)}{dt} = 0$$

$$\Rightarrow 3R^2\frac{dR}{dt} - 3r^2\frac{dr}{dt} = 0$$

Given that the rate of increase in inner radius of the hollow sphere, $\frac{dr}{dt} = 1cm/sec$, So the above equation becomes,

$$\Rightarrow 3R^2\frac{dR}{dt} - 3r^2(1) = 0$$

$$\Rightarrow 3R^2\frac{dR}{dt} = 3r^2$$

So when the radii are 4cm and 8 cm, the above equation becomes,

$$\Rightarrow 3(8)^2\frac{dR}{dt} = 3(4)^2$$

$$\Rightarrow \frac{dR}{dt} = \frac{3(4^2)}{3(8)^2}$$

$$\Rightarrow \frac{dR}{dt} = 0.25\ cm/sec$$

Therefore the rate of increase of the outer radius when the radii are 4 cm and 8cm respectively is 0.25 cm/sec

24. Question

Sand is being poured onto a conical pile at the constant rate of 50 cm^3/minute such that the height of the cone is always one half of the radius of its base. How fast is the height of the pile increasing when the sand is 5 cm deep?

Answer

Let the volume be V, height be h and radius be r of the cone at any instant of time.

We know, volume of the cone is

$$V = \frac{1}{3}\pi r^2 h$$

And its given height of the cone is always one half of the radius of its base, i.e., $h = \frac{r}{2} \Rightarrow r = 2h$

So the new volume becomes

$$V = \frac{1}{3}\pi(2h)^2 h$$

$$\Rightarrow V = \frac{4}{3}\pi h^3$$

Differentiate the above equation with respect to time, we get

$$\frac{dV}{dt} = \frac{d\left(\frac{4}{3}\pi h^3\right)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi\frac{d(h^3)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \times 3h^2\frac{dh}{dt}$$

And it is also given that the sand is being poured onto a conical pile at the constant rate of 50 cm^3/minute, so $\frac{dV}{dt} = 50cm^3/min$, so the above equation becomes,

$$\Rightarrow 50 = \frac{4}{3}\pi \times 3h^2\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{50 \times 3}{4 \times 3\pi h^2}$$

Now when height of the pile is 5cm, the above equation becomes

$$\Rightarrow \frac{dh}{dt} = \frac{50 \times 3}{12\pi(5)^2}$$

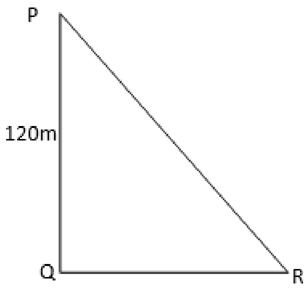
$$\Rightarrow \frac{dh}{dt} = \frac{1}{2\pi}$$

Therefore, the rate of height of the pile increasing when the sand is 5 cm deep is $\frac{1}{2\pi}$cm/min.

25. Question

A kite is 120 m high and 130 m of string is out. If the kite is moving away horizontally at the rate of 52 m/sec, find the rate at which the string is being paid out.

Answer



Let P be the position of the kite and PR be the position of the string. Let QR = x and PR = y. Then from figure by applying Pythagoras theorem, we get

$PR^2 = PQ^2 + QR^2$

$\Rightarrow y^2 = (120)^2 + x^2$........(i)

Differentiating the above equation with respect to time, we get

$$\frac{d(y^2)}{dt} = \frac{d((120)^2 + x^2)}{dt}$$

$$\Rightarrow 2y\frac{dy}{dt} = \frac{d((120)^2)}{dt} + \frac{d(x^2)}{dt}$$

$$\Rightarrow 2y\frac{dy}{dt} = 0 + 2x\frac{dx}{dt}$$

Now the kite is moving away horizontally at the rate of 52m/sec, so $\frac{dx}{dt} = 52 \text{ m/sec}$, so the above equation becomes

$\Rightarrow 2y\frac{dy}{dt} = 2x \times 52$.....(ii)

So when the string is 130m, y = 130m equation (i) become,

$y^2 = x^2 + (120)^2$

$\Rightarrow (130)^2 = x^2 + 14400$

$\Rightarrow x^2 = 16900 - 14400 = 2500$

Therefore x = 50m

Substituting the value of x and y in equation (ii), we get

$$\Rightarrow 2y\frac{dy}{dt} = 2x \times 52$$

$$\Rightarrow 2(130)\frac{dy}{dt} = 2(50) \times 52$$

$$\Rightarrow \frac{dy}{dt} = \frac{2(50) \times 52}{2(130)} = 20m/sec$$

Hence the rate at which the string is being paid out is 20m/sec

26. Question

A particle moves along the curve y = (2/3)x^3 + 1. Find the points on the curve at which the y - coordinate is changing twice as fast as the x - coordinate.

Answer

Given: a particle moves along the curve $y = \left(\frac{2}{3}\right)x^3 + 1$.

To find the points on the curve at which the y - coordinate is changing twice as fast as the x - coordinate.

Equation of curve is $y = \left(\frac{2}{3}\right)x^3 + 1$

Differentiating the above equation with respect to t, we get

$$\frac{dy}{dt} = \frac{d\left(\left(\frac{2}{3}\right)x^3 + 1\right)}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{d\left(\left(\frac{2}{3}\right)x^3\right)}{dt} + \frac{d(1)}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{2}{3} \times 3x^2\frac{dx}{dt} + 0 \ldots\ldots\ldots (i)$$

When y - coordinate is changing twice as fast as the x - coordinate, i.e.,

$$\frac{dy}{dt} = 2\frac{dx}{dt} \ldots\ldots (ii)$$

Equating equation (i) and equation (ii), we get

$$2x^2\frac{dx}{dt} = 2\frac{dx}{dt}$$

⇒ x^2 = 1 ⇒ x = ±1

When x = 1, $y = \left(\frac{2}{3}\right)(1)^3 + 1 \Rightarrow y = \frac{2+3}{3} \Rightarrow y = \frac{5}{3}$

When x = - 1, $y = \left(\frac{2}{3}\right)(-1)^3 + 1 \Rightarrow y = \frac{-2+3}{3} \Rightarrow y = \frac{1}{3}$

Hence the points on the curve at which the y - coordinate changes twice as fast as the x - coordinate are $\left(1, \frac{5}{3}\right)$ and $\left(-1, \frac{1}{3}\right)$

27. Question

Find the point on the curve y^2 = 8x for which the abscissa and ordinate change at the same rate.

Answer

Given: a point on the curve y^2 = 8x.

To find the point on the curve at which the abscissa and ordinate change at the same rate

Equation of curve is $y^2 = 8x$

Differentiating the above equation with respect to t, we get

$$\frac{d(y^2)}{dt} = \frac{d(8x)}{dt}$$

$$\Rightarrow 2y\frac{dy}{dt} = 8\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{y}{4}\frac{dy}{dt} \ldots\ldots\ldots (i)$$

When abscissa and ordinate change at the same rate, i.e.,

$$\frac{dy}{dt} = \frac{dx}{dt} \ldots\ldots (ii)$$

Equating equation (i) and equation (ii), we get

$$\frac{y}{4}\frac{dy}{dt} = \frac{dy}{dt}$$

$\Rightarrow y = 4$

When y, $y^2 = 8x \Rightarrow (4)^2 = 8x \Rightarrow x = 2$

Hence the point on the curve at which the abscissa and ordinate change at the same rate is (2,4)

28. Question

The volume of a cube is increasing at the rate of 9 cm^3/sec. How fast is the surface area increasing when the length of an edge is 10 cm?

Answer

Let x be the edge of the cube, V be the volume of the cube at any instant of the time.

We know,

$V = x^3$

Differentiating the above equation with respect to time, we get

$$\frac{dV}{dt} = \frac{d(x^3)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

But is is given that the volume of the cube is increasing at the rate of $9cm^3$/sec, so the above equation becomes,

$$\Rightarrow 9 = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{9}{3x^2} \ldots . (ii)$$

We also know that the surface area of the cube is

$S = 6x^2$

Again differentiating the above equation with respect to time is

$$\frac{dS}{dt} = \frac{d(6x^2)}{dt}$$

$$\Rightarrow \frac{dS}{dt} = 6 \times 2x\frac{dx}{dt}$$

Substitute equation (ii) in above equation, we get

$$\Rightarrow \frac{dS}{dt} = 6 \times 2x \times \frac{9}{3x^2}$$

$$\Rightarrow \frac{dS}{dt} = \frac{36}{x}$$

When the edge is of length 10 cm, we get

$$\Rightarrow \frac{dS}{dt} = \frac{36}{x} = \frac{36}{10} = 3.6cm^2/sec$$

Hence the rate at which the surface area increasing when the length of an edge is 10 cm is 3.6 cm^2/sec

29. Question

The volume of a spherical balloon is increasing at the rate of 25 cm^3/sec. Find the rate of change of its surface area at the instant when the radius is 5 cm.

Answer

Given: the volume of a spherical balloon is increasing at the rate of 25 cm^3/sec.

To find the rate of change of its surface area at the instant when the radius is 5 cm

Let the radius of the given spherical balloon be r cm, and V be its volume at any instant time.

Then according to the given criteria,

The rate of the volume of the spherical balloon is increasing is, $\frac{dV}{dt} = 25\ cm^3/sec \ldots (i)$

But volume of the spherical balloon is,

$$V = \frac{4}{3}\pi r^3$$

Applying derivative with respect to time on both sides we get,

$$\frac{dV}{dt} = \frac{d\left(\frac{4}{3}\pi r^3\right)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi\frac{d(r^3)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2\frac{dr}{dt}$$

Substituting the value from equation (i) in above equation, we get

$$\Rightarrow 25 = \frac{4}{3}\pi \times 3r^2\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{25 \times 3}{4\pi \times 3r^2}$$

$$\Rightarrow \frac{dr}{dt} = \frac{25}{4\pi r^2} \ldots . (ii)$$

Now the surface area of the spherical balloon at any time t will be

$S = 4\pi r^2\ cm^2$.

Applying derivative with respect to time on both sides we get,

$$\frac{dS}{dt} = \frac{d(4\pi r^2)}{dt}$$

$$\Rightarrow \frac{dS}{dt} = 4\pi \frac{d(r^2)}{dt}$$

$$\Rightarrow \frac{dS}{dt} = 4\pi \times 2r \frac{dr}{dt}$$

Substituting the value from equation (ii), we get

$$\Rightarrow \frac{dS}{dt} = 4\pi \times 2r \times \frac{25}{4\pi r^2}$$

$$\Rightarrow \frac{dS}{dt} = \frac{25 \times 2}{r}$$

So when the radius is 5cm, the rate of surface area will become,

$$\Rightarrow \frac{dS}{dt} = \frac{25 \times 2}{5}$$

$$\Rightarrow \frac{dS}{dt} = 10\ cm^2/sec$$

Hence the rate of change of its surface area at the instant when the radius is 5 cm is $10cm^2$/sec

30. Question

The length x of a rectangle is decreasing at the rate of 5 cm/minute, and the width y is increasing at the rate of 4cm/minute. When x = 8 cm and y = 6 cm, find the rates of change of (i) the perimeter (ii) the area of the rectangle.

Answer

Given the length of the rectangle is x cm and width of the rectangle is y cm.

As per given criteria, length is decreasing at the rate of 5cm/min

$$\therefore \frac{dx}{dt} = -5cm/min \ldots\ldots (i)$$

And width is increasing at the rate of 4cm/min

$$\therefore \frac{dx}{dt} = 4cm/min \ldots . (ii)$$

(i) Let P be the perimeter of the rectangle

And we know,

P = 2(x + y)

Differentiating both sides with respect to t, we get

$$\frac{dP}{dt} = \frac{d\big(2(x + y)\big)}{dt}$$

$$\Rightarrow \frac{dP}{dt} = 2\left[\frac{dx}{dt} + \frac{dy}{dt}\right]$$

Substituting the values from equation (i) and (ii),we get

$$\Rightarrow \frac{dP}{dt} = 2[-5 + 4] = -2cm/min$$

When x = 8 cm and y = 6 cm, the rates of change of the perimeter is - 2cm/min (it is decreasing in nature) and is independent on length and width of the rectangle.

(ii) Let A be the area of the rectangle

And we know,

A = xy

Differentiating both sides with respect to t, we get

$$\frac{dA}{dt} = \frac{d(xy)}{dt}$$

Now will apply the product rule of differentiation, i.e.,

$\frac{d(uv)}{dx} = v\frac{d(u)}{dx} + u\frac{d(v)}{dx}$, so the above equation becomes

$$\Rightarrow \frac{dA}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$$

Substituting the values from equation (i) and (ii),we get

$$\Rightarrow \frac{dA}{dt} = y(-5) + x(4)$$

When x = 8 cm and y = 6 cm, the above equation becomes,

$$\Rightarrow \frac{dA}{dt} = (6)(-5) + (8)(4) = -30 + 32$$

$$\Rightarrow \frac{dA}{dt} = 2cm^2/min$$

When x = 8 cm and y = 6 cm, the rates of change of the area is 2cm/min (it is increasing in nature) and is dependent on length and width of the rectangle

31. Question

A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/sec. Find the rate at which its area is increasing when the radius is 3.2 cm.

Answer

Let r be the radius of the circular disc and A be the area of the circular disc at any instant of time.

We know that, the area of the circle

$$A = \pi r^2$$

Differentiating both sides with respect to t, we get

$$\frac{dA}{dt} = \frac{d(\pi r^2)}{dt}$$

$$\frac{dA}{dt} = \pi \times 2r\frac{dr}{dt}$$

As per the given criteria, the circular disc expands on heating with the rate of change of radius is 0.05cm/s, i.e.,

$$\frac{dr}{dt} = 0.05cm/sec$$

Substituting this value in equation (i), we get

$$\frac{dA}{dt} = \pi \times 2r \times 0.05$$

When the radius is 3.2 cm, the above equation becomes

$$\frac{dA}{dt} = \pi \times 2(3.2) \times 0.05$$

$$\Rightarrow \frac{dA}{dt} = 0.32\pi\ cm^2/sec$$

So the rate at which its area is increasing when radius is 3.2 cm is 0.32 π cm^2/sec

MCQ

1. Question

Mark the correct alternative in the following:

If $V = \frac{4}{3}\pi r^3$, at what rate in cubic units is V increasing when $r = 10$ and $\frac{dr}{dt} = 0.001$?

A. π

B. 4π

C. 40π

D. $\frac{4\pi}{3}$

Answer

CORRECTION: taking $\frac{dr}{dt} = 0.01$ instead of $\frac{dr}{dt} = 0.001$

$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substituting values of r=10 and $\frac{dr}{dt} = 0.001$, we get

$$\frac{dV}{dt} = 4\pi \times 10^2 \times 0.01$$

$$= 4\pi$$

2. Question

Mark the correct alternative in the following:

Side of an equilateral triangle expands at the rate of 2 cm/sec. The rate of increase of its area when each side is 10 cm is

A. $10\sqrt{2}\ cm^2 / sec$

B. $10\sqrt{3}\ cm^2 / sec$

C. 10 cm^2/sec

D. 5 cm^2/sec

Answer

The area of an equilateral triangle, with side a, is defined as

$A(a) = \frac{\sqrt{3}}{4}a^2$ – (1)

Given that $\frac{da}{dt} = 2cm/sec$ and a=10cm, we have to calculate $\frac{dA}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dA}{dt}=\frac{\sqrt{3}}{2}a\frac{da}{dt}$$

Substituting the values, we get

$$\frac{dA}{dt}=\frac{\sqrt{3}}{2}\times 10\times 2$$

$=10\sqrt{3}\ cm^2$ /sec

3. Question

Mark the correct alternative in the following:

The radius of a sphere is changing at the rate of 0.1 cm/sec. The rate of change of its surface area when the radius is 200 cm is

A. $8\pi\ cm^2$/sec

B. $12\pi\ cm^2$/sec

C. $160\pi\ cm^2$/sec

D. 200 cm^2/sec

Answer

The surface area of a sphere, of radius r, is defined by

$A(r)=4\pi r^2$ – (1)

Given that $\frac{dr}{dt}=0.1cm/sec$ and r=200cm, we have to calculate $\frac{dA}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dA}{dt}=8\pi r\frac{dr}{dt}$$

Substituting values, we get

$$\frac{dA}{dt}=8\pi\times 200\times 0.1=160\pi\ cm^2/sec$$

4. Question

Mark the correct alternative in the following:

A cone whose height is always equal to its diameter is increasing in volume at the rate of 40 cm^3/sec. At what rate is the radius increasing when its circular base area is 1 m^2?

A. 1 mm/sec

B. 0.001 cm/sec

C. 2 mm/sec

D. 0.002 cm/sec

Answer

The volume of a cone, with height h and radius r, is defined by

$V(r,h)=\frac{1}{3}\pi r^2 h$ - (1)

Given that h=2r, we get $V(r)=\frac{2}{3}\pi r^3$- (1)

$\frac{dV}{dt}=40cm^3/sec$ and $\pi r^2=1m^2=10000cm^2$, we have to find out $\frac{dr}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}$$

Substituting the values, we get

$$40 = 2 \times 10000 \times \frac{dr}{dt}$$

$\Rightarrow \frac{dr}{dt} = 0.002 \text{ cm/sec}$

5. Question

Mark the correct alternative in the following:

A cylindrical vessel of radius 0.5 m is filled with oil at the rate of 0.25π m^3/minute. The rate at which the surface of the oil is rising, is

A. 1 m/minute

B. 2 m/minute

C. 5 m/minute

D. 125 m/minute

Answer

The volume of a cylinder, with radius r and height h, is defined by

$V(r,h)=\pi r^2 h$

Substituting r=0.5m, get

$V = \frac{\pi}{4} h$ - (1)

Given that $\frac{dV}{dt} = 0.25\pi \text{ m}^3/\text{min}$, we have to calculate $\frac{dh}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dV}{dt} = \frac{\pi}{4} \times \frac{dh}{dt}$$

Substituting values, we get

$$\frac{\pi}{4} = \frac{\pi}{4} \times \frac{dh}{dt}$$

$\Rightarrow \frac{dh}{dt} = 1 \text{m/min}$

6. Question

Mark the correct alternative in the following:

The distance moved by the particle in time is given by $x = t^3 - 12t^2 + 6t + 8$. At the instant when its acceleration is zero, the velocity is

A. 42

B. –42

C. 48

D. –48

Answer

If we are given the distance travelled(x) as a function of time, we can calculate velocity(v) and acceleration(a) by

$v(t)=\frac{dx}{dt}$ and $a(t)=\frac{d^2x}{dt^2}$

$x(t)= t^3 - 12t^2 + 6t + 8$

differentiating w.r.t. time, we get

$$v(t)=\frac{dx}{dt}=3t^2-24t+6$$

Differentiating again w.r.t. time, we get

$$a(t)=\frac{d^2x}{dt^2}=6t-24$$

Given that a=0 ⇒ 6t-24=0 or t=4 units

$v(4)=3\times4^2 -24\times4+6 =-42$

7. Question

Mark the correct alternative in the following:

The altitude of a cone is 20 cm and its semi-vertical angle is 30°. If the semi-vertical angle is increasing at the rate of 2° per second, then the radius of the base in increasing at the rate of

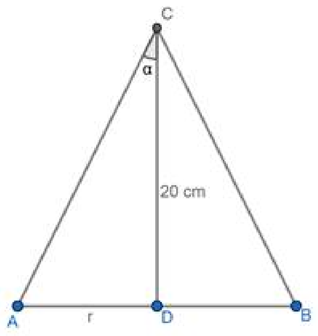
A. 30 cm/sec

B. $\frac{160}{3}$ cm/sec

C. 10 cm/sec

D. 160 cm/sec

Answer



The relation between height h, radius r and semi-vertical angle α is defined by

$$\tan\alpha=\frac{r}{h}$$

Given that h=20cm

$\tan\alpha=\frac{r}{20}$ -(1)

Given that α=30° and $\frac{d\alpha}{dt}=2°/\mathrm{sec}$, we have to find $\frac{dr}{dt}$

Differentiating (1) with respect to t, we get

$$\sec^2\alpha\frac{d\alpha}{dt}=\frac{1}{20}\frac{dr}{dt}$$

Substituting values, we get

$$(\sec^2 30°)\times 2°=\frac{1}{20}\frac{dr}{dt}$$

$\Rightarrow \frac{4}{3} \times 2 = \frac{1}{20}\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{160}{3}$ cm/sec

8. Question

Mark the correct alternative in the following:

For what values of x is the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x?

A. $-3, -\frac{1}{3}$

B. $-3, \frac{1}{3}$

C. $3, -\frac{1}{3}$

D. $3, \frac{1}{3}$

Answer

Let $P(x)=x^3-5x^2+5x+8$ – (1)

Given that $\frac{dP(x)}{dt} = 2\frac{dx}{dt}$, We have to calculate x

Differentiating (1) with respect to t, we get

$$\frac{dP(x)}{dt} = (3x^2 - 10x + 5)\frac{dx}{dt} = 2\frac{dx}{dt}$$

$3x^2-10x+5=2$

$\Rightarrow 3x^2-10x+3=0$

Factorizing the above quadratic equation, we get

$(3x-1)(x-3)=0$

$\Rightarrow x = \frac{1}{3}$ or x=3

9. Question

Mark the correct alternative in the following:

The coordinates of the point on the ellipse $16x^2+9y^2=400$ where the ordinate decreases at the same rate at which the abscissa increase, are:

A. $\left(3, \frac{16}{3}\right)$

B. $\left(-3, \frac{16}{3}\right)$

C. $\left(3, -\frac{16}{3}\right)$.

D. (3, –3)

Answer

Taking ellipse to be $16x^2+9y^2=400$ instead of $46x^2+9y^2=400$

Let $E(x,y):16x^2+9y^2=400$

Solving for y, we get

$y=\pm\frac{\sqrt{400-16x^2}}{3}$ - (1)

Given that $\frac{dx}{dt}=-\frac{dy}{dt}$, we have to calculate x,y

Differentiating (1) with respect to t, we get

$$\frac{dy}{dt}=\pm\frac{1}{3}\times\frac{1}{2\sqrt{400-46x^2}}\times-32x\frac{dx}{dt}=\mp\frac{16x}{3\sqrt{400-16x^2}}\frac{dx}{dt}$$

Substituting values, we get

$$-\frac{dx}{dt}=\mp\frac{16x}{3\sqrt{400-16x^2}}\frac{dx}{dt}$$

Simplifying the equation, we get

$$16x=\pm3\sqrt{400-16x^2}$$

Squaring both sides

$256x^2=9(400-16x^2)$

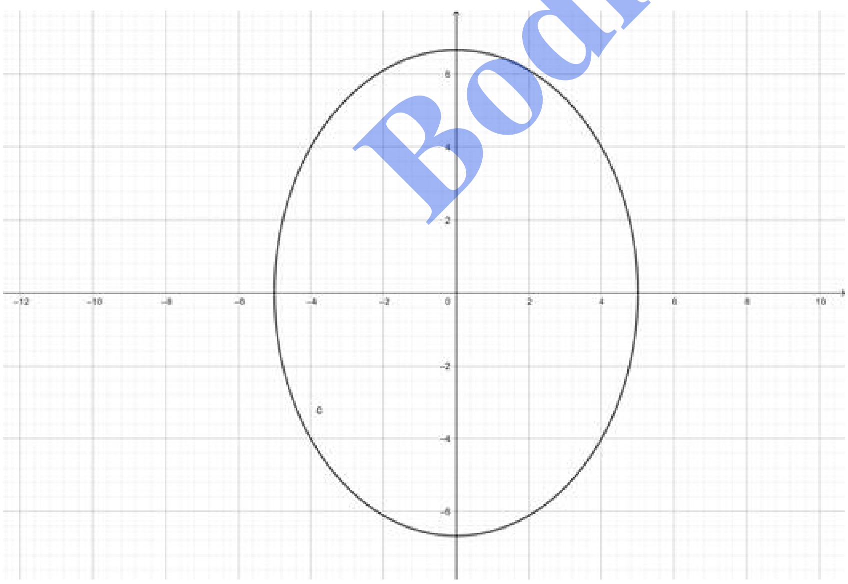
Solving the equation, we get

$x^2=9$

$x=\pm3$

Substituting in (1), we get

$$y=\pm\frac{\sqrt{400-16\times9}}{3}=\pm\frac{16}{3}$$



By seeing the graph of E(x,y), we can conclude that the point has to lie in I[st] or III[rd] quadrant, as only in these quadrants, the increase in abscissa leads to decrease in ordinate.

Hence the points are $\left(3,\frac{16}{3}\right)$ and $\left(3,-\frac{16}{3}\right)$

10. Question

Mark the correct alternative in the following:

The radius of the base of a cone is increasing at the rate of 3 cm/minute and the altitude is decreasing at the rate of 4 cm/minute. The rate of change of lateral surface when the radius = 7 cm and altitude 24 cm is:

A. $54\pi\ cm^2/min$

B. $7\pi\ cm^2/min$

C. $27\ cm^2/min$

D. none of these

Answer

The lateral surface area of a cone, with radius r and height h, is defined as

$L(r,h) = \pi r\sqrt{r^2 + h^2}$ -(1)

Given that r=7cm, h=24cm, $\frac{dr}{dt} = 3cm/min$ and $\frac{dh}{dt} = -4cm/min$, we have to calculate $\frac{dL}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dL}{dt} = \pi\left(\sqrt{r^2 + h^2}\,\frac{dr}{dt} + r \times \frac{1}{2\sqrt{r^2 + h^2}}\left(2r\frac{dr}{dt} + 2h\frac{dh}{dt}\right)\right)$$

Substituting values, we get

$$\frac{dL}{dt} = \pi\left(\sqrt{7^2 + 24^2} \times 3 + 7 \times \frac{1}{2\sqrt{7^2 + 24^2}}(2 \times 7 \times 3 + 2 \times 24 \times -4)\right)$$

Simplifying above equation, we get

$$\frac{dL}{dt} = 54\pi\ cm^2/min$$

11. Question

Mark the correct alternative in the following:

The radius of a sphere is increasing at the rate of 0.2 cm/sec. The rate at which the volume of the sphere increases when radius is 15 cm, is

A. $12\pi\ cm^3/sec$

B. $180\pi\ cm^3/sec$

C. $225\pi\ cm^3/sec$

D. $3\pi\ cm^3/sec$

Answer

The volume of a sphere, of radius r, is defined by

$V(r) = \frac{4}{3}\pi r^3$ - (1)

Given that r=15cm, $\frac{dr}{dt} = 0.2cm/sec$, we have to calculate $\frac{dV}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

Substituting values, we get

$$\frac{dV}{dt} = 4\pi \times 15^2 \times 0.2 = 180\pi\ cm^3/sec$$

12. Question

Mark the correct alternative in the following:

The volume of a sphere is increasing at 3 cm^2/sec. The rate at which the radius increases when radius is 2 cm, is

A. $\frac{3}{32\pi}$ cm/sec

B. $\frac{3}{16\pi}$ cm/sec.

C. $\frac{3}{48\pi}$ cm/sec

D. $\frac{1}{24\pi}$ cm/sec

Answer

The volume of a sphere, of radius r, is defined by

$V(r) = \frac{4}{3}\pi r^3$ – (1)

Given that r=2cm, $\frac{dV}{dt} = 3\ cm^3/sec$, we have to calculate $\frac{dr}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substituting values, we get

$$3 = 4\pi \times 2^2 \times \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{3}{16\pi}\ cm/sec$$

13. Question

Mark the correct alternative in the following:

The distance moved by a particle travelling in a straight line in t seconds is given by s = 45t + 11t^2 – t^3. The time taken by the particle to come to rest is

A. 9 sec

B. $\frac{5}{3}$ sec

C. $\frac{3}{5}$ sec

D. 2 sec

Answer

If we are given the distance travelled(s) as a function of time, we can calculate velocity(v) by

$$v(t) = \frac{ds}{dt}$$

s = 45t + 11t^2 – t^3

Differentiating with respect to time, we get

$v(t) = \frac{ds}{dt} = 45 + 22t - 3t^2$

If the particle is at rest, then its velocity will be 0

$v(t)=45+22t-3t^2=0$

Solving the quadratic equation, we get

$t = \frac{-5}{3}sec$ or t=9 sec

Since time can't be negative therefore t=9secs.

14. Question

Mark the correct alternative in the following:

The volume of a sphere is increasing at the rate of 4π cm^3/sec. The rate of increase of the radius when the volume is 288 π cm^3, is

A. $\frac{1}{4}$

B. $\frac{1}{12}$

C. $\frac{1}{36}$

D. $\frac{1}{9}$

Answer

The volume of a sphere, of radius r, is defined by

$V(r) = \frac{4}{3}\pi r^3$ – (1)

Given that V=288π cm^3,

$288\pi = \frac{4}{3}\pi r^3$

Solving for r, we get r=6cm

Given that $\frac{dV}{dt} = 4\pi\ cm^3/sec$, we have to calculate $\frac{dr}{dt}$

Differentiating (1) with respect to t, we get

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Substituting values, we get

$4\pi = 4\pi \times 6^2 \times \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{36} cm/sec$

15. Question

Mark the correct alternative in the following:

If the rate of change of volume of a sphere is equal to the rate of change of its radius, then its radius is equal to

A. 1 unit

B. $\sqrt{2\pi}$ units

C. $1/\sqrt{2\pi}$ unit

D. $1/2\sqrt{\pi}$ unit.

Answer

The volume of a sphere, of radius r, is defined by

$V(r) = \frac{4}{3}\pi r^3$

Given that $\frac{dV}{dt} = \frac{dr}{dt}$

We get $4\pi r^2 \frac{dr}{dt} = \frac{dr}{dt}$

$\Rightarrow 4\pi r^2=1$

$\Rightarrow r = \frac{1}{2\sqrt{\pi}}$ units

16. Question

Mark the correct alternative in the following:

If the rate of change of area of a circle is equal to the rate of change of its diameter, then its radius is equal to

A. $\frac{2}{\pi \text{ unit}}$

B. $\frac{1}{\pi \text{ unit}}$

C. $\frac{\pi}{2}$ units

D. π units

Answer

The area of a circle, of radius r, is defined by

$A(r)=\pi r^2$

Given that $\frac{dA}{dt} = \frac{d(2r)}{dt} = 2\frac{dr}{dt}$

We get $2\pi r\frac{dr}{dt} = 2\frac{dr}{dt}$

$\Rightarrow \pi r=1$

$\Rightarrow r = \frac{1}{\pi}$ units

17. Question

Mark the correct alternative in the following:

Each side of an equilateral triangle is increasing at the rate of 8 cm/hr. The rate of increase of its area when side is 2 cm, is

A. $8\sqrt{3}\ cm^2/hr$

B. $4\sqrt{3}\ \mathrm{cm}^2 / \mathrm{hr}$

C. $\sqrt{3}/8\ \mathrm{cm}^2 / \mathrm{hr}$

D. none of these

Answer

The area of an equilateral triangle, with side a, is defined as

$A(a) = \frac{\sqrt{3}}{4}a^2$ - (1)

Given that $\frac{da}{dt} = 8\mathrm{cm/hr}$ and a=2cm, we have to calculate $\frac{dA}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}a\frac{da}{dt}$$

Substituting the values, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} \times 2 \times 8 = 8\sqrt{3}\ \mathrm{cm}^2/\mathrm{hr}$$

18. Question

Mark the correct alternative in the following:

If s = $t^3 - 4t^2 + 5$ describes the motion of a particle, then its velocity when the acceleration vanishes, is

A. $\frac{16}{9}$ unit/sec

B. $-\frac{32}{3}$ unit/sec

C. $\frac{4}{3}$ unit/sec

D. $-\frac{16}{3}$ unit/sec

Answer

If we are given the distance travelled(s) as a function of time, we can calculate velocity(v) and acceleration(a) by

$v(t) = \frac{ds}{dt}$ and $a(t) = \frac{d^2s}{dt^2}$

$s(t) = t^3 - 4t^2 + 5$

differentiating w.r.t. time, we get

$$v(t) = \frac{ds}{dt} = 3t^2 - 8t$$

Differentiating again w.r.t. time, we get

$$a(t) = \frac{d^2s}{dt^2} = 6t - 8$$

Given that a=0 ⇒ 6t-8=0 or $t = \frac{4}{3}$ units

$$v\left(\frac{4}{3}\right) = 3 \times \frac{4^2}{3^2} - 8 \times \frac{4}{3} = -\frac{16}{3} \text{ unit/sec}$$

19. Question

Mark the correct alternative in the following:

The equation of motion of a particle is s = $2t^2$ + sin 2t, where s is in metre and t is in seconds. The velocity of the particle when its acceleration is 2m/sec^2, is

A. $\pi + \sqrt{3}$ m/sec

B. $\frac{\pi}{3} + \sqrt{3}$ m/sec

C. $\frac{2\pi}{3} + \sqrt{3}$ m/sec

D. $\frac{\pi}{3} + \frac{1}{\sqrt{3}}$ m/sec

Answer

If we are given the distance travelled(s) as a function of time, we can calculate velocity(v) and acceleration(a) by

$v(t) = \frac{ds}{dt}$ and $a(t) = \frac{d^2s}{dt^2}$

s(t)= $2t^2$ + sin2t

differentiating w.r.t. time, we get

$v(t) = \frac{ds}{dt} = 4t + 2\cos 2t$ - (1)

Differentiating again w.r.t. time, we get

$a(t) = \frac{d^2s}{dt^2} = 4 - 4\sin 2t$

Given that a=2 ⇒ 4-4sin2t=2 or $\sin 2t = \frac{1}{2} = \sin\frac{\pi}{6}$

$t = \frac{\pi}{12}$

$v\left(\frac{\pi}{12}\right) = 4 \times \frac{\pi}{12} + 2\cos\left(\frac{\pi}{6}\right)$

$\Rightarrow v\left(\frac{\pi}{12}\right) = \frac{\pi}{3} + \sqrt{3}$ m/sec

20. Question

Mark the correct alternative in the following:

The radius of a circular plate is increasing at the rate of 0.01 cm/sec. The rate of increases its area when the radius is 12 cm is

A. 144π cm^2/sec

B. 2.4π cm^2/sec

C. 0.24π cm^2/sec

D. 0.024π cm^2/sec

Answer

The circumference of a circle, with radius r, is defined as

$A(r)=\pi r^2$ – (1)

Given that r=12cm, $\frac{dr}{dt}=0.01\text{cm/sec}$, we have to calculate $\frac{dA}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dA}{dt}=2\pi r\frac{dr}{dt}=2\pi\times12\times0.01$$

$$=0.24\pi\ \text{cm}^2/\text{sec}$$

21. Question

Mark the correct alternative in the following:

The diameter of a circle is increasing at the rate of 1cm/sec. When its radius is π, the rate of increase of its area is

A. π cm^2/sec

B. 2π cm^2/sec

C. π^2 cm^2/sec

D. $2\pi^2$ cm^2/sec^2

Answer

The circumference of a circle, with radius r, is defined as

$A(r)=\pi r^2$ – (1)

Given that r=π cm, $\frac{d(2r)}{dt}=\frac{2dr}{dt}=1\text{cm/sec}$, we have to calculate $\frac{dA}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dA}{dt}=2\pi r\frac{dr}{dt}=\pi\times\pi\times1$$

$$=\pi^2\ \text{cm}^2/\text{sec}$$

22. Question

Mark the correct alternative in the following:

A man 2 metres tall walks away from a lamp post 5 metres height at the rate of 4.8 km/hr. The rate of increase of the length of his shadow is

A. 1.6 km/hr

B. 6.3 km/hr

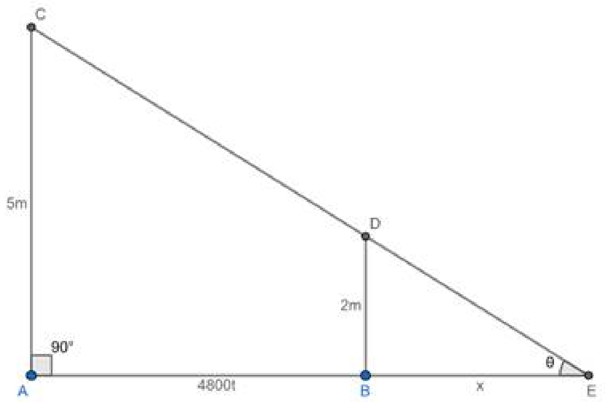
C. 5 km/hr

D. 3.2 km/hr

Answer

Given that h_m=2m, h_l=5m, v_m=4.8km/hr=4800m/hr. We have to calculate v_s(speed of shadow).

After time t hrs, the man would have moved distance 4800t m away from the lamp. Let the shadow move a distance of x m in time t hrs.



Consider ΔAEC and ΔBED

∠AED=∠BED=θ (common angle)

∠EAC=∠EBD=90°

Therefore, ΔAEC~ΔBED by AA criteria

$$\frac{\mathrm{AC}}{\mathrm{BD}}=\frac{\mathrm{AE}}{\mathrm{BE}}$$

Substituting values, we get

$$\frac{5}{2}=\frac{4800\mathrm{t}+\mathrm{x}}{\mathrm{x}}$$

Simplifying the equation, we get

x=3200t – (1)

Differentiating (1) with respect to t, we get

$$\mathrm{v_s}=\frac{\mathrm{dx}}{\mathrm{dt}}=3200\ \mathrm{m/hr}$$

v_s=3.2km/hr

23. Question

Mark the correct alternative in the following:

A man of height 6 ft walks at a uniform speed of 9 ft/sec from a lamp fixed at 15 ft height. The length of his shadow is increasing at the rate of

A. 15 ft/sec

B. 9 ft/sec

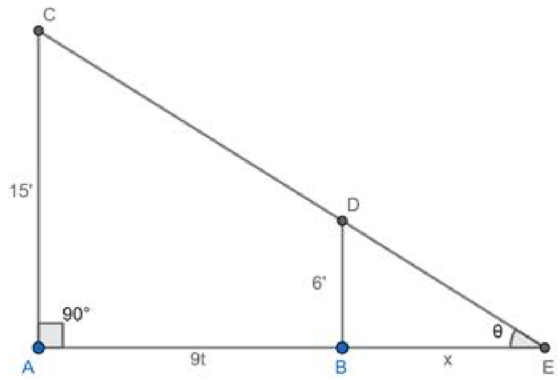
C. 6 ft/sec

D. none of these

Answer

Given that h_m=6ft, h_l=15ft, v_m=9ft/sec. We have to calculate v_s (speed of shadow).

After time t secs, the man would have moved distance 9t ft away from the lamp. Let the shadow move a distance of x ft in time t secs.



Consider ΔAEC and ΔBED

∠AED=∠BED=θ (common angle)

∠EAC=∠EBD=90°

Therefore, ΔAEC~ΔBED by AA criteria

$$\frac{AC}{BD} = \frac{AE}{BE}$$

Substituting values, we get

$$\frac{15}{6} = \frac{9t + x}{x}$$

Simplifying the equation, we get

x=6t – (1)

Differentiating (1) with respect to t, we get

$$v_s = \frac{dx}{dt} = 6 \text{ ft/sec}$$

24. Question

Mark the correct alternative in the following:

In a sphere the rate of change of volume is

A. π times the rate of change of radius

B. surface area times the rate of change of diameter

C. surface area times the rate of change of radius

D. none of these

Answer

The volume and surface area of a sphere, with radius r, is defined as

$V(r) = \frac{4}{3}\pi r^3$ - (1) and $A(r)=4\pi r^2$

Differentiating (1) with respect to t, we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = A \times \frac{dr}{dt}$$

=(Surface area of a sphere)×(rate of change of radius)

25. Question

Mark the correct alternative in the following:

In a sphere the rate of change of surface area is

A. 8π times the rate of change of diameter

B. 2π times the rate of change of diameter

C. 2π times the rate of change of radius

D. 8π times the rate of change of radius

Answer

The surface area of a sphere, with radius r, is defined as

$A(r)=4\pi r^2$ – (1)

Differentiating (1) with respect to t, we get

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

=8π×(current radius)×(rate of change of radius)

26. Question

Mark the correct alternative in the following:

A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of

A. 1 m/hr

B. 0.1 m/hr

C. 1.1 m/h

D. 0.5 m/hr

Answer

The volume of a cylinder, of radius r and height h, is defined as

$V(r,h)=\pi r^2 h$

Given that r=10m, we get

$V(h)=\pi\times 10^2\times h=100\pi h$ – (1)

Given that $\frac{dV}{dt} = 314 m^3/hr$, we have to calculate $\frac{dh}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dV}{dt} = 100\pi \times \frac{dh}{dt}$$

Substituting values and using π=3.14, we get

$$314 = 100 \times 3.14 \times \frac{dh}{dt}$$

$\Rightarrow \frac{dh}{dt} = 1 m/hr$

Very short answer

1. Question

If a particle moves in a straight line such that the distance travelled in time t is given by $s = t^3 - 6t^2 + 9t + 8$. Find the initial velocity of the particle.

Answer

The Velocity(v) of the particle, if the distance travelled(s) is given as a function of time, is defined as $\frac{ds}{dt}$.

Since $s(t)= t^3 - 6t^2 + 9t + 8$

Therefore $v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$

Substituting t=0 in the equation, we get

$v_{t=0}$ =9units/unit time

2. Question

The volume of a sphere is increasing at 3 cubic centimeter per second. Find the rate of increase of the radius, when the radius is 2 cms.

Answer

The volume of a sphere, with radius r, is defined as

$V(r) = \frac{4}{3}\pi r^3$ - (1)

Given that $\frac{dV}{dt} = 3cm^3/sec$ and r=2cm, we have to calculate $\frac{dr}{dt}$.

Differentiating (1) with respect to t, we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substituting the values, we get

$$3 = 4\pi \times 2^2 \times \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{3}{16\pi} \text{ cm/sec}$$

3. Question

The sides of an equilateral triangle are increasing at the rate of2 cm/sec. How far is the area increasing when the side is 10 cms?

Answer

The area of an equilateral triangle, with side a, is defined as

$A(a) = \frac{\sqrt{3}}{4}a^2$ - (1)

Given that $\frac{da}{dt} = 2cm/sec$ and a=10cm, we have to calculate $\frac{dA}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}a\frac{da}{dt}$$

Substituting the values, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3} \text{ cm}^2/\text{sec}$$

4. Question

The side of a square is increasing at the rate of 0.1 cm/sec. Find the rate of increase of its perimeter.

Answer

The perimeter of a square, with side a, is defined as

P(a)=4a – (1)

Given that $\frac{da}{dt} = 0.1cm/sec$, we have to calculate $\frac{dP}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dP}{dt} = 4\frac{da}{dt}$$

Substituting the values, we get

$$\frac{dP}{dt} = 4 \times 0.1 = 0.4cm/sec$$

5. Question

The radius of a circle is increasing at the rate of 0.5 cm/sec. Find the rate of increase of its circumference.

Answer

The circumference of a circle, with radius r, is defined as

C(r)=2πr – (1)

Given that $\frac{dr}{dt} = 0.5cm/sec$, we have to calculate $\frac{dC}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt} = 2\pi \times 0.5$$

$$= \pi\ cm/sec$$

6. Question

The side of an equilateral triangle is increasing at the rate of ..$\frac{1}{3}$.. cm/sec. Find the rate of increase of its perimeter.

Answer

The perimeter of an equilateral triangle, with side a, is defined as

P(a)=3a – (1)

Given that $\frac{da}{dt} = \frac{1}{3}cm/sec$, we have to calculate $\frac{dP}{dt}$

Differentiating (1) with respect to t, we get

$$\frac{dP}{dt} = 3\frac{da}{dt}$$

Substituting the values, we get

$$\frac{dP}{dt} = 3 \times \frac{1}{3} = 1cm/sec$$

7. Question

Find the surface area of a sphere when its volume is changing at the same rate as its radius.

Answer

The volume of a sphere, of radius r, and its surface area are defined by

$V(r) = \frac{4}{3}\pi r^3$ and $A(r)=4\pi r^2$

Given that $\frac{dV}{dt} = \frac{dr}{dt}$

We get $4\pi r^2 \frac{dr}{dt} = \frac{dr}{dt}$

$\Rightarrow 4\pi r^2 = A(r) = 1 unit^2$

8. Question

If the rate of change of volume of a sphere is equal to the rate of change of its radius, find the radius of the sphere.

Answer

The volume of a sphere, of radius r, is defined by

$$V(r) = \frac{4}{3}\pi r^3$$

Given that $\frac{dV}{dt} = \frac{dr}{dt}$

We get $4\pi r^2 \frac{dr}{dt} = \frac{dr}{dt}$

$\Rightarrow 4\pi r^2 = 1$

$\Rightarrow r = \frac{1}{2\sqrt{\pi}}$ units

9. Question

The amount of pollution content added in air in a city due to x diesel vehicles is given by $P(x) = 0.0005\ x^3 + 0.02\ x^2 + 30\ x$. Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above questions.

Answer

Given that $P(x) = 0.0005x^3 + 0.02x^2 + 30x$

The marginal increase in the pollution level is

On differentiating both sides, we get

$P'(x)=0.0005\times 3x^2 +0.02\times 2x +30$

$= 0.0015x^2 + 0.04x +30$

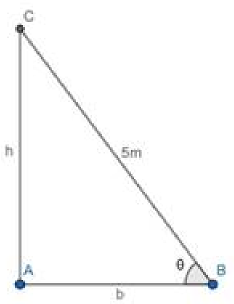
Substituting x=3, we get

$P'(3) = 0.0015\times 9 + 0.04\times 3 +30 = 30.255$ units

10. Question

A ladder, 5 meter long, standing on a horizontal floor, leans against a vertical wall. If the top to the ladder slides down wards at the rate of 10 cm/sec, then find the rate at which the angle between the floor and ladder is decreasing when lower end of ladder is 2 meters from the wall.

Answer



Given that $\frac{dh}{dt} = -0.1 m/sec$. We have to find $\frac{d\theta}{dt}$ when h=2m

Also $\sin\theta = \frac{h}{5}$ – (1)

At b=2m, $\cos\theta = \frac{2}{5}$

Differentiating (1) with respect to t, we get

$$\cos\theta\frac{d\theta}{dt}=\frac{1}{5}\times\frac{dh}{dt}$$

Substituting values, we get

$$\frac{2}{25}\times\frac{d\theta}{dt}=\frac{1}{5}\times 0.1$$

$$\Rightarrow\frac{d\theta}{dt}=\frac{1}{20}\text{ rad/sec}$$