14 Complex Numbers

14.1 COMPLEX NUMBERS

*A **complex number** z is defined as an ordered pair of real numbers and is written as* $z = (a, b)$ *or* $z = a + ib$*, where a, b are real numbers and* $i = \sqrt{-1}$.

Thus, a number of the form $a + ib$ where a, b are real numbers and $i = \sqrt{-1}$ is called a complex number.

For example, $3 + 5i$, $-2 + 3i$, $-2 + i\sqrt{5}$, $7 + i\left(-\frac{2}{3}\right)$ are all complex numbers.

The system of numbers $\mathbf{C} = \{z \,;\, z = a + ib \,;\, a, b \in \mathbf{R}\}$ *is called the set of **complex numbers.***

Real and imaginary parts of a complex number

If $z = a + ib$ $(a, b \in \mathbf{R})$ is a complex number, then a is called the **real part,** denoted by $\text{Re}(z)$ and b is called **imaginary part**, denoted by $\text{Im}(z)$.

For example :

(*i*) If $z = 2 + 3i$, then $\text{Re}(z) = 2$ and $\text{Im}(z) = 3$.

(*ii*) If $z = -3 + \sqrt{5}\,i$, then $\text{Re}(z) = -3$ and $\text{Im}(z) = \sqrt{5}$.

(*iii*) If $z = 7$, then $z = 7 + 0\,i$, so that $\text{Re}(z) = 7$ and $\text{Im}(z) = 0$.

(*iv*) If $z = -5i$, then $z = 0 + (-5)i$, so that $\text{Re}(z) = 0$ and $\text{Im}(z) = -5$.

Note that imaginary part is a real number.

In $z = a + ib$ $(a, b \in \mathbf{R})$, if $b = 0$ then $z = a$, which is a **real number.** If $a = 0$ and $b \neq 0$, then $z = ib$, which is called **purely imaginary number**. If $b \neq 0$, then $z = a + ib$ is **non-real** complex number. Since every real number a can be written as $a + 0\,i$, we see that $\mathbf{R} \subset \mathbf{C}$ *i.e.* the set of real numbers $\mathbf{R}$ is a **proper subset** of $\mathbf{C}$, the set of complex numbers.

Note that $\sqrt{3}$, 0, 2, π are real numbers ; $3 + 2\,i$, $3 - 2\,i$ etc. are non-real complex numbers ; $2\,i$, $-\sqrt{2}\,i$ etc. are purely imaginary numbers.

Equality of two complex numbers

Two complex numbers $z_1 = a + ib$ *and* $z_2 = c + id$ *are called **equal**, written as* $z_1 = z_2$*, if and only if* $a = c$ *and* $b = d$.

For example, if the complex numbers $z_1 = a + ib$ and $z_2 = -3 + 5i$ are equal, then $a = -3$ and $b = 5$.

14.2 ALGEBRA OF COMPLEX NUMBERS

In this section, we shall define the usual mathematical operations—addition, subtraction, multiplication, division, square, power etc. on complex numbers.

14.2.1 Addition of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then their sum $z_1 + z_2$ is defined as

$$z_1 + z_2 = (a + c) + i\,(b + d).$$

For example, let $z_1 = 2 + 3i$ and $z_2 = -5 + 4i$, then

$$z_1 + z_2 = (2 + (-5)) + (3 + 4)\,i = -3 + 7i.$$

If z_1, z_2, z_3 are any complex numbers, then it is easy to see that

(*i*) $z_1 + z_2 = z_2 + z_1$ (Commutative law)

(*ii*) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (Associative law)

(*iii*) $z + 0 = z = 0 + z$, so 0 acts as the ***additive identity*** for the set of complex numbers.

Negative of a complex number

For a complex number, $z = a + ib$, the negative is defined as $-z = (-a) + i\,(-b) = -a - ib$.

Note that $z + (-z) = (a - a) + i\,(b - b) = 0 + i0 = 0$.

Thus, $-z$ acts as the ***additive inverse*** of z.

14.2.2 Subtraction of complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then the difference of z_2 from z_1 is defined as

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (a + ib) + (-c - id) \\ &= (a - c) + i\,(b - d). \end{aligned}$$

For example, let $z_1 = 2 + 3i$ and $z_2 = -1 + 4i$, then

$$\begin{aligned} z_1 - z_2 &= (2 + 3i) - (-1 + 4i) \\ &= (2 + 3i) + (1 - 4i) \\ &= (2 + 1) + (3 - 4)\,i = 3 - i \end{aligned}$$

and

$$\begin{aligned} z_2 - z_1 &= (-1 + 4i) - (2 + 3i) \\ &= (-1 + 4i) + (-2 - 3i) \\ &= (-1 - 2) + (4 - 3)\,i = -3 + i. \end{aligned}$$

14.2.3 Multiplication of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then their product z_1z_2 is defined as

$$z_1z_2 = (ac - bd) + i(ad + bc).$$

Note that intuitively,

$$(a + ib)\,(c + id) = ac + ibc + iad + i^2bd\ ;\ \text{now put } i^2 = -1.$$

For example, let $z_1 = 3 + 7i$ and $z_2 = -2 + 5i$, then

$$\begin{aligned} z_1z_2 &= (3 + 7i)\,(-2 + 5i) \\ &= (3 \times (-2) - 7 \times 5) + i(3 \times 5 + 7 \times (-2)) \\ &= -41 + i. \end{aligned}$$

If z_1, z_2, z_3 are any complex numbers, then it is easy to see that

(*i*) $z_1z_2 = z_2z_1$ (Commutative law)

(*ii*) $(z_1z_2)\,z_3 = z_1(z_2z_3)$ (Associative law) (*iii*) $z.1 = z = 1.z$, so 1 acts as the ***multiplicative identity*** for the set of complex numbers.

(*iv*) ***Multiplicative inverse of a non-zero complex number***

For every non-zero complex number $z = a + ib$, we have the complex number $\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$ (denoted by z^{-1} or $\frac{1}{z}$) such that

$$z \cdot \frac{1}{z} = 1 = \frac{1}{z} \cdot z \qquad \text{(check it)}$$

$\frac{1}{z}$ is called the multiplicative inverse of z.

Note that intuitively, $\frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$.

(*v*) ***Multiplication of complex numbers is distributive over addition of complex numbers***

If z_1, z_2 and z_3 are any three complex numbers, then

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

and $$(z_1 + z_2)z_3 = z_1z_3 + z_2z_3.$$

These results are known as ***distributive laws***.

14.2.4 Division of complex numbers

Division of a complex number $z_1 = a + ib$ by $z_2 = c + id \neq 0$ is defined as

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 . z_2^{-1} = (a+ib) \cdot \frac{c-id}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}.$$

Note that intuitively,

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}.$$

For example, if $z_1 = 3 + 4i$ and $z_2 = 5 - 6i$, then

$$\frac{z_1}{z_2} = \frac{3+4i}{5-6i} = \frac{3+4i}{5-6i} \times \frac{5+6i}{5+6i} = \frac{(3\times5-4\times6)+(3\times6+4\times5)i}{5^2-6^2\times i^2}$$

$$= \frac{-9+38i}{25+36} = -\frac{9}{61} + \frac{38}{61}i.$$

14.2.5 Integral powers of a complex number

If z is any complex number, then positive integral powers of z are defined as

$$z^1 = z,\ z^2 = z.z,\ z^3 = z^2.z,\ z^4 = z^3.z \text{ and so on.}$$

If z is any non-zero complex number, then negative integral powers of z are defined as :

$$z^{-1} = \frac{1}{z},\ z^{-2} = \frac{1}{z^2},\ z^{-3} = \frac{1}{z^3} \text{ etc.}$$

If $z \neq 0$, then $z^0 = 1$.

14.2.6 Powers of i

Integral power of i are defined as :

$i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = i^2.i = (-1)\ i = -i,$

$i^4 = (i^2)^2 = (-1)^2 = 1,\ i^5 = i^4.i = 1.i = i,$

$i^6 = i^4.i^2 = 1.(-1) = -1$, and so on.

$$i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$$

Remember that $\frac{1}{i} = -i$

$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,\ i^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i,$

$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$, and so on.

Note that $i^4 = 1$ and $i^{-4} = 1$. It follows that for any integer k,

$\boldsymbol{i^{4k} = 1,\ i^{4k+1} = i,\ i^{4k+2} = i^2 = -1,\ i^{4k+3} = i^3 = -i.}$

14.2.7 Modulus of a complex number

Modulus of a complex number $z = a + ib$, denoted by mod(z) or $|z|$, is defined as

$|z| = \sqrt{a^2 + b^2}$, where $a = Re(z)$, $b = Im(z)$.

Sometimes, $|z|$ is called **absolute value** of z. Note that $|z| \geq 0$.

For example,

(*i*) If $z = -3 + 5i$, then $|z| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$.

(*ii*) If $z = 3 - \sqrt{7}\,i$, then $|z| = \sqrt{3^2 + (-\sqrt{7})^2} = \sqrt{9 + 7} = 4$.

Properties of modulus of a complex number

If z, z_1 and z_2 are complex numbers, then

(*i*) $|-z| = |z|$

(*ii*) $|z| = 0$ if and only if $z = 0$

(*iii*) $|z_1 z_2| = |z_1|\,|z_2|$

(*iv*) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$.

Proof. (*i*) Let $z = a + ib$, then $-z = -a - ib$.

$\therefore \quad |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$.

(*ii*) Let $z = a + ib$, then $|z| = \sqrt{a^2 + b^2}$.

Now $|z| = 0$ iff $\sqrt{a^2 + b^2} = 0$

i.e. iff $a^2 + b^2 = 0$ *i.e.* iff $a^2 = 0$ and $b^2 = 0$

i.e. iff $a = 0$ and $b = 0$ *i.e.* iff $z = 0 + i\,0$

i.e. iff $z = 0$.

(*iii*) Let $z_1 = a + ib$, and $z_2 = c + id$, *then*

$z_1 z_2 = (ac - bd) + i\,(ad + bc)$.

$$\begin{aligned} \therefore\ |z_1 z_2| &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 - 2abcd + a^2d^2 + b^2c^2 + 2abcd} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2}\,\sqrt{c^2 + d^2} \qquad (\because a^2 + b^2 \geq 0,\ c^2 + d^2 \geq 0) \\ &= |z_1|\,|z_2|. \end{aligned}$$

(*iv*) Here $z_2 \neq 0 \Rightarrow |z_2| \neq 0$.

Let $\frac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow |z_1| = |z_2 z_3|$

$\Rightarrow |z_1| = |z_2|\,|z_3|$ (using part (*iii*))

$\Rightarrow \frac{|z_1|}{|z_2|} = |z_3| \Rightarrow \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|$ $\left(\because z_3 = \frac{z_1}{z_2}\right)$

Remark. From (*iii*), on replacing both z_1 and z_2 by z, we get

$$|z\,z| = |z|\,|z| \text{ i.e. } |z^2| = |z|^2.$$

Similarly, $|z^3| = |z^2 z| = |z^2|\;|z| = |z|^2\;|z| = |z|^3$ etc.

14.2.8 Conjugate of a complex number

Conjugate of a complex number $z = a + ib$, denoted by $\bar{z}$, is defined as

$\bar{z} = a - ib$ i.e. $\overline{a+ib} = a - ib$.

For example,

(*i*) $\overline{2+5i} = 2 - 5i$, $\overline{2-5i} = 2 + 5i$

(*ii*) $\overline{-3-7i} = -3 + 7i$, $\overline{-3+7i} = -3 - 7i$.

Properties of conjugate of a complex number

If z, z_1 and z_2 are complex numbers, then

(*i*) $\overline{(\bar{z})} = z$

(*ii*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(*iii*) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(*iv*) $\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$

(*v*) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, provided $z_2 \neq 0$

(*vi*) $|\bar{z}| = |z|$

(*vii*) $z\,\bar{z} = |z|^2$

(*viii*) $z^{-1} = \dfrac{\bar{z}}{|z|^2}$, provided $z \neq 0$.

Proof. (*i*) Let $z = a + ib$, where $a, b \in \mathbf{R}$, so that $\bar{z} = a - ib$.

$$\therefore\ \overline{(\bar{z})} = \overline{a-ib} = a + ib = z.$$

(*ii*) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\overline{z_1 + z_2} = \overline{(a+ib)+(c+id)} = \overline{(a+c)+i(b+d)}$$

$$= (a + c) - i(b + d) = (a - ib) + (c - id) = \bar{z}_1 + \bar{z}_2.$$

(*iii*) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\overline{z_1 - z_2} = \overline{(a+ib)-(c+id)} = \overline{(a-c)+i(b-d)}$$

$$= (a - c) - i(b - d) = (a - ib) - (c - id) = \bar{z}_1 - \bar{z}_2.$$

(*iv*) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\overline{z_1 z_2} = \overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(ad+bc)}$$

$$= (ac - bd) - i(ad + bc).$$

Also $\bar{z}_1\,\bar{z}_2 = (a - ib)(c - id) = (ac - bd) - i(ad + bc)$.

Hence $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$.

(*v*) Here $z_2 \neq 0 \Rightarrow \bar{z}_2 \neq 0$.

Let $\dfrac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow \bar{z}_1 = \overline{z_2 z_3}$

$\Rightarrow \bar{z}_1 = \bar{z}_2\,\bar{z}_3$ (using part (*iv*))

$\Rightarrow \dfrac{\bar{z}_1}{\bar{z}_2} = \bar{z}_3 \Rightarrow \dfrac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\dfrac{z_1}{z_2}\right)}$ $\left(\because z_3 = \dfrac{z_1}{z_2}\right)$

(*vi*) Let $z = a + ib$, then $\bar{z} = a - ib$.

$$\therefore\ |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(*vii*) Let $z = a + ib$, then $\bar{z} = a - ib$.

$\therefore\ z\,\bar{z} = (a + ib)\,(a - ib)$

$= (aa - b(-b)) + i(a(-b) + ba)$ (Def. of multiplication)

$= (a^2 + b^2) + i\,.\,0$

$= a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2$.

Remember that (a + ib) (a − ib) = a² + b².

(*viii*) Let $z = a + ib \neq 0$, then $|z| \neq 0$.

$\therefore\ z\,\bar{z} = (a + ib)\,(a - ib) = a^2 + b^2 = |z|^2$

$\Rightarrow \dfrac{z\,\bar{z}}{|z|^2} = 1 \Rightarrow \dfrac{\bar{z}}{|z|^2} = \dfrac{1}{z} = z^{-1}$

Thus, $z^{-1} = \dfrac{\bar{z}}{|z|^2}$, provided $z \neq 0$.

Remark From (*iv*), on replacing both z_1 and z_2 by z, we get

$\overline{z\,z} = \bar{z}\,\bar{z}$ *i.e.* $\overline{z^2} = (\bar{z})^2$.

Similarly, $(\overline{z^3}) = (\overline{z^2 z}) = (\overline{z^2})\,\bar{z} = (\bar{z})^2\,\bar{z} = (\bar{z})^3$ etc.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following :*

(*i*) i^{50} (*ii*) i^{-57} (*iii*) $(-\sqrt{-1})^{31}$

(*iv*) $i + i^2 + i^3 + i^4$ (*v*) $\dfrac{i + i^2 + i^4}{1 + i^2 + i^4}$ (*vi*) $\dfrac{i^2 + i^4 + i^6 + i^7}{1 + i^2 + i^3}$.

Solution. (*i*) $i^{50} = i^{4.12 + 2} = i^2 = -1$.

(*ii*) $i^{-57} = i^{4.(-15) + 3} = i^3 = -i$.

(*iii*) $\left(-\sqrt{-1}\right)^{31} = (-i)^{31} = (-1)^{31}\,.\,i^{31} = -i^{4.7 + 3} = -i^3 = -(-i) = i$.

(*iv*) $i + i^2 + i^3 + i^4 = i + (-1) + (-i) + (1) = 0$.

(*v*) $\dfrac{i + i^2 + i^4}{1 + i^2 + i^4} = \dfrac{i + (-1) + (1)}{1 + (-1) + (1)} = \dfrac{i}{1} = i$.

(*vi*) $\dfrac{i^2 + i^4 + i^6 + i^7}{1 + i^2 + i^3} = \dfrac{(-1) + (1) + (-1) + (-i)}{1 + (-1) + (-i)} = \dfrac{-1 - i}{-i} = \dfrac{1}{i} + 1 = -i + 1 = 1 - i$.

Example 2. (*i*) *Show that* $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$ *for all* $n \in \mathbf{N}$.

(*ii*) *If* $z = \dfrac{3}{2} - 2i$, *find* $Re(z)$, $Im(z)$, $\bar{z}$, $|z|$, $z + \bar{z}$, $z - \bar{z}$.

(*iii*) *If* $Re(z) = \sqrt{2}$ *and* $Im(z) = -\sqrt{3}$, *find* z, $\bar{z}$, $|z|$, $|\bar{z}|$.

(*iv*) *Find values of x and y if* $(3x - 1) + (\sqrt{3} + 2y)\,i = 5$.

Solution. (*i*) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n\,(1 + i + i^2 + i^3)$

$= i^n\,(1 + i - 1 - i) = i^n \times 0 = 0$.

(*ii*) Given $z = \dfrac{3}{2} - 2i \Rightarrow Re(z) = \dfrac{3}{2}$, $Im(z) = -2$,

$\therefore\ \bar{z} = \dfrac{3}{2} + 2i$,

$$|z| = \sqrt{\left(\frac{3}{2}\right)^2 + (-2)^2} = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{9+16}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2},$$

$$z + \bar{z} = \left(\frac{3}{2} - 2i\right) + \left(\frac{3}{2} + 2i\right) = \frac{3}{2} + \frac{3}{2} = 3 \text{ and}$$

$$z - \bar{z} = \left(\frac{3}{2} - 2i\right) - \left(\frac{3}{2} + 2i\right) = -4i.$$

(*iii*) $z = Re(z) + i \,.\, Im(z) = \sqrt{2} - \sqrt{3}\, i,$

$\bar{z} = \sqrt{2} + \sqrt{3}\, i,$

$|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{3})^2} = \sqrt{2+3} = \sqrt{5}$ and

$|\bar{z}| = \sqrt{(\sqrt{2})^2 + (\sqrt{3})^2} = \sqrt{2+3} = \sqrt{5}.$

(*iv*) Equating real and imaginary parts on both sides, we get

$3x - 1 = 5, \ \sqrt{3} + 2y = 0 \ \Rightarrow x = 2,\ y = -\frac{\sqrt{3}}{2}.$

Example 3. (*i*) *Simplify* $(-2 + 3i) + 3\left(-\frac{1}{2}i + 1\right) - (2i)$.

(*ii*) *Simplify* $(-3 + \sqrt{-5})(3 + 4\sqrt{-5})$.

(*iii*) *Find the conjugate and modulus of* i^7.

Solution. (*i*) Given expression $= (-2 + 3) + (3 - \frac{3}{2} - 2)\, i$

$= 1 - \frac{1}{2}\, i.$

(*ii*) $(-3 + \sqrt{-5})(3 + 4\sqrt{-5}) = (-3 + \sqrt{5}\, i)(3 + 4\sqrt{5}\, i)$

$= -9 - 12\sqrt{5}\, i + 3\sqrt{5}\, i + \sqrt{5}.4\sqrt{5}\, i^2$

$= -9 - 9\sqrt{5}\, i + 20(-1)$

$= -29 - 9\sqrt{5}\, i.$

(*iii*) Let $z = i^7 = i^4 . i^3 = 1.(-i) = -i,$

so $\bar{z} = \overline{-i} = i$ and $|z| = |-i| = \sqrt{0^2 + (-1)^2} = 1.$

Example 4. *If* $i = \sqrt{-1}$, *prove the following :*

$(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i) = x^4 + 4.$ (*I.S.C. 2004*)

Solution. L.H.S. $= [(\overline{x+1} + i)(\overline{x+1} - i)]\,[(\overline{x-1} + i)(\overline{x-1} - i)]$

$= [(x+1)^2 - i^2]\,[(x-1)^2 - i^2]$

$= [x^2 + 2x + 1 - (-1)]\,[x^2 - 2x + 1 - (-1)]$

$= (\overline{x^2+2} + 2x)(\overline{x^2+2} - 2x)$

$= (x^2 + 2)^2 - (2x)^2$

$= x^4 + 4x^2 + 4 - 4x^2 = x^4 + 4 =$ R.H.S.

Example 5. *Express each of the following in the* ***standard form a + ib.***

(*i*) $\frac{i}{1+i}$ (*ii*) $(-1 + \sqrt{3}\, i)^{-1}$

(*iii*) $(2i - i^2)^2 + (1 - 3i)^3$ (*iv*) $\frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i}.$

Solution. (*i*) $\frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i - i^2}{1 - i^2} = \frac{i - (-1)}{1 - (-1)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}\, i.$

(*ii*) $\left(-1 + \sqrt{3}\, i\right)^{-1} = \frac{1}{-1 + \sqrt{3}\, i} = \frac{1}{-1 + \sqrt{3}\, i} . \frac{-1 - \sqrt{3}\, i}{-1 - \sqrt{3}\, i}$

$= \frac{-1 - \sqrt{3}\, i}{(-1)^2 - (\sqrt{3}\, i)^2} = \frac{-1 - \sqrt{3}\, i}{1 - (-3)} = -\frac{1}{4} - \frac{\sqrt{3}}{4}\, i.$

(*iii*) $(2i - i^2)^2 + (1 - 3i)^3 = (2i + 1)^2 + (1 - 3i)^3$

$= (4i^2 + 4i + 1) + (1 - 9i + 27i^2 - 27i^3)$

$= -4 + 4i + 1 + 1 - 9i - 27 + 27i = -29 + 22i.$

(*iv*) $$\frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i} = \frac{(1+i)(3+i)(3+i)-(1-i)(3-i)(3-i)}{(3-i)(3+i)}$$

$$= \frac{(1+i)(8+6i)-(1-i)(8-6i)}{9-i^2} = \frac{(2+14i)-(2-14i)}{9+1} = \frac{28i}{10} = 0 + \frac{14}{5}i.$$

Example 6. *(i) Find the conjugate of* $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

(ii) Find the modulus of $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$.

(iii) Find the modulus of $\dfrac{(2-3i)^2}{-1+5i}$.

Solution. (*i*) Let $$z = \frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{6+9i-4i+6}{2-i+4i+2}$$

$$= \frac{12+5i}{4+3i} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i}$$

$$= \frac{48-36i+20i+15}{4^2-(3i)^2} = \frac{63-16i}{16-9(-1)}$$

$$= \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i.$$

$\therefore$ Conjugate of $z = \dfrac{63}{25} + \dfrac{16}{25}i$.

(*ii*) $$z = \frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}$$

$$= \frac{(1+i^2+2i)-(1+i^2-2i)}{1^2-i^2} = \frac{4i}{1+1} = 2i$$

$= 0 + 2i.$

$\therefore$ Modulus of $z = \sqrt{0^2+2^2} = \sqrt{4} = 2.$

(*iii*) Let $z = \dfrac{(2-3i)^2}{-1+5i}$, then

$$|z| = \left|\frac{(2-3i)^2}{-1+5i}\right| = \frac{|(2-3i)^2|}{|-1+5i|} \qquad \left(\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right)$$

$$= \frac{|2-3i|^2}{|-1+5i|} \qquad (\because |z^2| = |z|^2)$$

$$= \frac{\left(\sqrt{(2)^2+(-3)^2}\right)^2}{\sqrt{(-1)^2+5^2}} = \frac{\left(\sqrt{13}\right)^2}{\sqrt{26}} = \frac{13}{\sqrt{26}} = \sqrt{\frac{13}{2}}.$$

Example 7. *If* $(-2+\sqrt{-3})(-3+2\sqrt{-3}) = a + ib$*, find the real numbers a and b. With these values of a and b, also find the modulus of* $a + ib$. (I.S.C. 2009)

Solution. Given $(-2+\sqrt{-3})(-3+2\sqrt{-3}) = a + ib$

$\Rightarrow \quad (-2+\sqrt{3}\,i)(-3+2\sqrt{3}\,i) = a + ib$

$\Rightarrow \quad 6 - 4\sqrt{3}\,i - 3\sqrt{3}\,i + \sqrt{3}.2\sqrt{3}\,i^2 = a + ib$

$\Rightarrow \quad 6 - 7\sqrt{3}\,i + 6(-1) = a + ib$

$\Rightarrow \quad 0 - 7\sqrt{3}\,i = a + ib.$

Equating real and imaginary parts on both sides, we get

$a = 0$ and $b = -7\sqrt{3}$.

$$|a + ib| = \sqrt{a^2 + b^2} = \sqrt{0^2 + (-7\sqrt{3})^2} = \sqrt{0 + 147}$$

$$= \sqrt{147} = 7\sqrt{3}.$$

Example 8. *Find the values of x and y given that (x + iy) (2 – 3i) = 4 + i.* (*I.S.C. 2008*)

Solution. Given $(x + iy)(2 - 3i) = 4 + i$

$$\Rightarrow \quad x + iy = \frac{4+i}{2-3i} = \frac{4+i}{2-3i} \times \frac{2+3i}{2+3i}$$

$$\Rightarrow \quad x + iy = \frac{8 + 12i + 2i + 3i^2}{2^2 - 9i^2} = \frac{8 + 14i + 3(-1)}{4 - 9(-1)}$$

$$\Rightarrow \quad x + iy = \frac{5}{13} + \frac{14}{13}i.$$

Equating real and imaginary parts on both sides, we get

$$x = \frac{5}{13} \text{ and } y = \frac{14}{13}.$$

Example 9. *Given that* $\dfrac{2\sqrt{3}\cos 30° - 2i\sin 30°}{\sqrt{2}(\cos 45° + i\sin 45°)} = A + iB$, *find the values of A and B.*

(*I.S.C. 2002*)

Solution. Given $\dfrac{2\sqrt{3}\cos 30° - 2i\sin 30°}{\sqrt{2}(\cos 45° + i\sin 45°)} = A + iB$

$$\Rightarrow \quad A + iB = \frac{2\sqrt{3}\cdot\frac{\sqrt{3}}{2} - 2i.\frac{1}{2}}{\sqrt{2}\left(\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)} = \frac{3-i}{1+i}$$

$$= \frac{3-i}{1+i} \times \frac{1-i}{1-i} = \frac{3 - 3i - i + i^2}{1^2 - i^2}$$

$$= \frac{3 + (-1) - 4i}{1 - (-1)} = \frac{2 - 4i}{2} = 1 - 2i.$$

Equating real and imaginary parts on both sides, we get

A = 1 and B = –2.

Example 10. *Find the real values of x and y satisfying the equality*

$$\frac{x - 2 + (y-3)i}{1+i} = 1 - 3i.$$ (*I.S.C. 2003*)

Solution. Given $\dfrac{x - 2 + (y-3)i}{1+i} = 1 - 3i$

$\Rightarrow \quad (x - 2) + (y - 3)i = (1 - 3i)(1 + i)$

$\Rightarrow \quad (x - 2) + (y - 3)i = (1 + 3) + (-3 + 1)i$

$\Rightarrow \quad (x - 2) + (y - 3)i = 4 - 2i.$

Equating real and imaginary parts on both sides, we get

$x - 2 = 4$ and $y - 3 = -2$

$\Rightarrow \quad x = 6$ and $y = 1.$

Example 11. *Find the real values of x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.*

Solution. According to given, $(x - iy)(3 + 5i) = \overline{-6 - 24i}$

$\Rightarrow \quad 3x + 5xi - 3yi - 5y(-1) = -6 + 24i$

$\Rightarrow \quad (3x + 5y) + (5x - 3y)i = -6 + 24i.$

Equating real and imaginary parts on both sides, we get

$3x + 5y = -6$ and $5x - 3y = 24.$

Solving these equations simultaneously, we get

$x = 3$ and $y = -3.$

Example 12. *For what real values of x and y are the following numbers equal*

(i) $(1 + i)y^2 + (6 + i)$ and $(2 + i)x$

(ii) $x^2 - 7x + 9yi$ and $y^2 i + 20i - 12$?

Solution. (*i*) $(1 + i)y^2 + (6 + i) = (2 + i)x$

$\Rightarrow (y^2 + 6) + i(y^2 + 1) = 2x + ix$

$\Rightarrow y^2 + 6 = 2x$ and $y^2 + 1 = x$

$\Rightarrow x = 5$ and $y^2 = 4 \Rightarrow x = 5$ and $y = \pm 2.$

Hence, $x = 5, y = 2 ; x = 5, y = -2.$

(*ii*) $x^2 - 7x + 9yi = y^2 i + 20i - 12$

$\Rightarrow \quad (x^2 - 7x) + i(9y) = (-12) + i(y^2 + 20)$

$\Rightarrow \quad x^2 - 7x = -12$ and $9y = y^2 + 20$

$\Rightarrow \quad x^2 - 7x + 12 = 0$ and $y^2 - 9y + 20 = 0$

$\Rightarrow \quad (x - 4)(x - 3) = 0$ and $(y - 5)(y - 4) = 0$

$\Rightarrow \quad x = 4, 3$ and $y = 5, 4.$

Hence, $x = 4, y = 5 ; x = 4, y = 4 ; x = 3, y = 5 ; x = 3, y = 4.$

Example 13. *If z is a complex number and $iz^3 + z^2 - z + i = 0$, then prove that $|z| = 1$.*

Solution. Given $iz^3 + z^2 - z + i = 0$

$\Rightarrow \quad iz^3 + z^2 + i^2 z + i = 0 \qquad (\because i^2 = -1)$

$\Rightarrow \quad z^2(iz + 1) + i(iz + 1) = 0$

$\Rightarrow \quad (z^2 + i)(iz + 1) = 0$

$\Rightarrow \quad z^2 = -i \;$ or $\; iz = -1$

$\Rightarrow \quad z^2 = -i \;$ or $\; z = -\frac{1}{i} = -(-i) = i.$

Now $z = i \Rightarrow |z| = |i| = \sqrt{0^2 + 1^2} = 1$

and $z^2 = -i \Rightarrow |z^2| = |-i| \Rightarrow |z|^2 = \sqrt{0 + (-1)^2} = 1 \Rightarrow |z| = 1.$

Hence, $|z| = 1$ in both cases.

Example 14. *Solve the equation $2z = |z| + 2i$.*

Solution. Let $z = x + iy$ where $x, y \in \mathbf{R}$ so that $|z| = \sqrt{x^2 + y^2}$.

Given $2z = |z| + 2i \Rightarrow 2(x + iy) = \sqrt{x^2 + y^2} + 2i$

$\Rightarrow \quad 2x = \sqrt{x^2 + y^2}$ and $2y = 2$

$\Rightarrow \quad 4x^2 = x^2 + y^2$ and $y = 1$

$\Rightarrow \quad 3x^2 = 1$ and $y = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$ and $y = 1.$

$$= \frac{|\beta - \alpha|\,|\bar{\beta}|}{|\overline{\beta - \alpha}|} \quad (\because\ |z_1 z_2| = |z_1|\,|z_2| \text{ and } \bar{z}_1 - \bar{z}_2 = \overline{z_1 - z_2})$$

$$= \frac{|\beta - \alpha|\,|\beta|}{|\beta - \alpha|} \quad (\because\ |\bar{z}| = |z|)$$

$$= |\beta| = 1 \quad (\because\ |\beta| = 1 \text{ given})$$

Hence, $\left|\dfrac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = 1.$

Example 26. *If z_1, z_2 are complex numbers such that $\left|\dfrac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right| = 1$ and $|z_2| \neq 1$, then find $|z_1|$.*

Solution. Given $\left|\dfrac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right| = 1 \Rightarrow \left|\dfrac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right|^2 = 1$

$$\Rightarrow \left(\frac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right)\left(\overline{\frac{z_1 - 3z_2}{3 - z_1\bar{z}_2}}\right) = 1 \quad (\because\ |z|^2 = z\bar{z})$$

$$\Rightarrow \frac{z_1 - 3z_2}{3 - z_1\bar{z}_2} \cdot \frac{\bar{z}_1 - 3\bar{z}_2}{3 - \bar{z}_1 z_2} = 1$$

$$\Rightarrow (z_1 - 3z_2)(\bar{z}_1 - 3\bar{z}_2) = (3 - z_1\bar{z}_2)(3 - \bar{z}_1 z_2)$$

$$\Rightarrow z_1\bar{z}_1 - 3z_1\bar{z}_2 - 3z_2\bar{z}_1 + 9z_2\bar{z}_2 = 9 - 3\bar{z}_1 z_2 - 3z_1\bar{z}_2 + z_1\bar{z}_1 z_2\bar{z}_2$$

$$\Rightarrow |z_1|^2 + 9\,|z_2|^2 = 9 + |z_1|^2\,|z_2|^2$$

$$\Rightarrow |z_1|^2\,|z_2|^2 - |z_1|^2 - 9\,|z_2|^2 + 9 = 0$$

$$\Rightarrow |z_1|^2\,(|z_2|^2 - 1) - 9\,(|z_2|^2 - 1) = 0$$

$$\Rightarrow (|z_2|^2 - 1)\,(|z_1|^2 - 9) = 0$$

$$\Rightarrow |z_1|^2 = 9 \quad (\because\ |z_2| \neq 1 \text{ given})$$

$$\Rightarrow |z_1| = 3 \quad (\because\ |z_1| \geq 0)$$

Example 27. *If $|z_1| = |z_2| = \ldots = |z_n| = 1$, prove that*

$$|z_1 + z_2 + \ldots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right|.$$

Solution. Given $|z_1| = |z_2| = \ldots = |z_n| = 1$

$$\Rightarrow |z_1|^2 = |z_2|^2 = \ldots = |z_n|^2 = 1$$

$$\Rightarrow z_1\bar{z}_1 = 1,\ z_2\bar{z}_2 = 1,\ \ldots,\ z_n\bar{z}_n = 1$$

$$\Rightarrow \frac{1}{z_1} = \bar{z}_1,\ \frac{1}{z_2} = \bar{z}_2,\ \ldots,\ \frac{1}{z_n} = \bar{z}_n.$$

$$\therefore \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right| = |\bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_n|$$

$$= |\overline{z_1 + z_2 + \ldots + z_n}| = |z_1 + z_2 + \ldots + z_n|. \quad (\because\ |\bar{z}| = |z|)$$

EXERCISE 14.1

1. Evaluate

(*i*) $\sqrt{-9}\,\sqrt{-4}$ (*ii*) $\sqrt{(-9)(-4)}$ (*iii*) $i + i^3 + i^5 + \ldots + i^{15}$

(*iv*) $\dfrac{1}{1 + i + i^2}$ (*v*) $(-\sqrt{-4})^3$ (*vi*) $\sum_{k=0}^{100} i^k$.

2. If $z = -3 - i$, find $Re(z)$, $Im(z)$, $\bar{z}$, $|z|$, z^{-1}.

3. Express each of the following in the standard form $a + ib$:

(*i*) $\dfrac{1}{3-4i}$ (*ii*) $\dfrac{1}{-2-\sqrt{-3}}$ (*iii*) $\dfrac{13i}{2-3i}$ (*I.S.C. 2000*)

(*iv*) $\dfrac{(1+i)^2}{3-i}$ (*v*) $\dfrac{3-4i}{(4-2i)(1+i)}$ (*vi*) $\dfrac{1-2i}{2+i}+\dfrac{3+i}{2-i}$. (*I.S.C. 2006*)

4. Find x and y if

(*i*) $2y + (3x - y)\,i = 5 - 2i$

(*ii*) $3 + ix^2y$ and $x^2 + y + 4i$ are conjugate complexes of each other.

(*iii*) $(x - yi)(2 + 3i) = \dfrac{x+2i}{1-i}$

(*iv*) $(x + iy)(2 - 3i)$ is the conjugate of $4 - i$.

5. If $z_1 = 1 - i$, $z_2 = -2 + 4i$, calculate the values of a and b if $a + bi = \dfrac{z_1 z_2}{\bar{z}_1}$.

6. Solve the equation :

$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i,\ x, y \in \mathbf{R},\ i=\sqrt{-1}.$$

7. (*i*) If $|z_1| = |z_2|$, does it follow that $z_1 = z_2$?

(*ii*) Show that $z = -\bar{z}$ iff z is either zero or purely imaginary.

8. If $z_1, z_2, z_3, \ldots, z_n$ are any complex numbers, prove that

(*i*) $|z_1 z_2 z_3 \ldots z_n| = |z_1|\ |z_2|\ |z_3| \ldots |z_n|$

(*ii*) $\overline{z_1 + z_2 + z_3 + \ldots + z_n} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \ldots + \bar{z}_n$

(*iii*) $\overline{z_1 z_2 \ldots z_n} = \bar{z}_1\ \bar{z}_2\ \bar{z}_3 \ldots \bar{z}_n$.

Hint. Use induction.

9. (*i*) Simplify $(3 + 2i)^3$. Also find its conjugate.

(*ii*) Simplify $(1 - 2i)^{-3}$. Also find its conjugate.

(*iii*) Separate into real and imaginary parts $\dfrac{3+\sqrt{-1}}{2-\sqrt{-1}}$. Also find its modulus.

(*iv*) Simplify $(\sqrt{5}+7i)(\sqrt{5}-7i)^2+(-2+7i)^2$.

(*v*) Express the square of $\dfrac{i}{1+i}$ in the form $x + iy$.

(*vi*) Show that $\dfrac{\sqrt{7}+\sqrt{3}\,i}{\sqrt{7}-\sqrt{3}\,i}+\dfrac{\sqrt{7}-\sqrt{3}\,i}{\sqrt{7}+\sqrt{3}\,i}$ is real.

(*vii*) If $z = 1 + i$, evaluate $z^3 - 2z^2 + 3z - 4$

(*viii*) If $z = 3 + 5i$, evaluate $z^3 + \bar{z} + 198$.

10. Find the modulus of the following numbers :

(*i*) $\dfrac{(1+3i)(2-5i)}{(2-\sqrt{6}i)(-3+2\sqrt{5}i)}$ (*ii*) $\dfrac{(3+4i)(4-5i)}{(4-3i)(6+7i)}$.

11. If z is a complex number and $|2z - 1| = |z - 2|$, show that $|z| = 1$.

12. Let $z = x + iy$ and $w = \dfrac{1-iz}{z-i}$, show that if $|w| = 1$, then z is purely real.

13. If x is real and $\dfrac{1-ix}{1+ix} = a - ib$, prove that $a^2 + b^2 = 1$.

Hint. $\left|\dfrac{1-ix}{1+ix}\right| = |a - ib|$.

14. Using $z_1 = 2 - 3i$, $z_2 = 4 + 5i$, $z_3 = -1 - i$, verify that complex number multiplication is left as well as right distributive over addition.

15. If $\left(\dfrac{1-i}{1+i}\right)^{500} = a + ib$, find the values of a and b.

Hint. First simplify $\dfrac{1-i}{1+i} = -i$.

16. Find the least positive integral value of n such that $\left(\dfrac{1+i}{1-i}\right)^n$ is real.

Hint. First simplify $\dfrac{1+i}{1-i} = i$.

17. Solve the equation $|1 - i|^x = 2^x$.

18. Solve the equation $|z| + z = 2 + i$.

19. Show that if $\left|\dfrac{z-5i}{z+5i}\right| = 1$, then z is a real number.

20. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = x + iy$, show that $2.5.10 \ldots\ldots (1 + n^2) = x^2 + y^2$.

Hint. Take modulus of both sides and use the fact that
$|z_1 z_2 \ldots z_n| = |z_1|\,|z_2| \ldots |z_n|$.

21. If $(x + iy)^{1/3} = a + ib$, prove that $\dfrac{x}{a} + \dfrac{y}{b} = 4(a^2 - b^2)$.

Hint. Cube both sides and find x, y in terms of a and b.

22. If $f(z) = z^4 + 9z^3 + 35z^2 - z + 4$, find $f(-5 + 2\sqrt{-4})$.

23. Find the complex numbers satisfying the equations
$\left|\dfrac{z-4}{z-8}\right| = 1$ and $\left|\dfrac{z-12}{z-8i}\right| = \dfrac{5}{3}$.

24. Prove that

(*i*) if $x + iy = \dfrac{a+ib}{a-ib}$, then $x^2 + y^2 = 1$

(*ii*) if $x + iy = \sqrt{\dfrac{1+i}{1-i}}$, then $x^2 + y^2 = 1$

(*iii*) if $x - iy = \sqrt{\dfrac{a-ib}{c-id}}$, then $(x^2 + y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$.

25. If $a + ib = \dfrac{c+i}{c-i}$, where c is real, then prove that

$a^2 + b^2 = 1$ and $\dfrac{b}{a} = \dfrac{2c}{c^2-1}$.

26. If z_1, z_2 are complex numbers such that $\left|\dfrac{z_1 - 2z_2}{2 - \bar{z}_1 z_2}\right| = 1$ and $|z_2| \neq 1$, then find $|z_1|$.

14.2 REPRESENTATIONS OF COMPLEX NUMBERS

A complex number can be represented geometrically or trigonometrically.

14.2.1 Geometric representation

We know that corresponding to every real number there exists a unique real number on the number line (called real axis) and conversely, corresponding to every point on the line there exists a unique real number *i.e.* there is a one-one corresponding between the set **R** of real numbers and the points on the real axis.

In a similar way, corresponding to every ordered pair (x, y) of real numbers there exists a unique point P in the co-ordinate plane with x as ***abscissa*** and y as ***ordinate*** of the point P and conversely, corresponding to every point P in the plane there exists a unique ordered pair of real numbers. Thus, there is a one-one correspondence between the set of ordered pairs $\{(x, y) \,;\, x, y \in \mathbf{R}\}$ and the points in the co-ordinate plane.

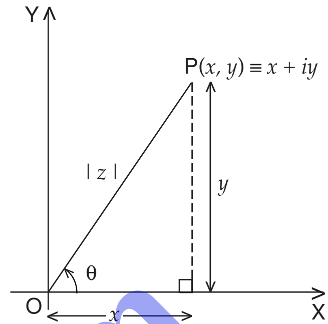

Fig. 14.1.

The point P with co-ordinates (x, y) is said to represent the complex number $z = x + iy$.

It follows that the complex number $z = x + iy$ can be uniquely represented by the point $P(x, y)$ in the co-ordinate plane and conversely, corresponding to the point $P(x, y)$ in the plane there exists a unique complex number $z = x + iy$. The plane is called the **complex plane** and the representation of complex numbers as points in the plane is called **Argand diagram**.

Notice that length OP $= \sqrt{x^2 + y^2} = |z|$.

Figure 14.2 represents the complex numbers $0, 2, -2, 3i, -3i, 2+3i, -2+3i, -2-3i, 2-3i$.

Note that every real number $x = x + 0i$ is represented by point $(x, 0)$ lying on x-axis, and every purely imaginary number iy is represented by point $(0, y)$ lying on y-axis. Consequently, x-axis is called the **real axis** and y-axis is called the **imaginary axis.**

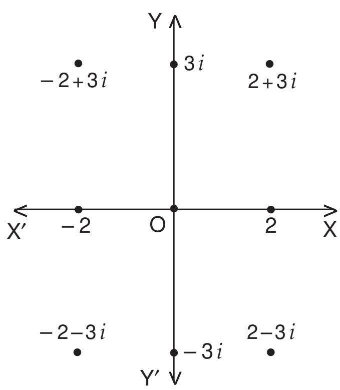

Fig. 14.2.

Since any complex number $z = x + iy$ is represented by point $P(x, y)$ in complex plane, many geometric ideas arise from this — distance between two points ; equation of line ; equation of a circle ; describing some region in complex plane *e.g.* all points lying inside a circle, and so on.

1. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then the distance between two corresponding points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2| = |z_2 - z_1|.$$

2. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ are two complex numbers and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are two corresponding points, then the point Q dividing $[P_1P_2]$ *i.e.* the segment P_1P_2 in the ratio $m : n$ is given by

$$Q\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right).$$

So corresponding complex number is

$$z_3 = \frac{mx_2 + nx_1}{m+n} + i\,\frac{my_2 + ny_1}{m+n}$$

$$= \frac{m(x_2 + i\,y_2) + n(x_1 + i\,y_1)}{m+n}$$

$$= \frac{mz_2 + nz_1}{m+n}.$$

In particular, putting $m = n = 1$, the **mid-point** of $[P_1P_2]$ is given by $\frac{z_1 + z_2}{2}$.

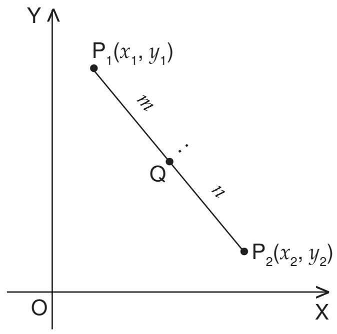

Fig. 14.3.

3. If $z = x + i\,y$, $x, y \in \mathbf{R}$, is represented by the point $P(x, y)$ in the complex plane, then the complex numbers $-z$, $\bar{z}$, $-\bar{z}$ are represented by the points $P'(-x, -y)$, $Q(x, -y)$ and $Q'(-x, y)$ respectively in the complex plane (see fig. 14.4.).

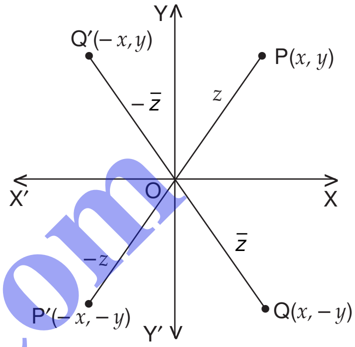

Fig. 14.4.

4. ***Geometric representation of sum of two complex numbers.***
Let the complex numbers z_1 and z_2 be represented by the points P and Q respectively in the complex plane. Complete the parallelogram OPRQ and let its diagonals meet at M. Since diagonals of a parallelogram bisect each other, M is mid-point of [PQ] and so it represents the complex number $\frac{z_1 + z_2}{2}$.

Let R represent the complex number z. As M is also mid-point of [OR], it will represent the complex number $\frac{0 + z}{2}$

$$\Rightarrow \quad \frac{z}{2} = \frac{z_1 + z_2}{2} \quad \Rightarrow \quad z = z_1 + z_2.$$

Hence, the point R represents the complex number $z_1 + z_2$.
*This is known as **parallelogram law.***

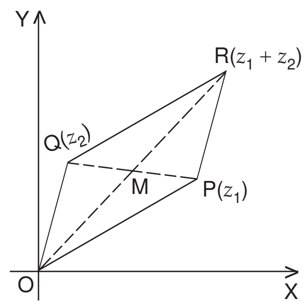

Fig. 14.5.

5. ***Geometric representation of difference of two complex numbers.***
Let the complex numbers z_1 and z_2 be represented by the points P and Q respectively in the complex plane. Join QO and produce it to Q′ such that O is mid-point of [QQ′] then Q′ represents the complex number $-z_2$. Complete the parallelogram OPRQ′ then the point R represents the complex number $z_1 + (-z_2)$ *i.e.* $z_1 - z_2$.

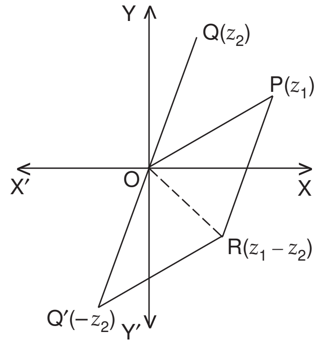

Fig. 14.6.

14.2.2 Trigonometric (polar) representation

We have already seen that a complex number $x + iy$ is uniquely represented by a point P(x, y) in the complex plane. *The angle XOP is called* ***amplitude*** *or* ***argument*** *of z and is written as amp(z) or arg(z).*

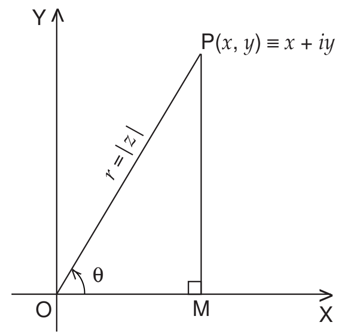

Fig. 14.7.

Let OP $= |z| = r > 0$, and amp(z) $= \theta$.

From fig. 14.7, we see that

$x = r\cos\theta$ and $y = r\sin\theta$

$\therefore\ z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$.

This form of z is called **trigonometric form or polar form.** The number $(\cos\theta + i\sin\theta)$ is usually abbreviated as cis θ. Thus, if modulus of z is r and amp(z) $= \theta$, then

$\boldsymbol{z = r(\cos\theta + i\sin\theta) = r \text{ cis } \theta}$.

The unique value of θ such that $-\pi < \theta \le \pi$ is called principal value of amplitude or argument.

Note that the number zero cannot be put into the form $r(\cos\theta + i\sin\theta)$. Therefore, if $z = 0$ then the arg (z) is not defined.

For example :

(*i*) Let $z = 1 + i$, then $r = \sqrt{1^2 + 1^2} = 2$

and $r\cos\theta = 1$, $r\sin\theta = 1$

$\Rightarrow\quad \cos\theta = \dfrac{1}{\sqrt{2}}$, $\sin\theta = \dfrac{1}{\sqrt{2}}$

$\Rightarrow\quad \theta = \dfrac{\pi}{4}$.

So the principal value of amplitude is $\dfrac{\pi}{4}$.

$\therefore\quad z = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$.

Conversely, if $r = \sqrt{2}$ and $\theta = \dfrac{\pi}{4}$

then $z = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

$= \sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right) = 1 + i$.

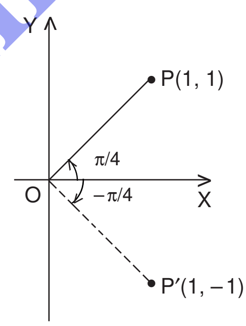

Fig. 14.8.

(*ii*) Let $z = 1 - i$, then $r = \sqrt{1^2 + (-1)^2} = 2$

and $r\cos\theta = 1$, $r\sin\theta = -1$

$\Rightarrow\quad \cos\theta = \dfrac{1}{\sqrt{2}}$, $\sin\theta = -\dfrac{1}{\sqrt{2}}$

$\Rightarrow\quad \theta = -\dfrac{\pi}{4}$.

So the principal value of amplitude is $-\dfrac{\pi}{4}$.

$\therefore\quad z = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right)$.

Conversely, if $r = \sqrt{2}$ and $\theta = -\dfrac{\pi}{4}$

then $z = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right)$

$= \sqrt{2}\left(\dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}\right) = 1 - i$.

26. If complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle at C, then show that

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2).$$

Hint. Take C as origin. Let BC = CA = k, then

$$z_1 = k + i\,0 = k,\ z_2 = 0 + ik = ik \text{ and } z_3 = 0 + i\,0 = 0.$$

27. If $\omega\,(\neq 1)$ is a cube root of unity, then find the conjugate of $2\omega - 3i$.

28. If z is a complex number, prove that $(z - 1)(\bar{z} - 1) = |z - 1|^2$.

ANSWERS

EXERCISE 14.1

1. (*i*) -6 (*ii*) 6 (*iii*) 0 (*iv*) $-i$ (*v*) $8i$ (*vi*) 1.

2. $-3, -1, -3 + i, \sqrt{10}, -\frac{3}{10} + \frac{1}{10}i$.

3. (*i*) $\frac{3}{25} + \frac{4}{25}i$ (*ii*) $-\frac{2}{7} + \frac{\sqrt{3}}{7}i$ (*iii*) $-3 + 2i$

(*iv*) $-\frac{1}{5} + \frac{3}{5}i$ (*v*) $\frac{1}{4} - \frac{3}{4}i$ (*vi*) $1 + 0i$.

4. (*i*) $x = \frac{1}{6}, y = \frac{5}{2}$ (*ii*) $x = \pm 2, y = -1$ (*iii*) $x = \frac{2}{21}, y = -\frac{8}{21}$

(*iv*) $x = \frac{5}{13}, y = \frac{14}{13}$.

5. $a = 4, b = 2$. **6.** $x = 3, y = -1$.

7. (*ii*) No; *e.g.* let $z_1 = 1 + i$, $z_2 = 1 - i$.

9. (*i*) $-9 + 46i$; $-9 - 46i$ (*ii*) $-\frac{11}{125} - \frac{2}{125}i$; $-\frac{11}{125} + \frac{2}{125}i$ (*iii*) $1 + i, \sqrt{2}$

(*iv*) $(54\sqrt{5} - 45) - 406i$ (*v*) $0 + \frac{1}{2}i$ (*vii*) $-3 + i$ (*viii*) $3 + 5i$.

10. (*i*) 1 (*ii*) $\sqrt{\frac{41}{85}}$. **15.** $a = 1, b = 0$. **16.** $n = 2$.

17. $x = 0$ is the only solution. **18.** $z = \frac{3}{4} + i$ is the only solution.

22. -160. **23.** Solutions are $6 + 8i$, $6 + 17i$. **26.** 2.

EXERCISE 14.2

1. (*i*) $2 \text{ cis } \frac{\pi}{6}$ (*ii*) $\sqrt{2} \text{ cis}\left(-\frac{3\pi}{4}\right)$ (*iii*) $\pi \text{ cis } 0$ (*iv*) $\text{cis } 0$

(*v*) $\sqrt{2} \text{ cis}\left(\frac{3\pi}{4}\right)$ (*vi*) $2 \text{ cis } \frac{\pi}{3}$ (*vii*) $\sqrt{2} \text{ cis } \frac{\pi}{4}$.

2. (*i*) $\sqrt{2}$; $\frac{3\pi}{4}$ (*ii*) 1 ; $\pi - \tan^{-1}\frac{4}{3}$ (*iii*) $\sqrt{2}$; $-\frac{3\pi}{4}$. **3.** $\frac{\pi}{10}$.

4. (*i*) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (*ii*) $1 - i$. **14.** Maximum value = $\sqrt{2} + 1$, minimum value = $\sqrt{2} - 1$.

EXERCISE 14.3

1. (*i*) $|z| = 2$ (*ii*) $|z + 1 + i| > \sqrt{2}$

(*iii*) $|z - 1 - i| = \sqrt{2}$ (*iv*) $Re(z) > 0$

(*v*) $Re(z) = 0$ (*vi*) $Re(z) > 0$ and $Im(z) > 0$.