6 Differentiation

6.1 RECAP

In class XI, you have read about (real) functions, their limits, continuity and derivatives. The following results and examples will refresh your memory. (Hopefully sweet !).

1. Derivative at any point

A function f is said to have a ***derivative*** *at any point x iff it is defined in some (undeleted) neighbourhood of the point x and* $\underset{\delta x \to 0}{\text{Lt}} \dfrac{f(x+\delta x)-f(x)}{\delta x}$ *exists (finitely).*

The value of this limit is called the ***derivative*** *of f at any point x and is denoted by f′(x) i.e.*

$$f'(x) = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}.$$

Other Notations :

If the function f is written as $y = f(x)$, then its derivative is written as $\frac{dy}{dx}$ (or y_1) and so

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x} \qquad \ldots(i)$$

But $\quad y = f(x) \Rightarrow y + \delta y = f(x + \delta x)$

$\therefore \quad \delta y = f(x+\delta x) - f(x) \Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{f(x+\delta x)-f(x)}{\delta x}$

$$\Rightarrow \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x} = \frac{dy}{dx} \qquad \text{(using } (i))$$

$$\therefore \quad \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x}.$$

2. $\frac{d}{dx}(x^n) = nx^{n-1}$, *n is a rational number.*

3. $\frac{d}{dx}((ax+b)^n) = n(ax+b)^{n-1}\,.\,a$, *n is a rational number.*

4. *The derivative of a constant is zero.*

5. *If f is a differentiable function at x and g is the function defined by g(x) = cf(x), then g′(x) = cf′(x) where c is a fixed real number.*

6. *If f and g are differentiable functions at x and if h is the function defined by h(x) = f(x) + g(x), then h′(x) = f′(x) + g′(x).*

7. *If f and g are differentiable functions at x and if h is the function defined by h (x) = f (x) g (x), then h′(x) = f (x) g′(x) + g (x) f ′(x)*

i.e. $$\frac{d}{dx}(f(x)\,g(x)) = f(x).\frac{d}{dx}(g(x)) + g(x).\frac{d}{dx}(f(x)).$$

Thus, the derivative of the product of two differentiable functions = first function × derivative of second function + second function × derivative of first function.

This is known as **product rule**.

Extension of the product rule :

If f, g and h are three differentiable functions at x, then

$$\frac{d}{dx}(f(x)g(x)h(x)) = f(x)g(x)\frac{d}{dx}(h(x)) + f(x)h(x)\frac{d}{dx}(g(x)) + g(x)h(x)\frac{d}{dx}(f(x)).$$

8. *If f is a differentiable function at x and f (x) ≠ 0, and g is a function defined by* $g(x) = \frac{1}{f(x)}$, *then* $g'(x) = -\frac{f'(x)}{(f(x))^2}$

i.e. $$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{(f(x))^2}.$$

9. *If f and g are differentiable functions at x and h is the function defined by* $h(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$, *then* $h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

i.e. $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).\frac{d}{dx}(f(x)) - f(x).\frac{d}{dx}(g(x))}{(g(x))^2},\ g(x) \neq 0.$$

Thus, the derivative of the quotient of two differentiable functions

$$= \frac{\text{deno.} \times \text{derivative of num.} - \text{num.} \times \text{derivative of deno.}}{(\text{denominator})^2}.$$

This is known as **quotient rule.**

ILLUSTRATIVE EXAMPLES

Example 1. *Differentiate the following functions :*

(*i*) $(2x^3 - 7)(9x^5 + 2x^2 - 3)$ (*ii*) $x(2x - 3)\sqrt{x - 2}$

(*iii*) $(2x + 1)^2 (3x - 2)^3 (4x + 5)^4$.

Solution. (*i*) Let $y = (2x^3 - 7)(9x^5 + 2x^2 - 3)$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = (2x^3 - 7).\frac{d}{dx}(9x^5 + 2x^2 - 3) + (9x^5 + 2x^2 - 3).\frac{d}{dx}(2x^3 - 7)$$

(using product rule)

$$= (2x^3 - 7).(9.5x^4 + 2.2x - 0) + (9x^5 + 2x^2 - 3).(2.3x^2 - 0)$$
$$= (2x^3 - 7)(45x^4 + 4x) + (9x^5 + 2x^2 - 3).6x^2$$
$$= 90x^7 + 8x^4 - 315x^4 - 28x + 54x^7 + 12x^4 - 18x^2$$
$$= 144x^7 - 295x^4 - 18x^2 - 28x.$$

(*ii*) Let $y = x(2x - 3)\sqrt{x - 2} = (2x^2 - 3x).(x - 2)^{1/2}$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = (2x^2 - 3x).\frac{1}{2}(x - 2)^{-1/2}.1 + (x - 2)^{1/2}.(2.2x^1 - 3.1)$$ (using product rule)

$$= \frac{2x^2 - 3x}{2\sqrt{x - 2}} + (4x - 3)\sqrt{x - 2} = \frac{2x^2 - 3x + 2(4x - 3)(x - 2)}{2\sqrt{x - 2}}$$

$$= \frac{2x^2 - 3x + 2(4x^2 - 11x + 6)}{2\sqrt{x - 2}} = \frac{10x^2 - 25x + 12}{2\sqrt{x - 2}}.$$

(*iii*) Let $y = (2x+1)^2 (3x-2)^3 (4x+5)^4$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = (2x+1)^2 (3x-2)^3 . \frac{d}{dx}((4x+5)^4) + (2x+1)^2 (4x+5)^4 \frac{d}{dx}((3x-2)^3)$$
$$+ (3x-2)^3 (4x+5)^4 \frac{d}{dx}((2x+1)^2) \quad \text{(using extension of product rule)}$$
$$= (2x+1)^2 (3x-2)^3 . 4(4x+5)^3 . 4 + (2x+1)^2 (4x+5)^4 . 3 (3x-2)^2 . 3$$
$$+ (3x-2)^3 (4x+5)^4 . 2(2x+1)^1 . 2$$
$$= (2x+1)(3x-2)^2 (4x+5)^3 [16(2x+1)(3x-2) + 9(2x+1)(4x+5)$$
$$+ 4(3x-2)(4x+5)]$$
$$= (2x+1)(3x-2)^2 (4x+5)^3 [16(6x^2 - x - 2) + 9(8x^2 + 14x + 5)$$
$$+ 4(12x^2 + 7x - 10)]$$
$$= (2x+1)(3x-2)^2 (4x+5)^3 (216x^2 + 138x - 27).$$

Example 2. *Differentiate the following functions:*

(*i*) $\frac{3x-2}{5x^2+7}$ (*ii*) $\frac{3x+2}{(x+5)(2x+1)+3}$ (*iii*) $\frac{(1-2x)^{5/2}}{2x^2+1}$.

Solution. (*i*) Let $y = \frac{3x-2}{5x^2+7}$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(5x^2+7) . \frac{d}{dx}(3x-2) - (3x-2) . \frac{d}{dx}(5x^2+7)}{(5x^2+7)^2} \quad \text{(using quotient rule)}$$
$$= \frac{(5x^2+7).(3.1+0) - (3x-2).(5.2x+0)}{(5x^2+7)^2}$$
$$= \frac{15x^2 + 21 - 30x^2 + 20x}{(5x^2+7)^2} = -\frac{15x^2 - 20x - 21}{(5x^2+7)^2}.$$

(*ii*) Let $y = \frac{3x+2}{(x+5)(2x+1)+3} = \frac{3x+2}{2x^2+11x+8}$, diff. w.r.t. x

$$\frac{dy}{dx} = \frac{(2x^2+11x+8).(3.1+0) - (3x+2).(2.2x+11.1+0)}{(2x^2+11x+8)^2}$$
$$= \frac{3(2x^2+11x+8) - (3x+2)(4x+11)}{(2x^2+11x+8)^2}$$
$$= \frac{6x^2 + 33x + 24 - (12x^2 + 41x + 22)}{(2x^2+11x+8)^2} = \frac{-6x^2 - 8x + 2}{(2x^2+11x+8)^2}$$
$$= -\frac{2(3x^2+4x-1)}{(2x^2+11x+8)^2}.$$

(*iii*) Let $y = \frac{(1-2x)^{5/2}}{2x^2+1}$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(2x^2+1). \frac{5}{2}(1-2x)^{3/2}.(-2) - (1-2x)^{5/2}.(2.2x+0)}{(2x^2+1)^2}$$
$$= \frac{(1-2x)^{3/2}[-5(2x^2+1) - 4x(1-2x)]}{(2x^2+1)^2}$$
$$= \frac{(1-2x)^{3/2}(-2x^2-4x-5)}{(2x^2+1)^2} = -\frac{(1-2x)^{3/2}(2x^2+4x+5)}{(2x^2+1)^2}.$$

Example 3. *Differentiate* $\frac{4x^2-1}{(5-2x)^3}$ *and find the value of the derivative at* $x = 2$.

Solution. Let $f(x) = \frac{4x^2-1}{(5-2x)^3}$, diff. w.r.t x, we get

$$f'(x) = \frac{(5-2x)^3.(4.2x-0)-(4x^2-1).3.(5-2x)^2.(-2)}{((5-2x)^3)^2}$$

$$= \frac{(5-2x)^2[(5-2x).8x+6(4x^2-1)]}{(5-2x)^6} = \frac{8x^2+40x-6}{(5-2x)^4}.$$

$$\therefore \quad f'(2) = \frac{8.2^2+40.2-6}{(5-2.2)^4} = \frac{32+80-6}{1^4} = 106.$$

Example 4. *If* $y = \sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}$, *prove that* $2xy\frac{dy}{dx} = \frac{x}{a} - \frac{a}{x}$.

Solution. Given $y = \frac{1}{\sqrt{a}}.x^{1/2} + \sqrt{a}.x^{-1/2}$, diff. w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{a}}\cdot\frac{1}{2}\cdot x^{-1/2} + \sqrt{a}\left(-\frac{1}{2}\right)\cdot x^{-3/2} = \frac{1}{2\sqrt{a}\sqrt{x}} - \frac{\sqrt{a}}{2x^{3/2}}$$

$$\Rightarrow \quad 2x\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{a}} - \frac{\sqrt{a}}{\sqrt{x}}$$

$$\Rightarrow \quad 2xy\frac{dy}{dx} = \left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)\left(\sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}\right) = \frac{x}{a} - \frac{a}{x}.$$

Example 5. *Find the coordinates of the points on the curve*

$y = \frac{x}{1-x^2}$ *for which* $\frac{dy}{dx} = 1$.

Solution. Given $y = \frac{x}{1-x^2} \Rightarrow \frac{dy}{dx} = \frac{(1-x^2).1-x.(-2x)}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$.

Now $\frac{dy}{dx} = 1 \Rightarrow \frac{1+x^2}{(1-x^2)^2} = 1$

$\Rightarrow \quad x^4 - 3x^2 = 0 \Rightarrow x^2(x^2-3) = 0 \Rightarrow x = 0, \pm\sqrt{3}$.

When $x = 0$, $y = 0$;

when $x = \sqrt{3}$, $y = \frac{\sqrt{3}}{1-3} = -\frac{\sqrt{3}}{2}$ and

when $x = -\sqrt{3}$, $y = \frac{-\sqrt{3}}{1-3} = \frac{\sqrt{3}}{2}$.

Hence, the points are $(0, 0)$, $\left(\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$, $\left(-\sqrt{3}, \frac{\sqrt{3}}{2}\right)$.

EXERCISE 6.1

Differentiate the following (1 to 6) functions :

1. (*i*) $(2x+3)(5x^2-7x+1)$ (*ii*) $(3x^4-5)(7x^3-11x+2)$.

2. (*i*) $x^3\sqrt{3x-4}$ (*ii*) $x(x-2)\sqrt{x-3}$.

3. (*i*) $(x+1)(5x+7)^2(2x+3)^3$ (*ii*) $x^2(3x+2)^3(1-2x)^4$.

4. (*i*) $\frac{x}{3x^2+5}$ (*ii*) $\frac{(2x+1)(3x-1)}{x+5}$.

5. (*i*) $\dfrac{ax^2 + bx + c}{px^2 + qx + r}$ (*ii*) $\dfrac{1}{ax^2 + bx + c}$.

6. (*i*) $\dfrac{(2 + 5x)^2}{x^3 - 1}$ (*ii*) $\sqrt{\dfrac{1 + x}{1 - x}}$.

7. Differentiate $\dfrac{2x^2 - 4}{3x^2 + 7}$ and find the value of the derivative at $x = 1$.

8. If $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, prove that $2x\dfrac{dy}{dx} + y = 2\sqrt{x}$.

9. If $y = \dfrac{1}{\sqrt{x}}\left(1 - \dfrac{1}{x}\right)$, prove that $x^{5/2}\left(2\dfrac{dy}{dx} - y\right) + x^2 - 3 = 0$.

10. If $y = \dfrac{x}{x + a}$, prove that $x\dfrac{dy}{dx} = y(1 - y)$.

11. Given $y = (3x - 1)^2 + (2x - 1)^3$, find $\dfrac{dy}{dx}$ and the points on the curve for which $\dfrac{dy}{dx} = 0$.

6.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. Derivative of sin *x*

Let $f(x) = \sin x$, then $f(x + \delta x) = \sin(x + \delta x)$.

By def., $f'(x) = \underset{\delta x \to 0}{\text{Lt}} \dfrac{f(x + \delta x) - f(x)}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \dfrac{\sin(x + \delta x) - \sin x}{\delta x}$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{2\cos\dfrac{x + \delta x + x}{2}\sin\dfrac{x + \delta x - x}{2}}{\delta x} \quad \text{(C – D formulae)}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{2\cos\left(x + \dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}}{\delta x} = \underset{\delta x/2 \to 0}{\text{Lt}} \cos\left(x + \frac{\delta x}{2}\right) . \underset{\delta x/2 \to 0}{\text{Lt}} \frac{\sin\dfrac{\delta x}{2}}{\dfrac{\delta x}{2}}$$

$$= \cos x . 1 = \cos x.$$

Thus, $\dfrac{d}{dx}(\sin x) = \cos x$, *for all* $x \in R$.

2. Derivative of cos *x*

Let $f(x) = \cos x$, then $f(x + \delta x) = \cos(x + \delta x)$.

By def., $f'(x) = \underset{\delta x \to 0}{\text{Lt}} \dfrac{f(x + \delta x) - f(x)}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \dfrac{\cos(x + \delta x) - \cos x}{\delta x}$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{2\sin\dfrac{x + \delta x + x}{2}\sin\dfrac{x - x + \delta x}{2}}{\delta x} \quad \text{(C – D formulae)}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{2\sin\left(x + \dfrac{\delta x}{2}\right)\sin\left(-\dfrac{\delta x}{2}\right)}{\delta x}$$

$$= -\underset{\delta x/2 \to 0}{\text{Lt}} \sin\left(x + \frac{\delta x}{2}\right) . \underset{\delta x/2 \to 0}{\text{Lt}} \frac{\sin\dfrac{\delta x}{2}}{\dfrac{\delta x}{2}}$$

$$= -\sin x . 1 = -\sin x.$$

Thus, $\dfrac{d}{dx}(\cos x) = -\sin x$, *for all* $x \in R$.

21. If $x = a \cos \theta + b \sin \theta$, $y = a \sin \theta - b \cos \theta$, prove that

$$y^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$$

Hint. On squaring and adding, we get $x^2 + y^2 = a^2 + b^2$.

22. If $x = (a + bt)\, e^{-nt}$, prove that $\frac{d^2x}{dt^2} + 2n \frac{dx}{dt} + n^2 x = 0$.

23. If $y = Ae^{-kt} \cos (pt + c)$, prove that $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + (p^2 + k^2)\, y = 0$.

24. If $\sin (x + y) = ky$, prove that $y_2 + y(1 + y_1)^3 = 0$.

ANSWERS

EXERCISE 6.1

1. (*i*) $30x^2 + 2x - 19$ (*ii*) $147x^6 - 165x^4 + 24x^3 - 105x^2 + 55$.

2. (*i*) $\dfrac{3x^2(7x-8)}{2\sqrt{3x-4}}$ (*ii*) $\dfrac{5x^2 - 18x + 12}{2\sqrt{x-3}}$.

3. (*i*) $(5x + 7)(2x + 3)^2(60x^2 + 151x + 93)$

(*ii*) $-x(3x + 2)^2(1 - 2x)^3(54x^2 + 9x - 4)$.

4. (*i*) $\dfrac{5 - 3x^2}{(3x^2+5)^2}$ (*ii*) $\dfrac{6(x^2 + 10x + 1)}{(x+5)^2}$, $x \neq -5$.

5. (*i*) $\dfrac{(aq - bp)x^2 + 2(ar - pc)x + br - cq}{(px^2 + qx + r)^2}$ (*ii*) $-\dfrac{2ax + b}{(ax^2 + bx + c)^2}$.

6. (*i*) $-\dfrac{(2+5x)(5x^3 + 6x^2 + 10)}{(x^3 - 1)^2}$ (*ii*) $\dfrac{1}{\sqrt{1+x}\,(1-x)^{3/2}}$.

7. $\dfrac{13}{25}$. **11.** $(0, 0)$, $\left(\dfrac{1}{4}, -\dfrac{1}{16}\right)$.

EXERCISE 6.2

1. (*i*) $5\sec^2 x - 3\cos x + 6\sqrt{x}$ (*ii*) $-3\,\text{cosec}^2 x - 20(1 - 2x)^{2/3}$.

2. (*i*) $x^2(x \cos x + 3 \sin x)$ (*ii*) $2x \cos x - (1 + x^2) \sin x$.

3. (*i*) $x^2 \,\text{cosec}\, x\,(3 - x \cot x)$ (*ii*) $4(1 - 2\tan x)\cos x - 2(5 + 4\sin x)\sec^2 x$.

4. (*i*) $\dfrac{x(\sin x + \cos x) + \cos x - \sin x + 1}{(x + \cos x)^2}$ (*ii*) $-\dfrac{2}{1 - \sin 2x}$.

5. (*i*) $\dfrac{2\sin x}{(1 + \cos x)^2}$ (*ii*) $\dfrac{(1 - a^2)\cos x}{(1 + a\sin x)^2}$.

6. (*i*) $\dfrac{x + \sin x}{1 + \cos x}$ (*ii*) $\dfrac{x^2}{(x\sin x + \cos x)^2}$.

7. (*i*) $-a\,\text{cosec}^2 x - c\,\text{cosec}\, x \cot x + x^{n-1}(n \cos x - x \sin x)$

(*ii*) $(\sec x + \tan x)\sec x$.

8. $6 - 2\sqrt{2}$. **9.** $\dfrac{\sqrt{3}}{2}$.

EXERCISE 6.3

1. (*i*) -1 (*ii*) 2. **2.** (*i*) $\log 7$ (*ii*) $2\sqrt{2}\,\log 3$.

3. (*i*) 2 (*ii*) $\dfrac{\log a}{\log b}$. **4.** (*i*) 0 (*ii*) $\dfrac{2}{3}$. **5.** (*i*) $\log \dfrac{9}{8}$ (*ii*) 1.