# 8 Mean Value Theorems, Maxima and Minima

## 8.1 ROLLE'S THEOREM

***If a function f (x) is***

**(*i*) *continuous in the closed interval* [*a, b*]**

**(*ii*) *derivable in the open interval (a, b) and***

**(*iii*) *f (a) = f (b),***

***then there exists atleast one real number c in (a, b) such that f′(c) = 0.***

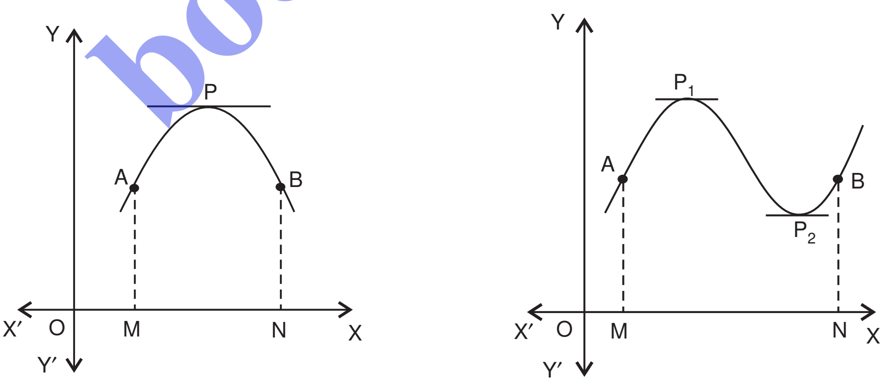
(We accept it without proof.)



### Geometrical Interpretation

Let A, B be the points on the curve $y = f(x)$ corresponding to the real numbers $a, b$ respectively.

Since $f(x)$ is continuous in $[a, b]$, the graph of the curve $y = f(x)$ is continuous from A to B. Again, as $f(x)$ is derivable in $(a, b)$, the curve $y = f(x)$ has a tangent at each point between A and B. Also as $f(a) = f(b)$ the ordinates of the points A and B are equal *i.e.* MA = NB (see fig. 8.1).




(*i*) (*ii*)

*Fig. 8.1.*

Then Rolle's *theorem* asserts that there is atleast one point lying between A and B such that the tangent at which is parallel to $x$-axis *i.e.* there exists atleast one real number $c$ in $(a, b)$ such that $f'(c) = 0$.

There may exist more than one point between A and B the tangents at which are parallel to $x$-axis (as shown in fig. 8.1 (*ii*)) *i.e.* there may exist more than one real number $c$ in $(a, b)$

such that $f'(c) = 0$. Rolle's theorem ensures the existence of atleast one real number $c$ in $(a, b)$ such that $f'(c) = 0$.

### Remarks

1. Rolle's theorem fails for the function which does not satisfy even one of the three conditions.
2. *The converse of Rolle's theorem may not be true,* for, $f'(x)$ may be zero at a point in $(a, b)$ without satisfying all the three conditions of Rolle's theorem.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Verify Rolle's theorem for the following functions:*

(*a*) $f(x) = x^2 + x - 6$ *in* $[-3, 2]$

(*b*) $f(x) = (x^2 - 1)(x - 2)$ *in* $[-1, 2]$.

**Solution.** (*a*) Given $f(x) = x^2 + x - 6$ ...(1)

(*i*) As $f(x)$ is a polynomial function, it is continuous in $[-3, 2]$,

(*ii*) $f(x)$ being a polynomial function is derivable in $(-3, 2)$ and

(*iii*) $f(-3) = (-3)^2 - 3 - 6 = 0, f(2) = 2^2 + 2 - 6 = 0 \Rightarrow f(-3) = f(2)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number $c$ in $(-3, 2)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. $x$, we get $f'(x) = 2x + 1$.

Now $f'(c) = 0 \Rightarrow 2c + 1 = 0 \Rightarrow c = -\frac{1}{2}$.

So there exists $-\frac{1}{2} \in (-3, 2)$ such that $f'\left(-\frac{1}{2}\right) = 0$.

Hence, Rolle's theorem is verified.

(*b*) Given $f(x) = (x^2 - 1)(x - 2)$ ...(1)

(*i*) Since $f(x)$ is a polynomial function, it is continuous in $[-1, 2]$,

(*ii*) $f(x)$ being a polynomial function is derivable in $(-1, 2)$ and

(*iii*) $f(-1) = (1 - 1)(1 - 2) = 0, f(2) = (4 - 1)(2 - 2) = 0 \Rightarrow f(-1) = f(2)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number $c$ in $(-1, 2)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = (x^2 - 1).1 + (x - 2).2x = 3x^2 - 4x - 1.$$

Now $f'(c) = 0 \Rightarrow 3c^2 - 4c - 1 = 0$

$$\Rightarrow \quad c = \frac{4 \pm \sqrt{16 - 4.3.(-1)}}{2.3} = \frac{2 \pm \sqrt{7}}{3}.$$

Also $-1 < \frac{2 - \sqrt{7}}{3} < \frac{2 + \sqrt{7}}{3} < 2 \Rightarrow \frac{2 - \sqrt{7}}{3}$ and $\frac{2 + \sqrt{7}}{3}$ both lie in $(-1, 2)$.

So there exist two real numbers $\frac{2 - \sqrt{7}}{3}$ and $\frac{2 + \sqrt{7}}{3}$ in $(-1, 2)$ such that

$$f'\left(\frac{2 - \sqrt{7}}{3}\right) = 0 \text{ and } f'\left(\frac{2 + \sqrt{7}}{3}\right) = 0.$$

Hence, Rolle's theorem is verified.

**Example 2.** *Using Rolle's theorem, find the point on the curve* $y = 16 - x^2, x \in [-1, 1]$ *where the tangent is parallel to x-axis.*

**Solution.** Given $y = 16 - x^2$ *i.e.* $f(x) = 16 - x^2$ ...(1)

(*i*) As $f(x)$ is a polynomial function, it is continuous in $[-1, 1]$,

(*ii*) $f(x)$ being a polynomial function is derivable in $(-1, 1)$ and

(*iii*) $f(-1) = 16 - (-1)^2 = 15, f(1) = 16 - 1^2 = 15 \Rightarrow f(-1) = f(1)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number $c$ in $(-1, 1)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. $x$, we get $f'(x) = -2x$.

Now $f'(c) = 0 \Rightarrow -2c = 0 \Rightarrow c = 0$.

So there exists $0 \in (-1, 1)$ where $f'(c) = 0$ *i.e.* the tangent is parallel to $x$-axis.

From (1), when $x = 0$, $y = 16 - 0^2 = 16$.

Hence, there exists the point (0, 16) on the given curve where the tangent is parallel to $x$-axis.

**Example 3.** *Verify Rolle's theorem for the following functions and find point (or points) in the given interval where derivative is zero :*

(a) $f(x) = \sin x + \cos x - 1$ in $\left[0, \frac{\pi}{2}\right]$

(b) $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

(c) $f(x) = e^{2x} (\sin 2x - \cos 2x)$ in $\left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$ (I.S.C. 2006)

(d) $f(x) = e^{1-x^2}$ in $[-1, 1]$.

**Solution.** (a) Given $f(x) = \sin x + \cos x - 1$ ...(1)

(i) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$,

(ii) $f(x)$ is derivable in $\left(0, \frac{\pi}{2}\right)$ and

(iii) $f(0) = \sin 0 + \cos 0 - 1 = 0 + 1 - 1 = 0$,

$f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} - 1 = 1 + 0 - 1 = 0 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number $c$ in $\left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = \cos x - \sin x$$

Now $f'(c) = 0 \Rightarrow \cos c - \sin c = 0 \Rightarrow \tan c = 1$

$\Rightarrow \quad c = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, -\frac{3\pi}{4}, \ldots$ but $c \in \left(0, \frac{\pi}{2}\right) \Rightarrow c = \frac{\pi}{4}$.

So there exists $\frac{\pi}{4}$ in $\left(0, \frac{\pi}{2}\right)$ such that $f'\left(\frac{\pi}{4}\right) = 0$.

Hence, Rolle's theorem is verified and $c = \frac{\pi}{4}$.

(b) Given $f(x) = \sin x - \sin 2x$ ...(1)

(i) $f(x)$ is continuous in $[0, \pi]$.

(ii) $f(x)$ is derivable in $(0, \pi)$ and

(iii) $f(0) = \sin 0 - \sin 0 = 0 - 0 = 0$, $f(\pi) = \sin \pi - \sin 2\pi = 0 - 0 = 0$

$\Rightarrow f(0) = f(\pi)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number $c$ in $(0, \pi)$ such that $f'(c) = 0$.

Diff. (1) w.r.t. $x$, we get $f'(x) = \cos x - \cos 2x \,.\, 2$

Now $f'(c) = 0 \Rightarrow \cos c - 2\cos 2c = 0$

$\Rightarrow \quad \cos c - 2(2\cos^2 c - 1) = 0 \Rightarrow 4\cos^2 c - \cos c - 2 = 0$

$\Rightarrow \quad \cos c = \frac{1 \pm \sqrt{1 - 4.4.(-2)}}{2.4} = \frac{1 \pm \sqrt{33}}{8} \Rightarrow c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$.

So there exist two real numbers '$c$' given by $c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$ in $(0, \pi)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified and $c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$.

(c) Given $f(x) = e^{2x}(\sin 2x - \cos 2x)$ ...(1)

(i) $f(x)$ is continuous in $\left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$,

(ii) $f(x)$ is derivable in $\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$ and

(iii) $f\left(\frac{\pi}{8}\right) = e^{\frac{\pi}{4}}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0,$

$$f\left(\frac{5\pi}{8}\right) = e^{\frac{5\pi}{4}}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(-\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$$

$\Rightarrow\ f\left(\frac{\pi}{8}\right) = f\left(\frac{5\pi}{8}\right).$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number $c$ in $\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$ such that $f'(c) = 0$.

Diff. (1) w.r.t. $x$, we get

$$f'(x) = e^{2x}(\cos 2x \,.\, 2 + \sin 2x \,.\, 2) + (\sin 2x - \cos 2x)\, e^{2x} \,.\, 2$$
$$= 4e^{2x} \sin 2x.$$

Now $f'(c) = 0 \Rightarrow 4e^{2c}\sin 2c = 0 \Rightarrow \sin 2c = 0$

$\Rightarrow\ 2c = 0,\ \pi,\ 2\pi,\ \dots,\ -\pi,\ -2\pi,\ \dots$

$\Rightarrow\ c = 0,\ \frac{\pi}{2},\ \pi,\ \dots,\ -\frac{\pi}{2},\ -\pi,\ \dots$ but $c \in \left(\frac{\pi}{8}, \frac{5\pi}{8}\right) \Rightarrow c = \frac{\pi}{2}$.

So there exists $\frac{\pi}{2}$ in $\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$ such that $f'\left(\frac{\pi}{2}\right) = 0$.

Hence, Rolle's theorem is verified and $c = \frac{\pi}{2}$.

(d) Given $f(x) = e^{1-x^2}$ ...(1)

(i) $f(x)$ is continuous in $[-1, 1]$, for,

since $g(x) = 1 - x^2$ and $h(x) = e^x$ are continuous in $[-1, 1]$ therefore,

$(hog)(x) = h(g(x)) = h(1 - x^2) = e^{1-x^2}$ is also continuous in $[-1, 1]$,

(ii) $f(x)$ is derivable in $(-1, 1)$ and

(iii) $f(-1) = e^{1-1} = e^0 = 1$, $f(1) = e^{1-1} = e^0 = 1 \Rightarrow f(-1) = f(1)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number $c$ in $(-1, 1)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = e^{1-x^2}(-2x) = -2x\, e^{1-x^2}.$$

Now $f'(c) = 0 \Rightarrow -2c\, e^{1-c^2} = 0 \Rightarrow c = 0$.

So there exists 0 in $(-1, 1)$ such that $f'(0) = 0$.

Hence, Rolle's theorem is verified and $c = 0$.

**Example 4.** *Verify Rolle's theorem for the following functions and find point (or points) in the given interval where derivative is zero :*

*(a)* $f(x) = (x - a)^m (x - b)^n$ *in* $[a, b]$, $m, n \in \mathbf{N}$.

*(b)* $f(x) = \log\left(\frac{x^2 + ab}{(a + b)x}\right)$ *in* $[a, b]$, $a > 0$. *(I.S.C. 2012)*

**Solution.** (a) Given $f(x) = (x - a)^m (x - b)^n$, $m, n \in \mathbf{N}$ ...(1)

Since $m, n \in \mathbf{N}$, $f(x)$ is a polynomial in $x$.

(i) $f(x)$ is continuous in $[a, b]$,

(ii) $f(x)$ is derivable in $(a, b)$ and

(iii) $f(a) = 0$, $f(b) = 0 \Rightarrow f(a) = f(b)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number $c$ in $(a, b)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = (x-a)^m\, n\,(x-b)^{n-1} + (x-b)^n\, m\,(x-a)^{m-1}$$
$$= (x-a)^{m-1}(x-b)^{n-1}(n(x-a) + m(x-b))$$
$$= (x-a)^{m-1}(x-b)^{n-1}((m+n)x - (na + mb)).$$

Now $f'(c) = 0 \Rightarrow (c-a)^{m-1}(c-b)^{n-1}((m+n)c - (na+mb)) = 0$.

But $c \neq a,\ c \neq b \Rightarrow (m+n)c - (na+mb) = 0$

$$\Rightarrow \quad c = \frac{mb+na}{m+n},$$

which is a point in $(a, b)$, for, it divides $[a, b]$ in the ratio $m : n$ internally.

Thus, there exists a real number $c = \frac{mb+na}{m+n}$ in $(a, b)$ such that $f'(c)$.

Hence, Rolle's theorem is verified and $c = \frac{mb+na}{m+n}$.

(*b*) Given $f(x) = \log\left(\frac{x^2+ab}{(a+b)x}\right) = \log(x^2 + ab) - \log(a+b) - \log x$ ...(1)

(*i*) Since $a > 0$ and $\log x$ is continuous for all $x > 0$, therefore, $f(x)$ is continuous in $[a, b]$,

(*ii*) $f(x)$ is derivable in $(a, b)$ and

(*iii*) $f(a) = \log\left(\frac{a^2+ab}{(a+b)a}\right) = \log 1 = 0$, $f(b) = \log\left(\frac{b^2+ab}{(a+b)b}\right) = \log 1 = 0$

$\Rightarrow f(a) = f(b)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number $c$ in $(a, b)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = \frac{1}{x^2+ab}\cdot 2x - 0 - \frac{1}{x} = \frac{2x}{x^2+ab} - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}.$$

Now $f'(c) = 0 \Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2 - ab = 0 \Rightarrow c = \pm\sqrt{ab}$.

But $c \in (a, b) \Rightarrow c = \sqrt{ab}$ ($\because$ Geometric mean lies between them)

So there exists a real number $c = \sqrt{ab}$ in $(a, b)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified and $c = \sqrt{ab}$.

**Example 5.** *It is given that for the function $f(x) = x^3 + bx^2 + ax + 5$ on $[1, 3]$ Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a and b.*

**Solution.** Given $f(x) = x^3 + bx^2 + ax + 5$ ...(1)

We note that $f(x)$ is continuous in $[1, 3]$ and derivable in $(1, 3)$ for all values of $a$ and $b$.

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = 3x^2 + 2bx + a \quad \text{...(2)}$$

Since Rolle's theorem holds for $f(x)$ in $[1, 3]$ with $c = 2 + \frac{1}{\sqrt{3}}$, therefore, we must have

$$f(1) = f(3) \text{ and } f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$$

$$\Rightarrow \quad 1 + b + a + 5 = 27 + 9b + 3a + 5 \Rightarrow 8b + 2a + 26 = 0$$

$$\Rightarrow \quad a + 4b + 13 = 0 \quad \text{...(3)}$$

and $$3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow \quad 3\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) + 4b + \frac{2b}{\sqrt{3}} + a = 0$$

$\Rightarrow \quad (a + 4b + 13) + \dfrac{12}{\sqrt{3}} + \dfrac{2b}{\sqrt{3}} = 0 \Rightarrow \dfrac{12}{\sqrt{3}} + \dfrac{2b}{\sqrt{3}} = 0$ (using (3))

$\Rightarrow \quad 2b + 12 = 0 \Rightarrow b = -6.$

From (3), $a - 24 + 13 = 0 \Rightarrow a = 11.$

Hence, $a = 11,\ b = -6.$

**Example 6.** *Discuss the applicability of Rolle's theorem for the function $f(x) = |x|$ in $[-2, 2]$.*

**Solution.** Given $f(x) = |x|,\ x \in [-2, 2]$ ...(1)

the graph of $f(x) = |x|$ in $[-2, 2]$ is shown in fig. 8.2.

(*i*) $f(x)$ is continuous in $[-2, 2]$.

(*ii*) Differentiating (1) w.r.t. $x$, we get

$$f'(x) = \frac{x}{|x|},\ x \neq 0$$

$\Rightarrow$ the derivative of $f(x)$ does not exist at $x = 0$

$\Rightarrow$ $f(x)$ is not derivable in $(-2, 2)$.

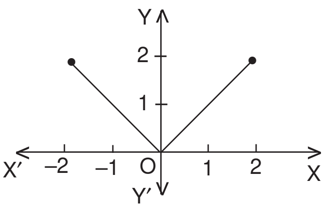


*Fig. 8.2.*

Thus, the condition (*ii*) of Rolle's theorem is not satisfied, therefore, Rolle's theorem is not applicable to the function $f(x) = |x|$ in $[-2, 2]$.

Moreover, $f(-2) = |-2| = 2$ and $f(2) = |2| = 2 \Rightarrow f(-2) = f(2)$, so the condition (*iii*) of Rolle's theorem is satisfied.

Further, it is clear from the graph that there is no point of the curve $y = |x|$ in $(-2, 2)$ at which the tangent is parallel to $x$-axis.



## EXERCISE 8.1

*Verify Rolle's theorem for the following (1 to 6) functions and find point (or points) in the interval where derivative is zero:*

**1.** (*i*) $f(x) = x^2 - 5x + 4$ in $[1, 4]$ (*ii*) $f(x) = x^2 + 5x + 6$ in $[-3, -2]$
(*iii*) $f(x) = x^2 - 8x + 12$ in $[2, 6]$.

**2.** (*i*) $f(x) = (x - 1)(x - 2)^2$ in $[1, 2]$ (*ii*) $f(x) = (x - 1)(x - 2)(x - 3)$ in $[1, 3]$
(*iii*) $f(x) = x^3 - 12x$ in $[0, 2\sqrt{3}]$ (*iv*) $f(x) = x^3 - 4x$ in $[-2, 2]$
(*v*) $f(x) = 2x^3 + x^2 - 4x - 2$ in $\left[-\frac{1}{2}, \sqrt{2}\right]$.

**3.** (*i*) $f(x) = \cos 2x$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (*ii*) $f(x) = \sin x - 1$ in $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.

**4.** (*i*) $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$ (*ii*) $f(x) = \sin 3x$ in $[0, \pi]$
(*iii*) $f(x) = \sin^2 x$ in $[0, \pi]$ (*iv*) $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ in $\left[0, \frac{\pi}{2}\right]$.

**5.** (*i*) $f(x) = \log(x^2 + 2) - \log 3$ on $[-1, 1]$
(*ii*) $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ (*iii*) $f(x) = \sin x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$
(*iv*) $f(x) = |9 - x^2|$ on $[-3, 3]$.

**6.** (*i*) $f(x) = e^x \sin x$ on $[0, \pi]$ (*I.S.C. 2005*)
(*ii*) $f(x) = e^x \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (*iii*) $f(x) = \dfrac{\sin x}{e^x}$ on $[0, \pi]$
(*iv*) $f(x) = e^x(\sin x - \cos x)$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

**7.** Apply Rolle's theorem to find point (or points) on the following curves where the tangent is parallel to $x$-axis:
(*i*) $y = x^2$ in $[-2, 2]$ (*ii*) $f(x) = -1 + \cos x$ in $[0, 2\pi]$.

**8.** Using Rolle's theorem, find a point on the curve $y = \sin x + \cos x - 1$, $x \in \left[0, \frac{\pi}{2}\right]$ where the tangent is parallel to the $x$-axis. *(I.S.C. 2010)*

**9.** If Rolle's theorem holds for the function $f(x) = x^3 + ax^2 + bx$ in [1, 2] at the point $x = \frac{4}{3}$, then find the values of $a$ and $b$. *(I.S.C. 2009)*

**10.** Considering the function $f(x) = (x - 3) \log x$, prove that there is atleast one real value of $x$ in (1, 3) which satisfies $x \log x = 3 - x$.

**11.** Discuss the applicability of Rolle's theorem for the following functions in the indicated intervals :

(*i*) $f(x) = x^{1/3}$ in $[-1, 1]$ (*ii*) $f(x) = x^{2/3}$ in $[-1, 1]$.

**12.** Discuss the applicability of Rolle's theorem for the following functions in the indicated intervals:

(*i*) $f(x) = \tan x$ in $[0, \pi]$ (*ii*) $f(x) = \sec x$ in $[0, 2\pi]$

(*iii*) $f(x) = 2 + (x - 1)^{2/3}$ in [0, 2] (*iv*) $f(x) = 1 + |x - 2|$ in [0, 4].

## 8.2 LAGRANGE'S MEAN VALUE THEOREM

***If a function f(x) is***

***(i) continuous in the closed interval [a, b] and***

***(ii) derivable in the open interval (a, b),***

***then there exists atleast one real number c in (a, b) such that***

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Proof.** Consider a function $g$ defined by $g(x) = f(x) + kx$ ...(1)

where $k$ is a constant (real number) to be chosen in such a way that $g(a) = g(b)$

$\Rightarrow f(c) + ka = f(b) + kb \Rightarrow k(a - b) = f(b) - f(a)$

$$\Rightarrow \quad k = -\frac{f(b) - f(a)}{b - a} \qquad ...(2)$$

Now (*i*) $g(x)$ is continuous in $[a, b]$

($\because f(x)$ is given to be continuous in $[a, b]$ and $kx$ being polynomial function is continuous for all $x \in \mathbf{R} \Rightarrow$ their sum *i.e.* $f(x) + kx$ is continuous in $[a, b]$)

(*ii*) $g(x)$ is derivable in $(a, b)$

($\because f(x)$ is given to be derivable in $(a, b)$ and $kx$ being polynomial function is derivable for all $x \in \mathbf{R} \Rightarrow$ their sum *i.e.* $f(x) + kx$ is derivable in $(a, b)$)

(*iii*) $g(a) = g(b)$.

Thus, all the three conditions of Rolle's theorem are satisfied by the function $g$ in $[a, b]$, therefore, there exists atleast one real number $c$ in $(a, b)$ such that $g'(c) = 0$.

Differentiating (1) w.r.t. $x$, we get

$$g'(x) = f'(x) + k.1$$

Now $g'(c) = 0 \Rightarrow f'(c) + k = 0 \Rightarrow k = -f'(c)$ ...(3)

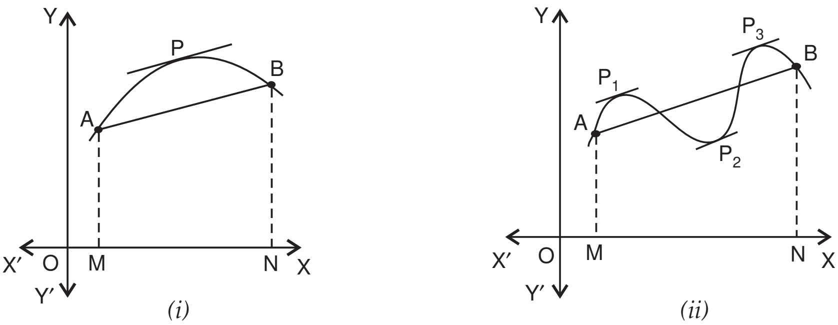
From (2) and (3), we get

$$-f'(c) = -\frac{f(b) - f(a)}{b - a} \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}.$$

### Geometrical Interpretation

Let A, B be the points on the curve $y = f(x)$ corresponding to the real numbers $a$, $b$ respectively.

Since $f(x)$ is continuous in $[a, b]$, the graph of the curve $y = f(x)$ is continuous from A to B. Again, as $f(x)$ is derivable in $(a, b)$ the curve $y = f(x)$ has a tangent at each point between A and B. Also as $a \neq b$, the slope of the chord AB exists and the slope of the chord AB $= \frac{f(b) - f(a)}{b - a}$.




*(i)* *(ii)*

*Fig. 8.3.*

Then *Lagrange's Mean Value Theorem* asserts that there is atleast one point lying between A and B such that the tangent at which is parallel to the chord AB. There may exist more than one point between A and B the tangents at which are parallel to the chord AB (as shown in fig. 8.3 (*ii*)). Lagrange's mean value theorem ensures the existence of atleast one real number $c$ in $(a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

## Remarks

1. Lagrange's mean value theorem fails for the function which does not satisfy even one of the two conditions.
2. *The converse of Lagrange's mean value theorem may not be true*, for, $f'(c)$ may be equal to $\dfrac{f(b) - f(a)}{b - a}$ at a point $c$ in $(a, b)$ without satisfying both the conditions of Lagrange's mean value theorem (see example 4 on page 339).

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Verify Lagrange's mean value theorem for the following functions in the given interval and find 'c' of this theorem.*

(*a*) $f(x) = 3x^2 - 5x + 1$ *in* $[2, 5]$ (*I.S.C. 2007*)

(*b*) $f(x) = (x - 1)(x - 2)(x - 3)$ *in* $[0, 4]$.

**Solution.** (*a*) Given $f(x) = 3x^2 - 5x + 1,\ x \in [2, 5]$ …(1)

(*i*) $f(x)$ being a polynomial function is continuous in [2, 5]

(*ii*) $f(x)$ being a polynomial function is derivable in (2, 5).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number $c$ in (2, 5) such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2}.$$

$$f(5) = 3.5^2 - 5.5 + 1 = 51,\ f(2) = 3.2^2 - 5.2 + 1 = 3.$$

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = 3.2x - 5.1 + 0 \Rightarrow f'(c) = 6c - 5.$$

$$\therefore \quad f'(c) = \frac{f(5) - f(2)}{5 - 2} \Rightarrow 6c - 5 = \frac{51 - 3}{3} \Rightarrow 6c - 5 = 16$$

$$\Rightarrow \quad 6c = 21 \Rightarrow c = \frac{7}{2}.$$

Thus, there exists $c = \dfrac{7}{2}$ in (2, 5) such that $f'\left(\dfrac{7}{2}\right) = \dfrac{f(5) - f(2)}{5 - 2}$.

Hence, Lagrange's mean value theorem is verified and $c = \dfrac{7}{2}$.

(*b*) Given $f(x) = (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$ …(1)

(*i*) $f(x)$ being a polynomial function is continuous in [0, 4]

(*ii*) $f(x)$ being a polynomial function is derivable in (0, 4).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number $c$ in (0, 4) such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}.$$

$$f(4) = 64 - 96 + 44 - 6 = 6,\ f(0) = 0 - 0 + 0 - 6 = -6.$$

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = 3x^2 - 12x + 11 \Rightarrow f'(c) = 3c^2 - 12c + 11$$

$$\therefore \quad f'(c) = \frac{f(4) - f(0)}{4 - 0} \Rightarrow 3c^2 - 12c + 11 = \frac{6 - (-6)}{4 - 0}$$

$$\Rightarrow \quad 3c^2 - 12c + 11 = 3 \Rightarrow 3c^2 - 12c + 8 = 0$$

$$\Rightarrow \quad c = \frac{12 \pm \sqrt{144 - 4.3.8}}{2.3} = 2 \pm \frac{\sqrt{48}}{6} = 2 \pm \frac{2}{\sqrt{3}}.$$

As $0 < 2 - \frac{2}{\sqrt{3}} < 2 + \frac{2}{\sqrt{3}} < 4 \Rightarrow 2 - \frac{2}{\sqrt{3}}$ and $2 + \frac{2}{\sqrt{3}}$ both lie in (0, 4).

Thus, there exist $c = 2 \pm \frac{2}{\sqrt{3}}$ in (0, 4) such that $f'(c) = \frac{f(4) - f(0)}{4 - 0}$.

Hence, Lagrange's mean value theorem is verified and $c = 2 \pm \frac{2}{\sqrt{3}}$.

**Example 2.** *Verify Lagrange's mean value theorem for the following functions in the given intervals :*

(*a*) $f(x) = \tan^{-1} x$ *in* [0, 1] (*b*) $f(x) = \sqrt{x^2 - x}$ *in* [1, 4]. (*I.S.C. 2013*)

**Solution.** (*a*) Given $f(x) = \tan^{-1} x,\ x \in [0, 1]$ ...(1)

(*i*) $f(x)$ is continuous in [0, 1]

(*ii*) $f(x)$ is derivable in (0, 1).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number $c$ in (0, 1) such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$f(1) = \tan^{-1}(1) = \frac{\pi}{4},\ f(0) = \tan^{-1}(0) = 0.$$

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = \frac{1}{1 + x^2} \Rightarrow f'(c) = \frac{1}{1 + c^2}$$

$$\therefore \quad f'(c) = \frac{f(1) - f(0)}{1 - 0} \Rightarrow \frac{1}{1 + c^2} = \frac{\frac{\pi}{4} - 0}{1} \Rightarrow \frac{1}{1 + c^2} = \frac{\pi}{4}$$

$$\Rightarrow 1 + c^2 = \frac{4}{\pi} \Rightarrow c^2 = \frac{4}{\pi} - 1 \Rightarrow c = \pm\sqrt{\frac{4 - \pi}{\pi}}.$$

But $c \in (0, 1) \Rightarrow c = \sqrt{\frac{4 - \pi}{\pi}}$.

Thus, there exists $c = \sqrt{\frac{4 - \pi}{\pi}}$ in (0, 1) such that $f'(c) = \frac{f(1) - f(0)}{1 - 0}$.

Hence, Lagrange's mean value theorem is verified.

(*b*) Given $f(x) = \sqrt{x^2 - x},\ x \in [1, 4]$ ...(1)

(*i*) Since $g(x) = x^2 - x$ is continuous on **R** and $h(x) = \sqrt{x}$ is continuous in $[0, \infty)$, therefore, $(hog)(x) = h(g(x)) = h(x^2 - x) = \sqrt{x^2 - x}$ is continuous for all $x$ such that $x^2 - x \geq 0$

$$\Rightarrow \quad \sqrt{x^2 - x} \text{ is continuous in } (-\infty, 0] \cup [1, \infty)$$

$$\Rightarrow \quad \sqrt{x^2 - x} \text{ is continuous in } [1, 4].$$

(*ii*) Differentiating (1) w.r.t. $x$, we get

$$f'(x) = \frac{1}{2}(x^2 - x)^{-1/2}(2x - 1) = \frac{2x-1}{2\sqrt{x^2 - x}}$$

which exists for all $x$ such that $x^2 - x > 0$ *i.e.* for $x$ in $(-\infty, 0) \cup (1, \infty)$

$\Rightarrow$ $f(x)$ is derivable for all $x$ in (1, 4).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number $c$ in (1, 4) such that

$$f'(c) = \frac{f(4) - f(1)}{4-1}$$

$$\Rightarrow \quad \frac{2c-1}{2\sqrt{c^2 - c}} = \frac{\sqrt{4^2 - 4} - \sqrt{1^2 - 1}}{4-1} = \frac{\sqrt{12}}{3} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad 3(2c-1)^2 = 16(c^2 - c) \Rightarrow 4c^2 - 4c - 3 = 0$$

$$\Rightarrow \quad (2c-3)(2c+1) = 0 \Rightarrow c = \frac{3}{2}, -\frac{1}{2}.$$

Thus, there exists $c = \frac{3}{2} \in (1, 4)$ such that $f'(c) = \frac{f(4) - f(1)}{4-1}$.

Hence, Lagrange's mean value theorem is verified.

**Example 3.** *Find a point on the graph of $y = x^3$ where the tangent is parallel to the chord joining (1, 1) and (3, 27).*

**Solution.** Let us apply Lagrange's mean value theorem to the function

$$f(x) = x^3 \text{ in the interval } [1, 3] \qquad \ldots(1)$$

(*i*) $f(x)$ being polynomial is continuous in [1, 3]

(*ii*) $f(x)$ being polynomial is derivable in (1, 3).

Thus, both the conditions of Lagrange's mean value theorem are satisfied by the function $f(x)$ in [1, 3], therefore, there exists atleast one real number $c$ in (1, 3) such that

$$f'(c) = \frac{f(3) - f(1)}{3-1}.$$

$$f(3) = 3^3 = 27 \text{ and } f(1) = 1^3 = 1.$$

Differentiating (1) w.r.t. $x$, we get

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2.$$

Now $f'(c) = \frac{f(3) - f(1)}{3-1} \Rightarrow 3c^2 = \frac{27-1}{3-1} \Rightarrow 3c^2 = 13$

$$\Rightarrow \quad c^2 = \frac{13}{3} = \frac{39}{9} \Rightarrow c = \pm\frac{\sqrt{39}}{3}.$$

But $c \in (1, 3) \Rightarrow c = \frac{\sqrt{39}}{3}$.

When $x = \frac{\sqrt{39}}{3}$, from (1), $y = \frac{13\sqrt{39}}{9}$.

Hence, there exists a point $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$ on the given curve $y = x^3$ where the tangent is parallel to the chord joining the points (1, 1) and (3, 27).

**Example 4.** *Does the Lagrange's mean value theorem apply to $f(x) = x^{1/3}, -1 \le x \le 1$? What conclusions can be drawn?*

**Solution.** Given $f(x) = x^{1/3}, x \in [-1, 1]$ $\qquad \ldots(1)$

(*i*) $f(x)$ is continuous in [– 1, 1].

(*ii*) Differentiating (1) w.r.t. $x$, we get

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}, \; x \ne 0 \qquad \ldots(2)$$

$\Rightarrow$ the derivative of $f(x)$ does not exist at $x = 0$

$\Rightarrow f(x)$ is not derivable in $(-1, 1)$.

Thus, the condition (*ii*) of Lagrange's mean value theorem is not satisfied by the function $f(x) = x^{1/3}$ in $[-1, 1]$ and hence Lagrange's mean value theorem is not applicable to the given function $f(x) = x^{1/3}$ in $[-1, 1]$.

**Conclusion.** However, from (2), $f'(c) = \frac{1}{3c^{2/3}}, \ c \neq 0$

Also $f(-1) = (-1)^{1/3} = -1, f(1) = 1^{1/3} = 1$

(we have taken only real values)

$$\therefore \quad f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} \Rightarrow \frac{1}{3c^{2/3}} = \frac{1 - (-1)}{1 - (-1)} = \frac{2}{2} = 1$$

$$\Rightarrow \quad c^{2/3} = \frac{1}{3} \Rightarrow c^2 = \frac{1}{27} \Rightarrow c = \pm \frac{1}{3\sqrt{3}}.$$

As $-1 < -\frac{1}{3\sqrt{3}} < \frac{1}{3\sqrt{3}} < 1 \Rightarrow c = \pm \frac{1}{3\sqrt{3}}$ both lie in $(-1, 1)$.

Thus, we find that there exist two real numbers $c = \pm \frac{1}{3\sqrt{3}}$ in $(-1, 1)$ such that $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$. It follows that the converse of Lagrange's mean value theorem may not be true.

## EXERCISE 8.2

*Verify Lagrange's mean value theorem for the following (1 to 5) functions in the given intervals. Also find 'c' of this theorem :*

**1.** (*i*) $f(x) = x(x - 2)$ in $[1, 3]$ (*ii*) $f(x) = x^2 + x - 1$ in $[0, 4]$

(*iii*) $f(x) = 2x^2 - 10x + 29$ in the interval $[2, 7]$

(*iv*) $f(x) = x(x + 4)^2$ in $[0, 4]$.

**2.** (*i*) $f(x) = x^3 - 3x - 1$ in $\left[-\frac{11}{7}, \frac{13}{7}\right]$

(*ii*) $f(x) = x(x - 1)(x - 2)$ in $\left[0, \frac{1}{2}\right]$ (*I.S.C. 2000*)

(*iii*) $f(x) = (x - 3)(x - 6)(x - 9)$ in $[3, 5]$.

**3.** (*i*) $f(x) = x + \frac{1}{x}$ in $[1, 3]$ (*ii*) $f(x) = \frac{1}{4x - 1}$ in $[1, 4]$

(*iii*) $f(x) = \frac{2x + 3}{3x - 2}$ in $[1, 5]$ (*iv*) $f(x) = x^{2/3}$ in $[0, 1]$.

**4.** (*i*) $f(x) = \sin x$ in $\left[0, \frac{\pi}{2}\right]$ (*ii*) $f(x) = x - 2\sin x$ in $[-\pi, \pi]$

(*iii*) $f(x) = 2\sin x + \sin 2x$ in $[0, \pi]$.

**5.** (*i*) $f(x) = \log x$ on $[1, 2]$ (*ii*) $f(x) = \sqrt{x^2 - 4}$ on $[2, 4]$

(*iii*) $f(x) = \sin x - \sin 2x$ on $[0, \pi]$ (*I.S.C. 2011*)

**6.** Find a point on the parabola $y = (x - 3)^2$ where the tangent is parallel to the chord joining $(3, 0)$ and $(4, 1)$.

**7.** Find a point on the curve $y = x^3 - 3x$ where the tangent to curve is parallel to the chord joining $(1, -2)$ and $(2, 2)$.

**8.** Show that the function $f(x) = x^2 - 6x + 1$ satisfies the Lagrange's mean value theorem. Also find the co-ordinates of a point at which the tangent to the curve represented by the above function is parallel to the chord joining A $(1, -4)$ and B $(3, -8)$. (*I.S.C. 2004*)

**Hint.** Check Lagrange's mean value theorem for the given function in the interval $[1, 3]$.

**9.** Using Lagrange's mean value theorem, find a point on the curve $y = \sqrt{x-2}$ defined in the interval [2, 3] where the tangent is parallel to the chord joining the end points of the curve. (*I.S.C. 2008*)

**10.** Discuss the applicability of Lagrange's mean value theorem for the following functions in the indicated intervals.

(*i*) $f(x) = |x|$ in $[-1, 1]$ (*ii*) $f(x) = 1 - (2 - x)^{2/3}$ in $[1, 3]$.

## 8.3 MAXIMA AND MINIMA

### 8.3.1 Absolute maxima and absolute minima

*Let f be a real valued function defined on D (subset of* **R**), *then*

(*i*) *f is said to have* ***absolute maxima*** *at $x = c$ (in D) iff $f(x) \le f(c)$ for all $x \in D$, and c is called point of* ***absolute maxima*** *and $f(c)$ is called* ***absolute maximum*** *(or* ***greatest****)* ***value*** *of f on D.*

(*ii*) *f is said to have* ***absolute minima*** *at $x = d$ (in D) iff $f(d) \le f(x)$ for all $x \in D$, and d is called point of* ***absolute minima*** *and $f(d)$ is called* ***absolute minimum*** *(or* ***smallest****)* ***value*** *of f on D.*

Obviously, *absolute maximum and absolute minimum values* of a function (if they exist) are unique. However, absolute maximum or absolute minimum values of a given function *f* on D (subset of **R**) may be obtained at more than one points. Also it is not essential that a given function must have *maxima* or *minima* in its domain.

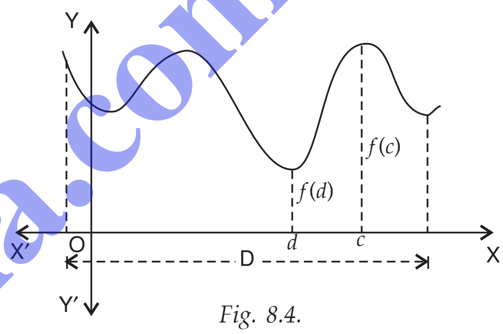


*Fig. 8.4.*

For example :

(*i*) Consider the function $f(x) = \sin x$ in $\left[-\pi, \frac{3\pi}{2}\right]$.

Its graph is shown in fig. 8.5. Here $c = \frac{\pi}{2}$ is a point of maxima and maximum value $= f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$.

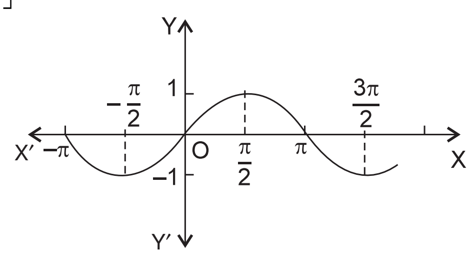


*Fig. 8.5.*

There are two points of minima, $d = -\frac{\pi}{2}$ or $\frac{3\pi}{2}$ and the minimum value

$$= f\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

$$\text{or } f\left(\frac{3\pi}{2}\right) = \sin\frac{3\pi}{2} = -\sin\frac{\pi}{2} = -1.$$

(*ii*) Consider the function

$f(x) = \cos x$ in $\left[-\frac{\pi}{2}, 2\pi\right]$.

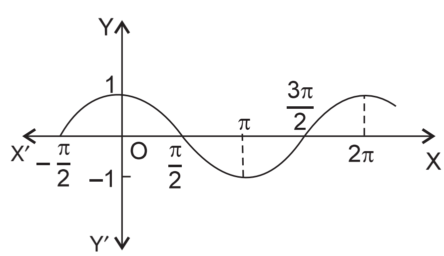


*Fig. 8.6.*

Its graph is shown in fig. 8.6.

Here $d = \pi$ is a point of minima and minimum value $= f(\pi) = \cos \pi = -1$.

There are two points of maxima, $c = 0, 2\pi$ and the maximum value

$$= f(0) = \cos 0 = 1 \text{ or } f(2\pi) = \cos 2\pi = \cos 0 = 1.$$

(*iii*) Consider the function $f(x) = x^2$,

$D_f = \mathbf{R}$ and $R_f = [0, \infty)$.

The graph of $f$ is shown in fig. 8.7.

The function $f$ has minima at $x = 0$ and minimum value $= f(0) = 0$. Note that $f$ has no maxima.

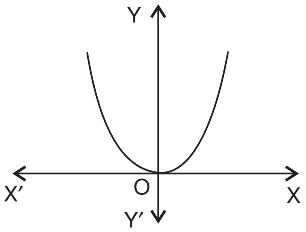



*Fig. 8.7.*

(*iv*) Consider the function $f(x) = 2x - x^2$,

$D_f = \mathbf{R}$. It can be written as

$y = 2x - x^2 = 1 - (x - 1)^2$,

$R_f = (-\infty, 1]$.

The graph of $f$ is shown in fig. 8.8.

The function $f$ has maxima at $x = 1$ and maximum value $= f(1) = 2 . 1 - 1^2 = 1$.

Note that $f$ has no minima.

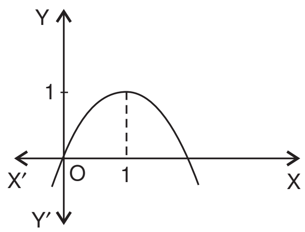



*Fig. 8.8.*

## 8.3.2 Local maxima and local minima

*Let f be a real valued function defined on D (subset of* ***R****), then*

(*i*) *f is said to have a* ***local*** *(or* ***relative****)* ***maxima*** *at $x = c$ (in D) iff there exists a positive real number $\delta$ such that $f(x) \le f(c)$ for all x in $(c - \delta, c + \delta)$ i.e. $f(x) \le f(c)$ for all x in the immediate neighbourhood of c, and c is called point of* ***local maxima*** *and $f(c)$ is called* ***local maximum value****.*

(*ii*) *f is said to have* ***local*** *(or* ***relative****)* ***minima*** *at $x = d$ (in D) iff there exists some positive real number $\delta$ such that $f(d) \le f(x)$ for all $x \in (d - \delta, d + \delta)$ i.e. $f(d) \le f(x)$ for all x in the immediate neighbourhood of d, and d is called point of* ***local minima*** *and $f(d)$ is called* ***local minimum value.***

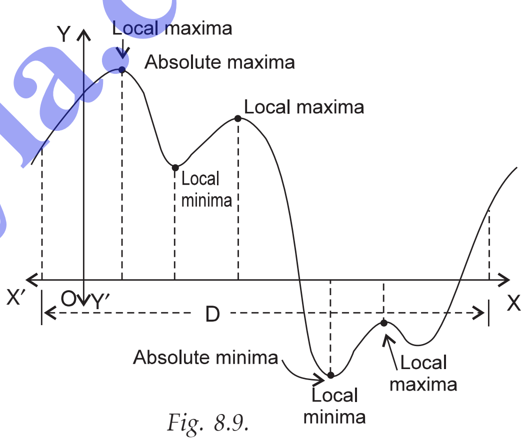



*Fig. 8.9.*



Geometrically, a point $c$ in the domain of the given function $f$ is a point of *local maxima* or *local minima* according as the graph of $f$ has a **peak** or **trough** (**cavity**) at $c$.

(*iii*) *a point (in D) which is either a point of local maxima or a point of local minima is called an* ***extreme point****, and the value of the function at this point is called an* ***extreme value****.*

## Remarks

1. A local maximum (minimum) value may not be the absolute maximum (minimum) value.
2. A local maximum value at some point may be less than a local minimum value of the function at another point.

### Stationary (or Turning) Point

*Let f be a real valued function defined on D (subset of* ***R****), then a point c (in D) is called a* ***stationary*** *(or* ***turning*** *or* ***critical****)* ***point*** *of f iff f is differentiable at $x = c$ and $f'(c) = 0$.*

*However, it is not essential that an extreme point is a stationary point, and a stationary point is an extreme point.*

For example :

(*i*) Consider the function $f(x) = \sin x$, $D_f = \mathbf{R}$. A portion of the graph of this function is shown in fig. 8.5.

**11.** Find the area of the largest isosceles triangle having perimeter 18 metres.

**Hint.** Let $x$ be one of two equal sides, then base = 18 – 2 $x$, then

area = A = $\sqrt{9(9-x)(9-x)(9-(18-2x))}$, $\frac{9}{2} < x < 9$, for, if $x < \frac{9}{2}$, then the sum of two sides is smaller than the third side.

**12.** Find a point on the hypotenuse of a given right-angled triangle from which the perpendiculars can be dropped on the other sides to form a rectangle of maximum area.

**Hint.** Let $l$ be the length of the hypotenuse of the given right-angled triangle ABC at B and $\angle CAB = \alpha$ (in radian measure), $0 < \alpha < \frac{\pi}{2}$.
As $\Delta$ABC is given, $l$ and $\alpha$ are fixed. Let P be a point on AC and AP = $x$, then PC = $l - x$.

**13.** Divide the number 4 into two positive numbers such that the sum of square of one and the cube of other is minimum.

**14.** The perimeter of a sector of a circle is $p$. Show that its area is maximum when its radius is $\frac{p}{4}$.

**15.** Find the minimum distance from the point (4, 2) to the parabola $y^2 = 8x$.

**16.** Find the point on the curve $y^2 = 2x$ which is nearest to the point (1, –4).

**17.** Find the dimensions of the rectangle of maximum area that can be inscribed in the portion of the parabola $y^2 = 4px$ intercepted by the line $x = a$.

**18.** A point on the hypotenuse of a right angled triangle is at distances $a$ and $b$ from the sides. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$.

**19.** Show that the maximum volume of a cylinder which can be inscribed in a cone of height $h$ and semi-vertical angle 30° is $\frac{4}{81}\pi h^3$.

**20.** A cylinder is such that the sum of its height and circumference of its base is 10 metres. Find the maximum volume of the cylinder.

**Hint.** If $h$ metres be the height and $r$ metres be the radius of the base of the cylinder, then $2\pi r + h = 10 \Rightarrow h = 10 - 2\pi r$.
V (volume of cylinder) = $\pi r^2 h = \pi r^2 (10 - 2\pi r)$.

**21.** Find the semi-vertical angle of the cone of maximum curved surface that can be inscribed in a sphere of radius R.



## ANSWERS

### EXERCISE 8.1

**1.** (*i*) $\frac{5}{2}$ (*ii*) $-\frac{5}{2}$ (*iii*) 4.

**2.** (*i*) $\frac{4}{3}$ (*ii*) $2 \pm \frac{1}{\sqrt{3}}$ (*iii*) 2 (*iv*) $\pm\frac{2}{\sqrt{3}}$ (*v*) $\frac{2}{3}$.

**3.** (*i*) 0 (*ii*) $\frac{3\pi}{2}$.

**4.** (*i*) $\frac{\pi}{4}$ (*ii*) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ (*iii*) $\frac{\pi}{2}$ (*iv*) $\frac{\pi}{4}$.

**5.** (*i*) 0 (*ii*) 0 (*iii*) $\frac{\pi}{4}$ (*iv*) 0.

**6.** (*i*) $\frac{3\pi}{4}$ (*ii*) $\frac{\pi}{4}$ (*iii*) $\frac{\pi}{4}$ (*iv*) $\pi$.

**7.** (*i*) (0, 0) (*ii*) ($\pi$, – 2).

**8.** $\left(\frac{\pi}{4}, \sqrt{2} - 1\right)$.

**9.** $a = -5$, $b = 8$.

**11.** Not applicable in both parts (*i*) and (*ii*), for :
(*i*) not derivable at $x = 0$ (*ii*) not derivable at $x = 0$.

**12.** Not applicable in parts (*i*) to (*iv*), for:

(*i*) discontinuous at $\frac{\pi}{2}$ (*ii*) discontinuous at $\frac{\pi}{2}, \frac{3\pi}{2}$

(*iii*) not derivable at $x = 1$ (*iv*) not derivable at $x = 2$.

## EXERCISE 8.2

**1.** (*i*) 2 (*ii*) 2 (*iii*) $\frac{9}{2}$ (*iv*) $\frac{-8+4\sqrt{13}}{3}$.

**2.** (*i*) $\pm 1$ (*ii*) $1 - \frac{\sqrt{21}}{6}$ (*iii*) $6 - \frac{\sqrt{39}}{3}$.

**3.** (*i*) $\sqrt{3}$ (*ii*) $\frac{1+3\sqrt{5}}{4}$ (*iii*) $\frac{1}{3}\left(2+\sqrt{13}\right)$ (*iv*) $\frac{8}{27}$.

**4.** (*i*) $\cos^{-1}\left(\frac{2}{\pi}\right)$ (*ii*) $\pm\frac{\pi}{2}$ (*iii*) $\frac{\pi}{3}$.

**5.** (*i*) $\log_2 e$ (*ii*) $\sqrt{6}$ (*iii*) $\cos^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right)$.

**6.** $\left(\frac{7}{2}, \frac{1}{4}\right)$. **7.** $\left(\sqrt{\frac{7}{3}}, -\frac{2}{3}\sqrt{\frac{7}{3}}\right)$. **8.** $(2, -7)$. **9.** $\left(\frac{9}{4}, \frac{1}{2}\right)$.

**10.** Not applicable in both (*i*) and (*ii*), for:

(*i*) not derivable at $x = 0$ (*ii*) not derivable at $x = 2$.

## EXERCISE 8.3

**1.** (*i*) Minimum value = 3, no maximum value.
(*ii*) Maximum value = 7, no minimum value.

**2.** (*i*) Minimum value = – 2, no maximum value.
(*ii*) Maximum value = 9, no minimum value.

**3.** (*i*) Neither maximum nor minimum.
(*ii*) Minimum value = 0, no maximum value.

**4.** (*i*) Maximum value = 3, no minimum value.
(*ii*) Maximum value = 6, minimum value = 4.

**5.** (*i*) Maximum value = 4, minimum value = 2.
(*ii*) Maximum value = –2, minimum value = –3.

**6.** (*i*) Maximum value = 5, minimum value = –1.
(*ii*) Maximum value = 5, minimum value = –5.

**7.** –2. **8.** 1. **9.** $x = \frac{\pi}{4}$.

**11.**

| | Maximum value | Minimum value | Point of maxima | Point of minima |
|---|---|---|---|---|
| (*i*) | 8 | –8 | $x = 2$ | $x = -2$ |
| (*ii*) | 19 | 3 | $x = -3$ | $x = 1$ |
| (*iii*) | 738 | 8 | $x = 10$ | $x = 0$ |
| (*iv*) | 25 | –39 | $x = 0$ | $x = 2$ |
| (*v*) | $2\pi$ | 0 | $x = 2\pi$ | $x = 0$ |
| (*vi*) | $\sqrt{2}$ | –1 | $x = \frac{\pi}{4}$ | $x = \pi$ |
| (*vii*) | 7 | 3 | $x = 3$ | $x = -1$ |
| (*viii*) | $\frac{3}{2}$ | –3 | $x = \frac{\pi}{6}, \frac{5\pi}{6}$ | $x = \frac{\pi}{2}$. |

**12.** 18; –9. **13.** $\frac{\pi}{4}, \frac{5\pi}{4}$.

**14.** Maximum value = $2\pi$, minimum value = 0.