7 Indeterminate Forms

INTRODUCTION

We know that if $\underset{x\to c}{\text{Lt}}\, f(x)$, $\underset{x\to c}{\text{Lt}}\, g(x)$ both exist and $\underset{x\to c}{\text{Lt}}\, g(x) \neq 0$, then $\underset{x\to c}{\text{Lt}}\, \frac{f(x)}{g(x)} = \frac{\underset{x\to c}{\text{Lt}}\, f(x)}{\underset{x\to c}{\text{Lt}}\, g(x)}$. The question arises, what happens if $\underset{x\to c}{\text{Lt}}\, g(x) = 0$. It is easy to see that if $\underset{x\to c}{\text{Lt}}\, g(x) = 0$, then the necessary condition for $\underset{x\to c}{\text{Lt}}\, \frac{f(x)}{g(x)}$ to exist (finitely) is that $\underset{x\to c}{\text{Lt}}\, f(x) = 0$.

In fact, if $\underset{x\to c}{\text{Lt}}\, \frac{f(x)}{g(x)}$ exists, say l, then

$$\underset{x\to c}{\text{Lt}}\, f(x) = \underset{x\to c}{\text{Lt}}\, \frac{f(x)}{g(x)} . g(x) = \underset{x\to c}{\text{Lt}}\, \frac{f(x)}{g(x)} \cdot \underset{x\to c}{\text{Lt}}\, g(x) = l.\ 0 = 0.$$

If $\underset{x\to c}{\text{Lt}}\, f(x) = 0 = \underset{x\to c}{\text{Lt}}\, g(x)$, then $\frac{f(x)}{g(x)}$ is said to assume **indeterminate form** $\frac{0}{0}$ as $x \to c$.

We also have some other *indeterminate forms* such as $\frac{\infty}{\infty}$, $\infty - \infty$, $0.\infty$, 0°, ∞° and 1^{∞} etc.

7.1 INDETERMINATE FORM $\frac{0}{0}$

L' Höpital's rule

If $f(x)$, $g(x)$ are differentiable and $g'(x) \neq 0$ for all x in $(c - \delta, c + \delta)$ except possibly at $x = c$, $\underset{x\to c}{Lt}\, f(x) = 0 = \underset{x\to c}{Lt}\, g(x)$ and $\underset{x\to c}{Lt}\, \frac{f'(x)}{g'(x)}$ exists (finitely or infinitely), then $\underset{x\to c}{Lt}\, \frac{f(x)}{g(x)} = \underset{x\to c}{Lt}\, \frac{f'(x)}{g'(x)}$.

(We accept it without proof.)

Remark. L' Höpital's rule remains valid when $\underset{x\to c}{Lt}$ is replaced by one sided limits $\underset{x\to c^-}{\text{Lt}}$ or $\underset{x\to c^+}{\text{Lt}}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following limits :*

(*i*) $\underset{x\to 4}{Lt}\, \frac{x^4 - 256}{x^2 - 16}$ (*ii*) $\underset{x\to \frac{\pi}{2}}{Lt}\, \frac{\cos x}{\frac{\pi}{2} - x}$ (*iii*) $\underset{x\to \frac{\pi}{4}}{Lt}\, \frac{1 - \tan x}{\cos 2x}$. *(I.S.C. 2009)*

Solution. (*i*) $\displaystyle \operatorname*{Lt}_{x \to 4} \frac{x^4 - 256}{x^2 - 16}$ $\left(\frac{0}{0} \text{ form, use L'Hôpital's rule}\right)$

$$= \operatorname*{Lt}_{x \to 4} \frac{4x^3 - 0}{2x - 0} = \operatorname*{Lt}_{x \to 4} 2x^2 = 2.4^2 = 32.$$

(*ii*) $\displaystyle \operatorname*{Lt}_{x \to \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x}$ $\left(\frac{0}{0} \text{ form, use L'Hôpital's rule}\right)$

$$= \operatorname*{Lt}_{x \to \frac{\pi}{2}} \frac{-\sin x}{-1} = \operatorname*{Lt}_{x \to \frac{\pi}{2}} \sin x = \sin \frac{\pi}{2} = 1.$$

(*iii*) $\displaystyle \operatorname*{Lt}_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{\cos 2x}$ $\left(\frac{0}{0} \text{ form, use L'Hôpital's rule}\right)$

$$= \operatorname*{Lt}_{x \to \frac{\pi}{4}} \frac{0 - \sec^2 x}{-\sin 2x . 2} = \operatorname*{Lt}_{x \to \frac{\pi}{4}} \frac{\sec^2 x}{2 \sin 2x}$$

$$= \frac{\sec^2 \frac{\pi}{4}}{2 \sin \frac{\pi}{2}} = \frac{(\sqrt{2})^2}{2.1} = \frac{2}{2} = 1.$$

Example 2. *Evaluate the following limits :*

(*i*) $\displaystyle \operatorname*{Lt}_{x \to 0} \frac{\sin x - x + \frac{1}{6} x^3}{x^3}$ (*I.S.C. 2001*) (*ii*) $\displaystyle \operatorname*{Lt}_{x \to 0} \frac{x e^x - \log (1 + x)}{x^2}$.

Solution. (*i*) $\displaystyle \operatorname*{Lt}_{x \to 0} \frac{\sin x - x + \frac{1}{6} x^3}{x^3}$ $\left(\frac{0}{0} \text{ form, use L'Hôpital's rule}\right)$

$$= \operatorname*{Lt}_{x \to 0} \frac{\cos x - 1 + \frac{1}{2} x^2}{3x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \operatorname*{Lt}_{x \to 0} \frac{-\sin x + x}{6x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \operatorname*{Lt}_{x \to 0} \frac{-\cos x + 1}{6} = \frac{-1 + 1}{6} = 0.$$

(*ii*) $\displaystyle \operatorname*{Lt}_{x \to 0} \frac{x e^x - \log (1 + x)}{x^2}$ $\left(\frac{0}{0} \text{ form, use L'Hôpital's rule}\right)$

$$= \operatorname*{Lt}_{x \to 0} \frac{x . e^x + e^x . 1 - \frac{1}{1 + x}}{2x}$$

$$= \operatorname*{Lt}_{x \to 0} \frac{(x + 1) e^x - \frac{1}{1 + x}}{2x} \qquad \left(\frac{0}{0} \text{ form, use L'Hôpital's rule}\right)$$

$$= \operatorname*{Lt}_{x \to 0} \frac{(x + 1) . e^x + e^x . 1 + \frac{1}{(1 + x)^2}}{2} = \frac{(0 + 1) . 1 + 1 + \frac{1}{1}}{2} = \frac{3}{2}.$$

Example 3. *Evaluate the following limits :*

(*i*) $\displaystyle \operatorname*{Lt}_{x \to 1} \frac{1 + \log x - x}{1 - 2x + x^2}$ (*ii*) $\displaystyle \operatorname*{Lt}_{x \to 0} \frac{e^x + e^{-x} + 2 \cos x - 4}{x^4}$.

Solution. (*i*) $\displaystyle \operatorname*{Lt}_{x \to 1} \frac{1 + \log x - x}{1 - 2x + x^2}$ $\left(\frac{0}{0} \text{ form, use L'Hôpital's rule}\right)$

$$= \operatorname*{Lt}_{x \to 1} \frac{0 + \frac{1}{x} - 1}{0 - 2 + 2x} = \operatorname*{Lt}_{x \to 1} \frac{1 - x}{-2x (1 - x)} = \operatorname*{Lt}_{x \to 1} \frac{1}{-2x} = -\frac{1}{2}.$$

(*ii*) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x + e^{-x} + 2\cos x - 4}{x^4}$ $\left(\dfrac{0}{0} \text{ form, use L'Höpital's rule}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2\sin x}{4x^3}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{e^x + e^{-x} - 2\cos x}{12x^2}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{e^x - e^{-x} + 2\sin x}{24x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{e^x + e^{-x} + 2\cos x}{24} = \dfrac{1+1+2}{24} = \dfrac{1}{6}.$

Example 4. *Evaluate the following limits :*

(*i*) $\underset{x\to\pi^+}{Lt} \dfrac{\sin x}{\sqrt{x-\pi}}$ (*ii*) $\underset{x\to 0}{Lt} \dfrac{\log(1+x^3)}{\sin^3 x}.$

Solution. (*i*) $\underset{x\to\pi^+}{\text{Lt}} \dfrac{\sin x}{\sqrt{x-\pi}}$ $\left(\dfrac{0}{0} \text{ form, use L'Höpital's rule}\right)$

$= \underset{x\to\pi^+}{\text{Lt}} \dfrac{\cos x}{\frac{1}{2}(x-\pi)^{-1/2}\cdot 1} = \underset{x\to\pi^+}{\text{Lt}} 2\sqrt{x-\pi}\ \cos x = 0.$

It may be noted that $\sqrt{x-\pi}$ is not defined on the left of π so that the left limit does not exist.

(*ii*) $\underset{x\to 0}{\text{Lt}} \dfrac{\log(1+x^3)}{\sin^3 x} = \underset{x\to 0}{\text{Lt}} \dfrac{\log(1+x^3)}{x^3}\cdot\left(\dfrac{x}{\sin x}\right)^3$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\log(1+x^3)}{x^3} \cdot 1^3$ $\left(\because \underset{x\to 0}{\text{Lt}} \dfrac{x}{\sin x} = 1\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\log(1+x^3)}{x^3}$ $\left(\dfrac{0}{0} \text{ form, use L'Höpital's rule}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\frac{1}{1+x^3}\cdot 3x^2}{3x^2} = \underset{x\to 0}{\text{Lt}} \dfrac{1}{1+x^3} = \dfrac{1}{1+0} = 1.$

Example 5. *Evaluate the following limits :*

(*i*) $\underset{x\to 0}{Lt} \dfrac{x - \tan^{-1} x}{x - \sin x}$ *(I.S.C. 2011)* (*ii*) $\underset{x\to 0}{Lt} \dfrac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}.$

Solution. (*i*) $\underset{x\to 0}{\text{Lt}} \dfrac{x - \tan^{-1} x}{x - \sin x}$ $\left(\dfrac{0}{0} \text{ form, use L'Höpital's rule}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{1 - \frac{1}{1+x^2}}{1-\cos x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{0 - (-1)(1+x^2)^{-2}.2x}{\sin x} = \underset{x\to 0}{\text{Lt}} \dfrac{2x}{(1+x^2)^2 \sin x}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{2}{(1+x^2)^2}\cdot\left(\dfrac{x}{\sin x}\right) = \dfrac{2}{(1+0)^2}.1 = 2.$

(*ii*) $\underset{x\to 0}{\text{Lt}} \dfrac{1+\sin x - \cos x + \log(1-x)}{x\tan^2 x} = \underset{x\to 0}{\text{Lt}} \dfrac{1+\sin x - \cos x + \log(1-x)}{x^3}\cdot\left(\dfrac{x}{\tan x}\right)^2$

$= \underset{x\to 0}{\text{Lt}} \dfrac{1+\sin x - \cos x + \log(1-x)}{x^3}.1^2$ $\left(\because \underset{x\to 0}{\text{Lt}} \dfrac{x}{\tan x} = 1\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{1+\sin x - \cos x + \log(1-x)}{x^3}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\cos x + \sin x + \frac{1}{1-x}(-1)}{3x^2}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{-\sin x + \cos x - \frac{1}{(1-x)^2}}{6x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x\to 0}{\text{Lt}} \dfrac{-\cos x - \sin x - \frac{2}{(1-x)^3}}{6} = \dfrac{-1-0-2}{6} = -\dfrac{1}{2}.$

Example 6. *What is the fallacy in the following use of L' Höpital's rule ?*

$$\underset{x \to 1}{Lt} \frac{x^3 + 3x - 4}{2x^2 + x - 3} = \underset{x \to 1}{Lt} \frac{3x^2 + 3}{4x + 1} = \underset{x \to 1}{Lt} \frac{6x}{4} = \frac{3}{2}$$

Solution. The function $\frac{3x^2 + 3}{4x + 1}$ is not of the form $\frac{0}{0}$ as $x \to 1$, therefore, L'Höpital's rule is not applicable to evaluate $\underset{x \to 1}{\text{Lt}} \frac{3x^2 + 3}{4x + 1}$.

In fact, we have

$$\underset{x \to 1}{\text{Lt}} \frac{3x^2 + 3}{4x + 1} = \frac{3.1^2 + 3}{4.1 + 1} = \frac{6}{5} \text{ and hence } \underset{x \to 1}{\text{Lt}} \frac{x^3 + 3x - 4}{2x^2 + x - 3} = \frac{6}{5}.$$

EXERCISE 7.1

Evaluate the following (1 to 13) limits :

1. (*i*) $\underset{x \to 3}{\text{Lt}} \frac{x^4 - 81}{x - 3}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{(1 + x)^n - 1}{x}$.

2. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{\sin ax}{\sin bx}$ (*ii*) $\underset{x \to 2}{\text{Lt}} \frac{e^x - e^2}{x - 2}$.

3. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{x\, e^x}{1 - e^x}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{\tan 2x}$.

4. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{e^x - (1 + x)}{x^2}$ (*ii*) $\underset{x \to 1}{\text{Lt}} \frac{x^2 - x \log x + \log x - 1}{x - 1}$.

5. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{\cos x - 1}{\cos 2x - 1}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{8^x - 2^x}{4x}$.

6. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{x - \tan x}{x - \sin x}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{2 \tan^{-1} x - x}{2x - \sin^{-1} x}$.

7. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{\log \sec 2x}{\log \sec x}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{\cos 2x - \cos x}{\sin^2 x}$.

8. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{x - \sin x}{x^3}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2 \log(1 + x)}{x \sin x}$.

9. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{e^x + e^{-x} + 2 \cos x - 4}{x^3}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{x^2 + 2 \cos x - 2}{x \sin^3 x}$.

10. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{\log(1 - x)}{\tan \frac{\pi}{2} x}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{(\tan^{-1} x)^2}{\log(1 + x^2)}$.

11. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ (*I.S.C. 2013*) (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{e^x - e^{\sin x}}{x - \sin x}$.

12. (*i*) $\underset{x \to 0}{\text{Lt}} \frac{\log(1 - x^2)}{\log \cos x}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{e^x \sin x - x - x^2}{x^3}$.

13. (*i*) $\underset{x \to 0^+}{\text{Lt}} \frac{3^x - 2^x}{\sqrt{x}}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{(1 + x)^n - nx - 1}{x^2}$, $n > 1$.

14. What is the fallacy in the following use of L'Höpital's rule ?

$$\underset{x \to 2}{\text{Lt}} \frac{x^3 - x^2 - x - 2}{x^3 - 3x^2 + 3x - 2} = \underset{x \to 2}{\text{Lt}} \frac{3x^2 - 2x - 1}{3x^2 - 6x + 3} = \underset{x \to 2}{\text{Lt}} \frac{6x - 2}{6x - 6} = \underset{x \to 2}{\text{Lt}} \frac{6}{6} = 1$$

7.2 INDETERMINATE FORM $\frac{\infty}{\infty}$

L' Höpital's rule

If $f(x)$, $g(x)$ are differentiable and $g'(x) \neq 0$ for all x in $(c - \delta, c + \delta)$ except possibly at $x = c$, $\underset{x \to c}{Lt} f(x) \to \infty$, $\underset{x \to c}{Lt} g(x) \to \infty$ and $\underset{x \to c}{Lt} \frac{f'(x)}{g'(x)}$ exists (finitely or infinitely), then $\underset{x \to c}{Lt} \frac{f(x)}{g(x)} = \underset{x \to c}{Lt} \frac{f'(x)}{g'(x)}$.

(We accept it without proof.)

Analogously, we have L'Höpital's rule when $x \to -\infty$.

7.2.1 Indeterminate forms $\infty - \infty$ and $0 . \infty$

These may be handled by first transforming to one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. See examples 3 and 4 (below).

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following limits :*

(*i*) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \dfrac{\tan x}{\tan 3x}$ (*ii*) $\underset{x \to \infty}{\text{Lt}} \dfrac{e^x + 3x^3}{4e^x + 2x^2}$.

Solution. (*i*) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \dfrac{\tan x}{\tan 3x}$ $\left(\frac{\infty}{\infty}\text{ form, use L'Höpital's rule}\right)$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\sec^2 x}{\sec^2 3x \,.\, 3} = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\cos^2 3x}{3\cos^2 x} = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{(4\cos^3 x - 3\cos x)^2}{3\cos^2 x}$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{(4\cos^2 x - 3)^2}{3} = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{(4.0 - 3)^2}{3} = \frac{9}{3} = 3.$$

(*ii*) $\underset{x \to \infty}{\text{Lt}} \dfrac{e^x + 3x^3}{4e^x + 2x^2}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \underset{x \to \infty}{\text{Lt}} \frac{e^x + 9x^2}{4e^x + 4x} \quad \left(\tfrac{\infty}{\infty}\text{ form}\right)$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{e^x + 18x}{4e^x + 4} \quad \left(\tfrac{\infty}{\infty}\text{ form}\right)$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{e^x + 18}{4e^x} \quad \left(\tfrac{\infty}{\infty}\text{ form}\right)$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{e^x}{4e^x} = \frac{1}{4}.$$

Example 2. *Evaluate the following limits :*

(*i*) $\underset{x \to c^+}{\text{Lt}} \dfrac{\log (x - c)}{\log (e^x - e^c)}$ (*ii*) $\underset{x \to 0^+}{\text{Lt}} \log_{\sin 2x} \sin x$.

Solution. (*i*) $\underset{x \to c^+}{\text{Lt}} \dfrac{\log (x - c)}{\log (e^x - e^c)}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \underset{x \to c^+}{\text{Lt}} \frac{\frac{1}{x - c}}{\frac{1}{e^x - e^c} \cdot e^x} = \underset{x \to c^+}{\text{Lt}} \frac{e^x - e^c}{(x - c)\, e^x} \quad \left(\tfrac{0}{0}\text{ form}\right)$$

$$= \underset{x \to c^+}{\text{Lt}} \frac{e^x}{e^x.1 + (x - c)\, e^x} = \underset{x \to c^+}{\text{Lt}} \frac{1}{1 + x - c} = \frac{1}{1} = 1.$$

(*ii*) $\underset{x \to 0^+}{\text{Lt}} \log_{\sin 2x} \sin x = \underset{x \to 0^+}{\text{Lt}} \dfrac{\log \sin x}{\log \sin 2x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \underset{x \to 0^+}{\text{Lt}} \frac{\frac{1}{\sin x} \cdot \cos x}{\frac{1}{\sin 2x} \cdot \cos 2x \cdot 2} = \underset{x \to 0^+}{\text{Lt}} \frac{\cos x}{\sin x} \cdot \frac{2 \sin x \cos x}{2 \cos 2x}$$

$$= \underset{x \to 0^+}{\text{Lt}} \frac{\cos^2 x}{\cos 2x} = \frac{1}{1} = 1.$$

Example 3. *Evaluate the following limits :*

(*i*) $\underset{x \to 1}{\text{Lt}} \left(\dfrac{x}{x - 1} - \dfrac{1}{\log x}\right)$ (*ii*) $\underset{x \to 0}{\text{Lt}} \left(\text{cosec}\, x - \dfrac{1}{x}\right)$

(*iii*) $\underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{x^2} - \dfrac{\cot x}{x}\right)$ (*iv*) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(x \tan x - \dfrac{\pi}{2} \sec x\right)$. *(I.S.C. 2010)*

7. Evaluate the following limits :

(*i*) $\underset{x \to 1}{\text{Lt}} (2-x)^{\tan \frac{\pi x}{2}}$ (*ii*) $\underset{x \to 1}{\text{Lt}} x^{\frac{1}{x-1}}$.

Hint. (*ii*) Let $x = 1 + t$, so that when $x \to 1$, $t \to 0$.

$$\therefore \underset{x \to 1}{\text{Lt}}\ x^{\frac{1}{x-1}} = \underset{t \to 0}{\text{Lt}}\ (1+t)^{\frac{1}{t}}.$$

ANSWERS

EXERCISE 7.1

1. (*i*) 108 (*ii*) n. **2.** (*i*) $\frac{a}{b}$ (*ii*) e^2. **3.** (*i*) -1 (*ii*) $\frac{1}{2}$.

4. (*i*) $\frac{1}{2}$ (*ii*) 2. **5.** (*i*) $\frac{1}{4}$ (*ii*) $\frac{1}{2}\log 2$. **6.** (*i*) -2 (*ii*) 1.

7. (*i*) 4 (*ii*) $-\frac{3}{2}$. **8.** (*i*) $\frac{1}{6}$ (*ii*) 1. **9.** (*i*) 0 (*ii*) $\frac{1}{12}$.

10. (*i*) $-\frac{2}{\pi}$ (*ii*) 1. **11.** (*i*) 2 (*ii*) 1. **12.** (*i*) 2 (*ii*) $\frac{1}{3}$.

13. (*i*) 0 (*ii*) $\frac{n(n-1)}{2}$. **14.** $\frac{3x^2-2x-1}{3x^2-6x+3}$ is not of the form $\frac{0}{0}$ as $x \to 2$.

EXERCISE 7.2

1. (*i*) 0 (*ii*) $-\infty$. **2.** (*i*) 0 (*ii*) 0. **3.** (*i*) 0 (*ii*) 1.

4. (*i*) 0 (*ii*) 5. **5.** (*i*) 0 (*ii*) 2. **6.** (*i*) $-\frac{1}{\pi}$ (*ii*) 0.

7. (*i*) $\frac{1}{2}$ (*ii*) $\frac{1}{2}$. **8.** (*i*) $\frac{1}{2}$ (*ii*) $\frac{1}{2}$. **9.** (*i*) $\frac{1}{2}$ (*ii*) $\frac{\pi}{4}$.

10. (*i*) 0 (*ii*) $-\frac{2}{3}$. **11.** (*i*) 0 (*ii*) 0. **12.** (*i*) 0 (*ii*) 0.

13. (*i*) 0 (*ii*) 0. **14.** (*i*) $\frac{2}{\pi}$ (*ii*) $\frac{2}{\pi}$. **15.** (*i*) 1 (*ii*) $\frac{2c}{\pi}$.

EXERCISE 7.3

1. (*i*) 1 (*ii*) e. **2.** (*i*) 1 (*ii*) 1. **3.** (*i*) $\frac{1}{e}$ (*ii*) 1.

4. (*i*) 1 (*ii*) 1. **5.** (*i*) 1 (*ii*) $\frac{1}{e}$. **6.** (*i*) e (*ii*) e^5.

EXERCISE 7.4

1. (*i*) 2 (*ii*) 1. **2.** -2; -1. **3.** $a = \frac{1}{2}$, $b = -\frac{1}{2}$.

4. $a = 1$, $b = 2$, $c = 1$. **5.** (*i*) 0 (*ii*) a. **6.** (*i*) 1 (*ii*) $-\frac{2}{\pi}$.

7. (*i*) $e^{1/3}$ (*ii*) e^{10}. **8.** e^{-1}.

CHAPTER TEST

1. (*i*) $\frac{2}{\pi}$ (*ii*) $-\frac{1}{\sqrt{2}}$. **2.** (*i*) $-\frac{2}{3}$ (*ii*) -2. **3.** (*i*) $-\frac{1}{4}$ (*ii*) $-\frac{1}{3}$.

4. (*i*) 0 (*ii*) $\frac{1}{3}$. **5.** (*i*) $-\frac{1}{3}$ (*ii*) $-\frac{1}{6}$. **6.** (*i*) $\frac{1}{3}$ (*ii*) 0.

7. (*i*) $e^{2/\pi}$ (*ii*) e.