15 Differential Equations

INTRODUCTION

In engineering, physics, chemistry and sometimes in subjects like economics, biology etc., it becomes necessary to build a mathematical model to represent certain problems. It is often the case that these mathematical models involve the search of the unknown function (or functions) that satisfies the equation which contains the derivatives of unknown function (or functions). Such equations are called **differential equations.** In this chapter, our main aim is to solve the differential equations *i.e.* to find the unknown function (or functions) that satisfies the given differential equation.

15.1 DIFFERENTIAL EQUATION

An equation which involves unknown functions and their derivatives with respect to one or more independent variables is called a ***differential equation.***

An equation which contains unknown function or functions, containing only one independent variable and derivatives w.r.t. that independent variable is called an ***ordinary differential equation****.*

In the present book, we shall be dealing only with the ordinary differential equations.

15.1.1 Order and Degree of a Differential Equation

The ***order of a differential equation*** *is the order of the highest order derivative of the dependent variable with respect to independent variable appearing in the equation. If each term involving derivatives of a diferential equation is a polynomial (or can be expressed as polynomial), then the exponent of the highest order derivative is called the* ***degree of the differential equation.***

If any term of a differential equation cannot be expressed as a polynomial in the derivative (or derivatives), then the degree of the differential equation is not defined.

For example, consider the equations :

(*i*) $\frac{dy}{dx} = 2x^3 + 7\sqrt{x}$ (*ii*) $3\left(\frac{dy}{dx}\right)^2 = \sin 2x - \frac{2}{\frac{dy}{dx}}$

(*iii*) $5\frac{d^2y}{dx^2} + 8\left(\frac{dy}{dx}\right)^2 = \log x$ (*iv*) $y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(*v*) $\left(1-\left(\frac{dy}{dx}\right)^2\right)^{3/2} = k\frac{d^2y}{dx^2}$ (*vi*) $(3x^2 + 5y)\, dy - 5x\, dx = 0$

(*vii*) $(y')^2 + \cos^2 y = 0$ (*viii*) $(y'')^2 - \sin y' = 0$

(*ix*) $y''' - 3(y'')^2 + (y')^3 + \log y = 5x.$

These are all ordinary differential equations.

(*i*) The highest order derivative present in the given differential equation is $\frac{dy}{dx}$, so its order is 1. Here the term in the derivative is a polynomial, so its degree is the highest exponent of $\frac{dy}{dx}$, which is 1. Thus, its degree is 1.

(*ii*) The highest order derivative present in the given differential equation is $\frac{dy}{dx}$, so its order is 1. It can be written as $3\left(\frac{dy}{dx}\right)^3 = \sin 2x.\ \frac{dy}{dx} - 2$. Here each term in the derivatives is a polynomial, so its degree is the highest exponent of $\frac{dy}{dx}$, which is 3. Thus, its degree is 3.

(*iii*) The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$, so its order is 2. Here each term in the derivatives is a polynomial, so its degree is the highest exponent of $\frac{d^2y}{dx^2}$, which is 1. Thus, its degree is 1.

(*iv*) The highest order derivative present in the given differential equation is $\frac{dy}{dx}$, so its order is 1. It can be written as

$$\left(y - x\frac{dy}{dx}\right)^2 = a^2\left(1+\left(\frac{dy}{dx}\right)^2\right) \text{ or } (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2 - a^2 = 0.$$

Here each term in the derivatives is a polynomial, so its degree is the highest exponent of $\frac{dy}{dx}$, which is 2. Thus, its degree is 2.

(*v*) The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$, so its order is 2. It can be written as $\left(1-\left(\frac{dy}{dx}\right)^2\right)^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$. Here each term in the derivative is a polynomial, so its degree is the highest exponent of $\frac{d^2y}{dx^2}$, which is 2. Thus, its degree is 2.

(*vi*) The given differential equation can be written as $(3x^2 + 5y)\frac{dy}{dx} = 5x$, so this equation is of order 1 and degree 1.

(*vii*) The highest order derivative present in the given differential equation is y', so its order is 1. Here the term containing the derivative is a polynomial, so its degree is the highest exponent of y', which is 2. Thus, its degree is 2. Note that the term $\cos^2 y$ is not a polynomial in y.

(*viii*) The highest order derivative present in the given differential equation is y'', so its order is 2. Here the term $\sin y'$ is not a polynomial in y', so the degree of the given differential equation is not defined.

(*ix*) The highest order derivative present in the given differential equation is y''', so its order is 3. Here each term containing derivatives y''', y'' and y' are polynomials, so its degree is the highest exponent of y''', which is 1. Thus, its degree is 1. Note that the term $\log y$ is not a polynomial in y.

In general, an ordinary differential equation of order one is of the form $F\left(x,\ y,\ \frac{dy}{dx}\right) = 0$ where F is a function of the variables x, y, $\frac{dy}{dx}$; and an ordinary differential equation of order 2 is of the form $F\left(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}\right) = 0$ etc.

More generally, an equation of the form $F(x, y, y_1, y_2, \ldots, y_n) = 0$ is called an ordinary differential equation of order n.

15.1.2 Linear Differential Equation

*An ordinary differential equation $F(x, y, y_1, y_2, \ldots, y_n) = 0$ is called **linear** iff the function F is a linear function of the variables $y, y_1, y_2, \ldots, y_n$ i.e. iff the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Otherwise, it is called **non-linear.***

Thus, a general linear differential equation of order n may be written as

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_n y = Q \quad \ldots(i)$$

where $P_0, P_1, P_2, \ldots, P_n$, Q are all functions of x only and $P_0 \neq 0$.

*The linear differential equation (i) is called **homogeneous** iff Q is a zero function and it is called **non-homogeneous** iff Q is non-zero function.*

Note that a linear differential equation is always of degree one but every differential equation of degree one need not be linear. For example, the differential equation $x \frac{d^2y}{dx^2} - 5 \sin x \left(\frac{dy}{dx}\right)^3 = 3y$ is of degree 1 but it is not linear.

EXERCISE 15.1

Determine the order and the degree (if any) of each of the following differential equations. Also state if these are linear or non-linear.

1. $\frac{dy}{dx} + 5y = 0.$
2. $\frac{dy}{dx} + 2y = 7x^3.$
3. $(x^2y - 3x)\, dy + (x^3 - 3y^2)\, dx = 0.$
4. $\sqrt{1 - y^2}\, dx + \sqrt{1 - x^2}\, dy = 0.$
5. $\frac{1}{x} \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} = \sin 2x.$
6. $y' + 6y^2 + y = 0.$
7. $(y')^2 + y^2 - 1 = 0.$
8. $(y')^2 + y^3 + y = 0.$
9. $y'' + y^2 = 0.$
10. $\left(\frac{d^2x}{dt^2}\right)^2 - 7t \left(\frac{dx}{dt}\right)^3 = \log t.$
11. $3x \frac{dy}{dx} + \frac{5}{\frac{dy}{dx}} = y^3.$
12. $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2x - \frac{dy}{dx}.$
13. $y' + \cos y' = 0.$
14. $y'' + 2y' + \sin y = 0.$
15. $\frac{d^3y}{dx^3} + 7x^2 \frac{dy}{dx} = 5x^2 - 2y.$
16. $\frac{d^2y}{dx^2} = \sin x + \cos x.$
17. $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} = 5 \frac{d^2y}{dx^2}.$
18. $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0.$
19. $y^{(iv)} + y''' + y'' + y' + y = 0.$
20. $y^{(v)} + y^2 + e^{y'} = 0.$
21. $y = px + \sqrt{a^2 p^2 + b^2}$ where $p = \frac{dy}{dx}.$

15.2 SOLUTION OF A DIFFERENTIAL EQUATION

*A **solution** of a differential equation is a relation between the variables, by means of which and the derivatives obtained therefrom, the equation is satisfied.*

For example :

(*i*) Consider the differential equation $\frac{dy}{dx} = \frac{1}{x}$, $x > 0$, then $y = \log x$ is a solution of this equation, and a more general solution of this differential equation is $y = A + \log x$, where A is arbitrary constant.

(*ii*) Consider the differential equation $\frac{d^2y}{dx^2} + y = 0$ ...(1)

Let $y = A \sin x + B \cos x$ ...(2)

where A, B are arbitrary constants.

Differentiating (2) twice w.r.t. x, we get

$$\frac{dy}{dx} = A \cos x - B \sin x \text{ and } \frac{d^2y}{dx^2} = -A \sin x - B \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y \text{ [using (2)]} \quad \Rightarrow \frac{d^2y}{dx^2} + y = 0.$$

Therefore, $y = A \sin x + B \cos x$ is a solution of the differential equation (1).

It can be easily seen that $y = 2 \sin x - 3 \cos x$, $y = A \sin x$, $y = B \cos x$ are all solutions of (1). The most general of these solutions is (2); all others are particular solutions of the differential equation (1).

General Solution

Let the equation involving the variables x, y and n independent arbitrary constants be

$$f(x, y, c_1, c_2, \ldots, c_n) = 0 \quad \ldots(i)$$

and the differential equation obtained from (i) be

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0 \quad \ldots(ii)$$

*then (i) is called the **general solution** (or **complete primitive** or **complete solution**) of (ii).*

Note that the general solution of an ordinary differential equation of nth order contains n independent arbitrary constants. Thus the general solution of an ordinary differential equation of order one contains one arbitrary constant and of second order contains two independent arbitrary constants and so on.

Particular Solution

*Any solution obtained from the general solution of a differential equation by giving particular values to some or all the arbitrary constants is called a **particular solution** (or **particular primitive**).*

The problem of finding the particular solution of a differential equation that satisfies the given condition (s) is called **initial value problem**.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that* $y = \frac{A}{x + A}$ *is a solution of the differential equation* $xy_1 + y = y^2$.

Solution. $y = \frac{A}{x + A} \Rightarrow y(x + A) = A$ ...(*i*)

Differentiating w.r.t. x, we get

$$y \cdot 1 + (x + A) y_1 = 0 \quad \Rightarrow x + A = -\frac{y}{y_1}$$

$$\Rightarrow \quad A = -x - \frac{y}{y_1}.$$

Substituting the values of $x + A$ and A in (*i*), we get

$$y\left(-\frac{y}{y_1}\right) = -x - \frac{y}{y_1} \quad \Rightarrow y^2 = xy_1 + y.$$

Hence, $y = \frac{A}{x + A}$ is a solution of the given differential equation.

Example 2. *Verify that* $y = A\cos x - B\sin x$ *is a solution of the differential equation* $\frac{d^2y}{dx^2} + y = 0$.

Solution. Given $y = A\cos x - B\sin x$ ...(*i*)

Differentiating (*i*) twice w.r.t. x, we get

$$\frac{dy}{dx} = -A\sin x - B\cos x \text{ and}$$

$$\frac{d^2y}{dx^2} = -A\cos x - B(-\sin x)$$

$$= -(A\cos x - B\sin x) = -y \quad \text{(using (i))}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0.$$

Therefore, $y = A\cos x - B\sin x$ is a solution of the given differential equation.

Example 3. *Show that* $y = Ax + \frac{B}{x}$ *is a solution of the differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

Solution. $y = Ax + \frac{B}{x} \Rightarrow xy = Ax^2 + B.$

Differentiating w.r.t. x, we get $x\frac{dy}{dx} + y = 2Ax \Rightarrow \frac{dy}{dx} + \frac{y}{x} = 2A.$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} + \frac{x\frac{dy}{dx} - y.1}{x^2} = 0 \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

Hence, $y = Ax + \frac{B}{x}$ is a solution of $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

Example 4. *Show that* $y = a e^{2x} + b e^{-x}$ *is a solution of the differential equation*

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$$

Solution. Given $y = a e^{2x} + b e^{-x}$ ...(*i*)

Differentiating (*i*) w.r.t. x, we get $\frac{dy}{dx} = ae^{2x}.2 + be^{-x}(-1)$ ...(*ii*)

Adding (*i*) and (*ii*), we get $\frac{dy}{dx} + y = 3ae^{2x}$ ...(*iii*)

Differentiating (*iii*) w.r.t. x, we get $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3a.e^{2x}.2$ ...(*iv*)

Multiplying (*iii*) by 2 and subtracting from (*iv*), we get $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$

Hence, $y = a e^{2x} + b e^{-x}$ is a solution of the given differential equation.

Example 5. *Show that* $y = (a + bx)e^{2x}$ *is a solution of the differential equation*

$$y_2 - 4y_1 + 4y = 0.$$

Solution. Given $y = (a + bx)e^{2x}$ ...(*i*)

Differentiating (*i*) w.r.t. x, we get

$$y_1 = (a + bx)e^{2x}.2 + e^{2x}.(0 + b.1)$$

$$\Rightarrow \quad y_1 = 2y + be^{2x} \quad \text{...(ii) (using (i))}$$

Differentiating again w.r.t. x, we get

$$y_2 = 2y_1 + be^{2x}.2$$

$\Rightarrow \quad y_2 = 2y_1 + 2(y_1 - 2y)$ (using (*ii*))

$\Rightarrow \quad y_2 - 4y_1 + 4y = 0.$

Hence, $y = (a + bx)\, e^{2x}$ is a solution of the given differential equation.

Example 6. *Show that* $y = e^{m \cos^{-1} x}$ *is a solution of the differential equation*

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0.$$

Solution. Given $y = e^{m \cos^{-1} x}$ ...(*i*)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{m\cos^{-1}x} . m . \left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{m}{\sqrt{1-x^2}} . y \qquad \text{(using } (i)\text{)}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = -my \quad \Rightarrow \quad (1 - x^2)\left(\frac{dy}{dx}\right)^2 = m^2 y^2.$$

Differentiating again w.r.t. x, we get

$$(1 - x^2) . 2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 . (-2x) = m^2 . 2y . \left(\frac{dy}{dx}\right).$$

Dividing both sides by $2\frac{dy}{dx}$, we get

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0.$$

Hence $y = e^{m \cos^{-1} x}$ is a solution of $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0.$

EXERCISE 15.2

In each of the following (1 to 4), show that the given function is a solution of the differential equation :

1. $\frac{dy}{dx} + y = 0 ; y = A e^{-x}.$
2. $y_2 + 9y = 0 ; y = 4 \sin 3x.$
3. $y_2 + 4y = 0 ; y = A \cos 2x - B \sin 2x.$
4. $y = x\frac{dy}{dx} + a\frac{dy}{dx} ; y = cx + \frac{a}{c}.$
5. Show that $y = (A + Bx)\, e^{3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$
6. Show that $y = e^{-x} + ax + b$ is a solution of the differential equation $e^x \frac{d^2y}{dx^2} = 1.$
7. Show that $y = b e^x + c e^{2x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$
8. Show that $y = a \cos (\log x) + b \sin (\log x)$ is a solution of the differential equation
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$
9. Show that $y = e^x \cos x$ is a solution of the differential equation
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$
10. Show that $xy = ae^x + be^{-x} + x^2$ is a solution of the differential equation
$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0.$$
11. Show that $y = \sin (\sin x)$ is a solution of the differential equation
$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y \cos^2 x = 0.$$

ANSWERS

EXERCISE 15.1

1. 1, 1 ; linear. **2.** 1, 1 ; linear. **3.** 1, 1 ; non-linear.
4. 1, 1; non-linear. **5.** 2, 1; linear. **6.** 1, 1; non-linear.
7. 1, 2; non-linear. **8.** 1, 2; non-linear. **9.** 2, 1; non-linear.
10. 2, 2; non-linear. **11.** 1, 2; non-linear. **12.** 1, 1; linear.
13. 1, not defined; non-linear. **14.** 2, 1; non-linear.
15. 3, 1; linear. **16.** 2, 1; linear. **17.** 2, 2; non-linear.
18. 3, 2; non-linear. **19.** 4, 1; linear.
20. 5, not defined ; non-linear. **21.** 1, 2; non-linear.

EXERCISE 15.3

1. (*i*) $y = \frac{1}{2}\log(1 + x^2) + C$ (*ii*) $y = \frac{x^6}{6} + \frac{x^3}{3} - 2\log|x| + C.$

2. (*i*) $y = \frac{x^3}{3} - \frac{1}{3}\cos 3x + C$ (*ii*) $y = \frac{1}{3}e^{3x} - x^2 + C.$

3. (*i*) $y = \tan^{-1}(x + 1) + C$ (*ii*) $y = \frac{x^2}{2} + 2x - 13\log|x + 2| + C.$

4. (*i*) $y = \log|e^x + e^{-x}| + C$ (*ii*) $y = \frac{1}{2}\sin^{-1}(x^2) + C.$

5. (*i*) $y = x\log x - x + C$ (*ii*) $y = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + (x - 1)e^x + C.$

6. (*i*) $y = -\frac{1}{3}\operatorname{cosec}^3 x + C$ (*ii*) $y = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{2}}\right) + C.$

7. (*i*) $y = -x e^{-x} + C$ (*ii*) $y = \frac{1}{2}(x^2 + 1)\tan^{-1}x - \frac{1}{2}x + C.$

8. $y = \frac{1}{4}x^4 - \frac{2}{3}x^3 + 2.$

EXERCISE 15.4

1. (*i*) $\log|y| = e^x + x + C$ (*ii*) $x + \frac{1}{2}\log(1 + y^2) = C.$

2. (*i*) $x + \cot y = C$ (*ii*) $x + \log|1 - y| = C.$

3. (*i*) $\tan^{-1}y = x + \frac{1}{2}x^2 + C$ (*ii*) $-\frac{5}{3y^3} = e^x + C.$

4. (*i*) $(1 + e^y)\sin x = C$ (*ii*) $x + y + \log|xy| = C.$

5. (*i*) $y - 1 = Axy$ (*ii*) $\log|1 + y| = x + \frac{x^2}{2} + C.$

6. (*i*) $\tan^{-1}x + \tan^{-1}y + \frac{1}{2}\log(1 + x^2)(1 + y^2) = C$

(*ii*) $\sqrt{1 - x^2} + \sqrt{1 - y^2} = C.$

7. (*i*) $\tan x \tan y = A$ (*ii*) $\sin y = A\,e^{-\sin x}.$

8. (*i*) $\sin y = A\cos x$ (*ii*) $(1 + \sin x)(1 + \cos y) = A.$

9. (*i*) $e^y + e^{-y} + \frac{1}{2}x^2 - \log|x| = C$ (*ii*) $\log|y| = e^x + x + C.$

10. (*i*) $-\sqrt{1 - y^2} + (x - 1)e^x = C$ (*ii*) $y(x + 1)^2 = A\,e^{2x}.$