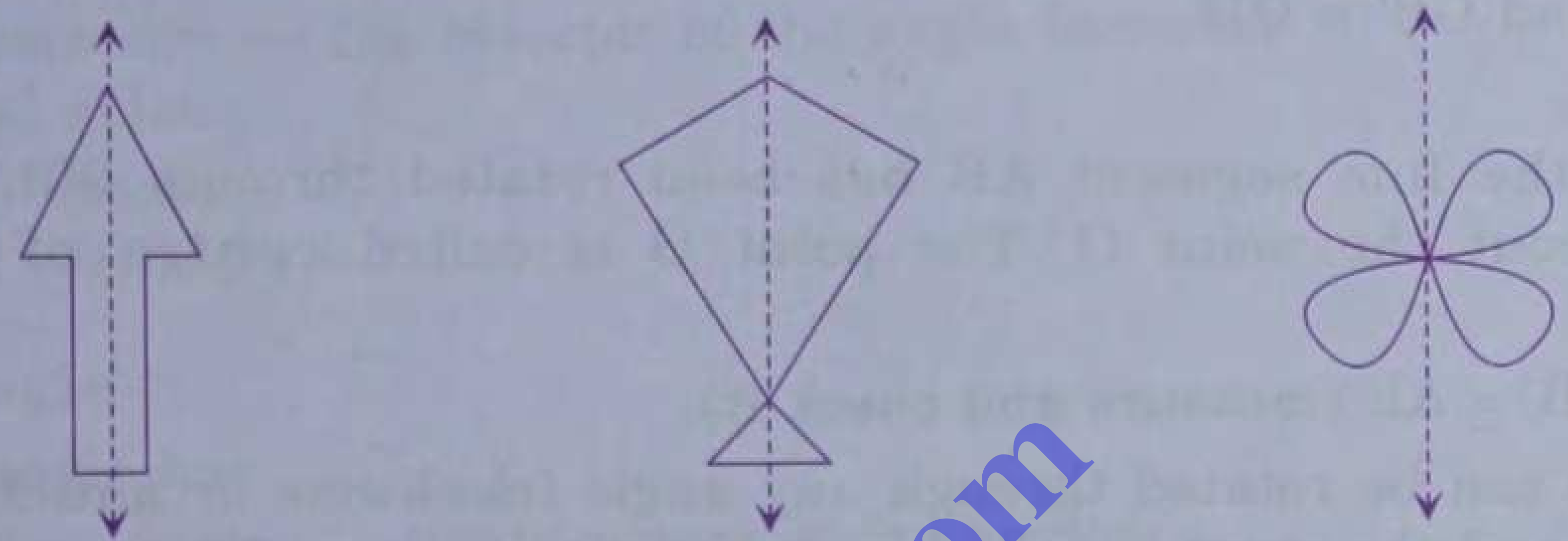
Chapter 27

SYMMETRY, REFLECTION

John Keats wrote 'A thing of beauty is joy forever'. When we describe an object as beautiful, perhaps we mean that it is balanced and has regularity in shape.

LINE SYMMETRY

Look at the following plane figures :



We observe that if these figures are folded along a specific line (shown dotted), each figure on the left hand side of the dotted line fits exactly on the top of the figure on the right hand side of the dotted line *i.e.* each figure is divided into two coincident parts about the dotted line. This leads to :

If a figure is divided into two coincident parts by a line then the figure is called ***symmetrical*** *about that line. The line which divides the figure into two coincident parts is called the* ***line of symmetry*** *or* ***axis of symmetry*** *or* ***mirror line****.*

A simple test to determine whether a figure has line symmetry is to fold the figure along the supposed line of symmetry and to see if the two halves of the figure coincide.

Each one of the figures (*i*), (*ii*) and (*iii*) shown above is symmetrical about the dotted line, and the dotted line is its axis (or line) of symmetry.

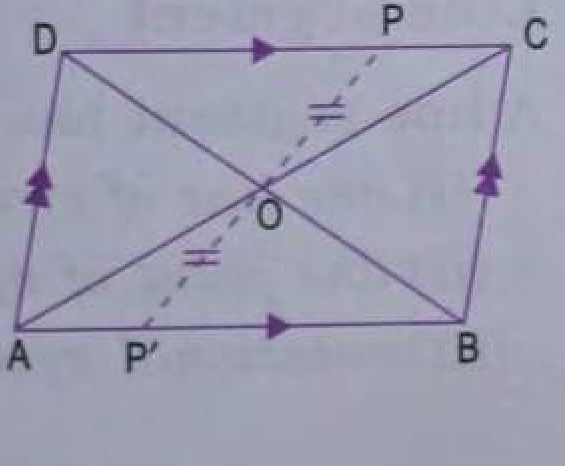
A two dimensional (plane) figure may or may not have a line of symmetry. Moreover, a plane figure may have more than one line of symmetry.

POINT SYMMETRY

Point symmetry exists when a figure is built around a single point called the ***centre*** *of the figure. For every point of the figure, there is another point found directly opposite it at the same distance on the other side of the centre.*

A simple test to determine whether a figure has point symmetry is to turn it upside-down and see if it looks the same. A figure that has point symmetry is unchanged in appearance by 180 degree rotation.

Parallelogram



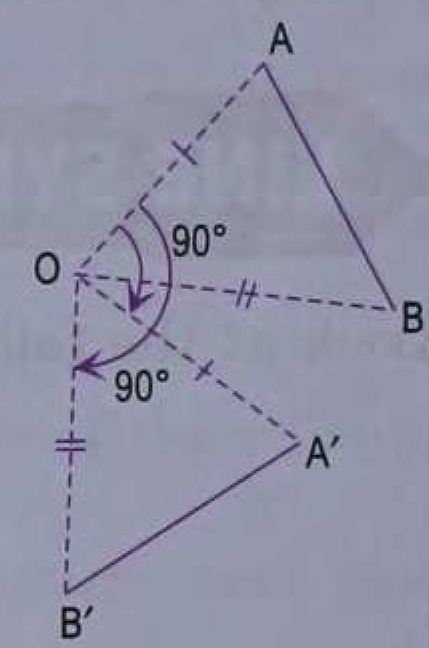
Let ABCD be a parallelogram and O be the point of intersection of its diagonals. It has a point symmetry about the point O because every point P on the figure has a point P′ directly opposite it on the other side of O.

Letter S

The letter 'S' has a point symmetry about the point O because every point P on the figure has a point P′ directly opposite it on the other side of O.

ROTATION

Look at the adjoining sketch. Let us rotate the line segment AB through 90° (clockwise) about the point O.



Steps

1. Join OA and OB.
2. Draw ∠AOA′ = 90° and ∠BOB′ = 90° (clockwise) such that OA′ = OA and OB′ = OB.
3. Join A′B′.

We say that the line segment AB has been rotated through 90° (clockwise) about the point O. The point O is called **centre of rotation**.

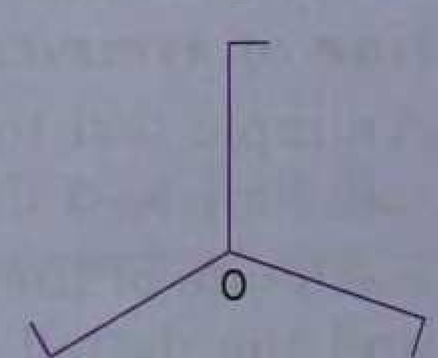
Notice that A′B′ = AB (measure and check it).

Thus, a figure can be rotated through any angle (clockwise or anticlockwise direction) about a point, and the point is called centre of rotation.

On rotation, the size of the figure remains the same.

ROTATIONAL SYMMETRY

Look at the adjoining sketch. It does not have any line of symmetry or point of symmetry. Yet it seems balanced and has regularity of shape. Let this figure be rotated through one complete turn (clockwise or anticlockwise) about the point O. There are three occasions when it looks the same as it did in its starting position. These are when it has been rotated through 120°, 240° and 360°. We say that this figure has a **rotational symmetry of order 3**.



Note that if A° is the smallest angle through which a figure can be rotated and still look the same, then it has a rotational symmetry of order = $\frac{360}{A}$.

Remember that for a figure to have a rotational symmetry, A° must be less than or equal to 180°.

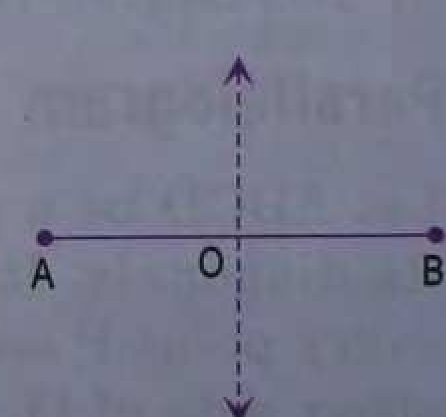
SYMMETRY OF SOME FIGURES

In each of the following case, the dotted line (or lines) is the axis of symmetry:

Line segment

A line segment has

(*i*) *one line of symmetry* — the perpendicular bisector of the segment.
(*ii*) *one point of symmetry* — the mid-point of the segment.
(*iii*) *rotational symmetry of order 2.*



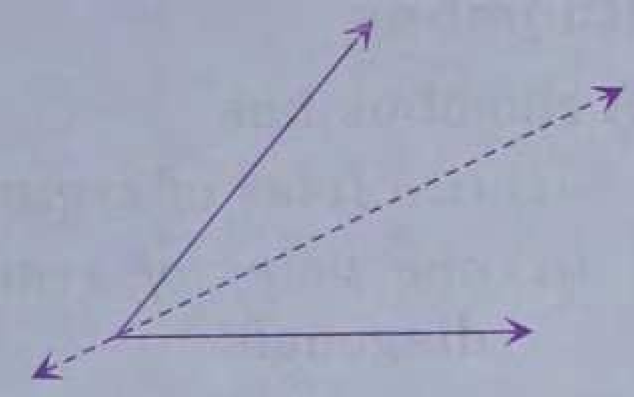
Angle

An angle (with equal arms) has

(*i*) *one line of symmetry* — the bisector of the angle.

(*ii*) *no point of symmetry*.

(*iii*) *no rotational symmetry*.



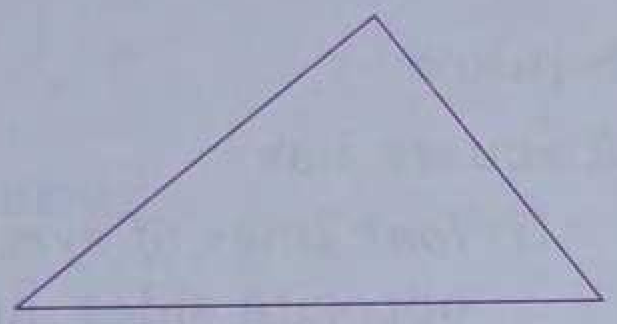
Scalene triangle

A scalene triangle has

(*i*) *no line of symmetry*.

(*ii*) *no point of symmetry*.

(*iii*) *no rotational symmetry*.



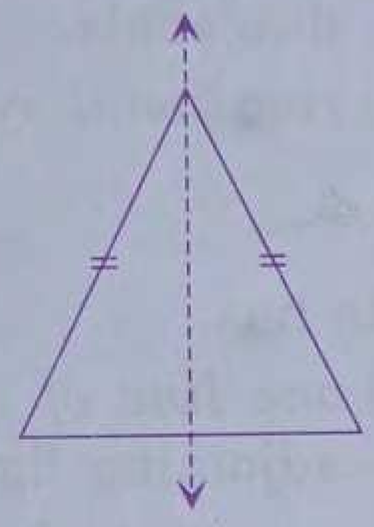
Isosceles triangle

An isosceles triangle has

(*i*) *one line of symmetry* — the bisector of the angle included between equal sides.

(*ii*) *no point of symmetry*.

(*iii*) *no rotational symmetry*.



Equilateral triangle

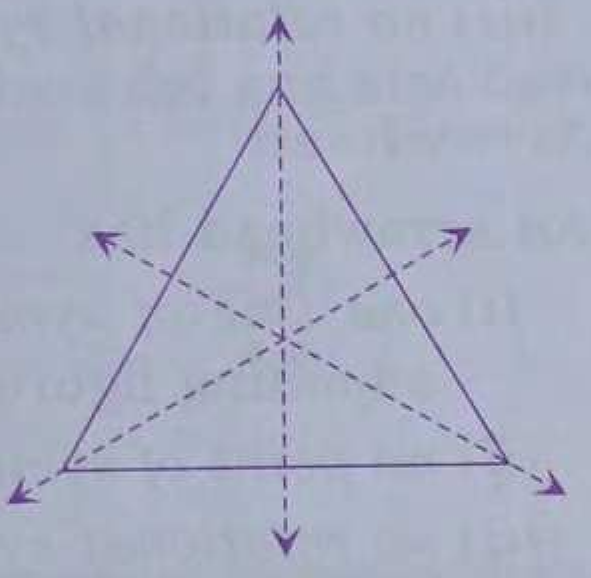
An equilateral triangle has

(*i*) *three lines of symmetry* — the bisectors of the angles.

(*ii*) *no point of symmetry*.

(*iii*) *rotational symmetry of order 3*.

The centre of rotation is the point of intersection of the bisectors of the angles.



Parallelogram

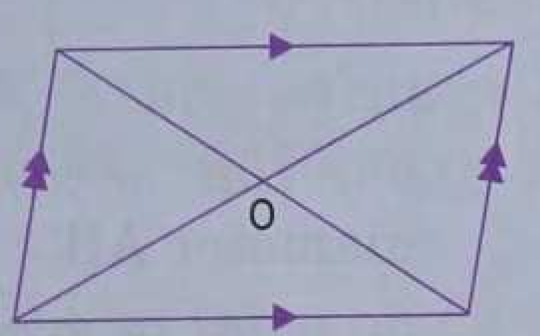
A parallelogram has

(*i*) *no line of symmetry*.

(*ii*) *one point of symmetry* — the point of intersection of the diagonals.

(*iii*) *rotational symmetry of order 2*.

The centre of rotation is the point of intersection of the diagonals.



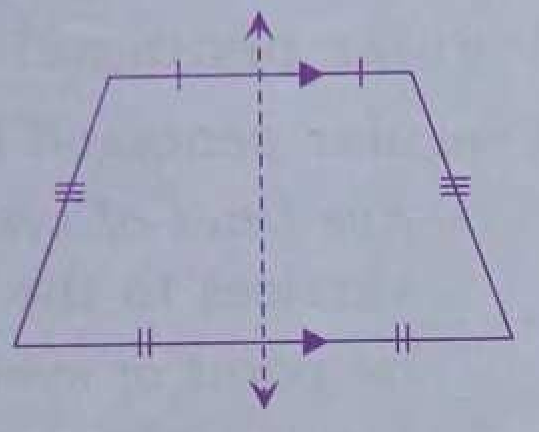
Isosceles trapezium

As isosceles trapezium has

(*i*) *one line of symmetry* — the line joining the mid-points of the bases of the trapezium.

(*ii*) *no point of symmetry*.

(*iii*) *no rotational symmetry*.



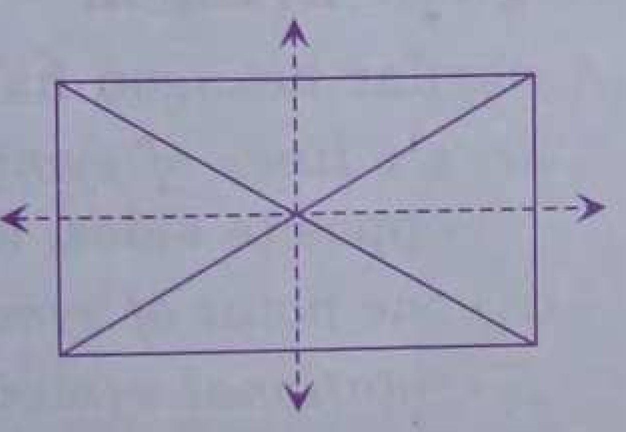
Rectangle

A rectangle has

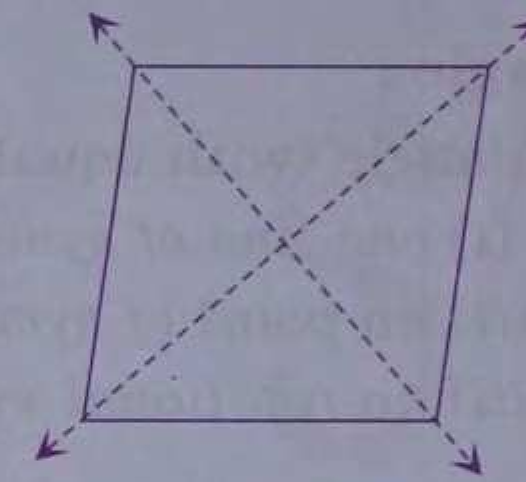
(*i*) *two lines of symmetry* — the lines joining the mid-points of opposite sides.

(*ii*) *one point of symmetry* — the point of intersection of the diagonals.

(*iii*) *rotational symmetry of order 2*.



Rhombus



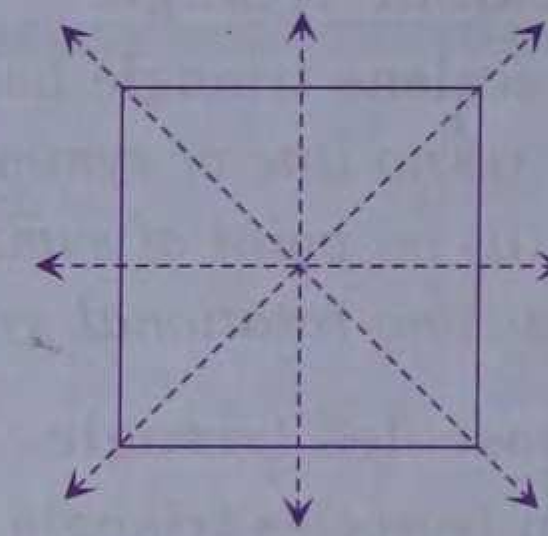
A rhombus has

(*i*) *two lines of symmetry* — the diagonals of the rhombus.

(*ii*) *one point of symmetry* — the point of intersection of the diagonals.

(*iii*) *rotational symmetry of order 2.*

Square



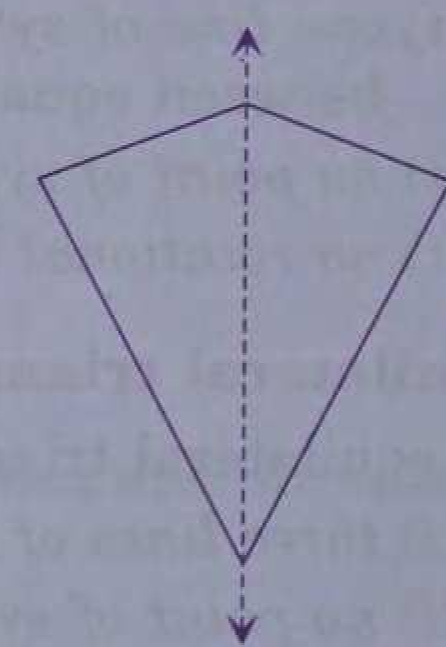
A square has

(*i*) *four lines of symmetry* — the lines joining the mid-points of opposite sides, and the diagonals.

(*ii*) *one point of symmetry* — the point of intersection of the diagonals.

(*iii*) *rotational symmetry of order 4.*

Kite



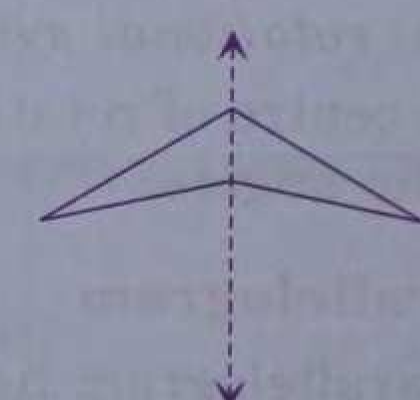
A kite has

(*i*) *one line of symmetry* — the diagonal shown dotted in the adjoining figure.

(*ii*) *no point of symmetry.*

(*iii*) *no rotational symmetry.*

Arrowhead



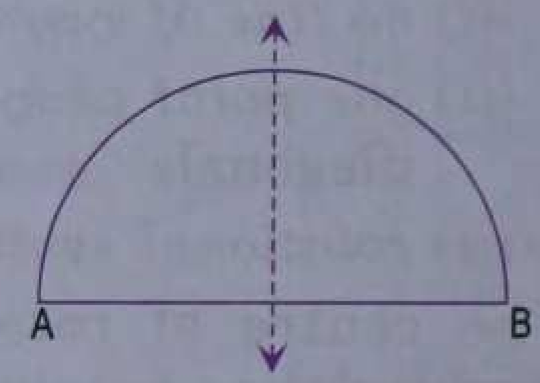
An arrowhead has

(*i*) *one line of symmetry* — the diagonal shown dotted in the adjoining figure.

(*ii*) *no point of symmetry.*

(*iii*) *no rotational symmetry.*

Semicircle



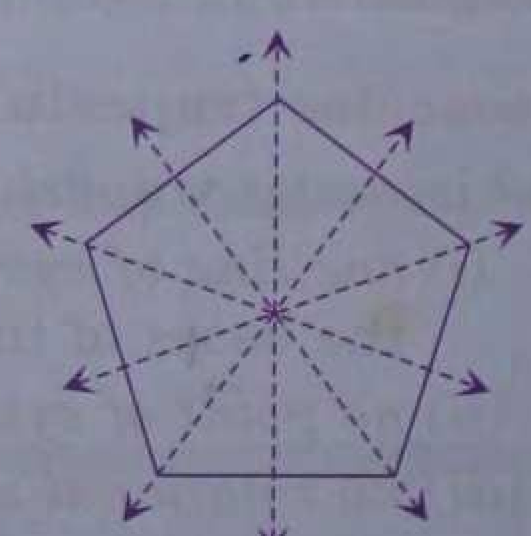
A semicircle has

(*i*) *one line of symmetry* — the perpendicular bisector of the diameter AB.

(*ii*) *no point of symmetry.*

(*iii*) *no rotational symmetry.*

Regular pentagon



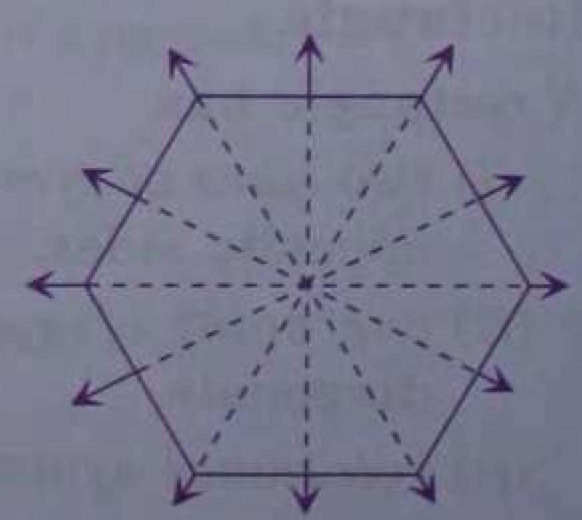
A regular pentagon has

(*i*) *five lines of symmetry* — the perpendiculars drawn from the vertices to the opposite sides.

(*ii*) *no point of symmetry.*

(*iii*) *rotational symmetry of order 5.*

Regular hexagon



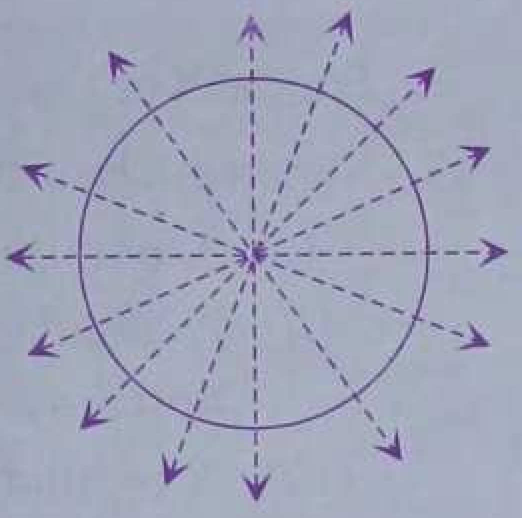
A regular hexagon has

(*i*) *six lines of symmetry* — the lines joining the mid-points of opposite sides, and the diagonals through the centre.

(*ii*) *one point of symmetry — the centre of the regular hexagon.*

(*iii*) *rotational symmetry of order 6.*

Circle



A circle has

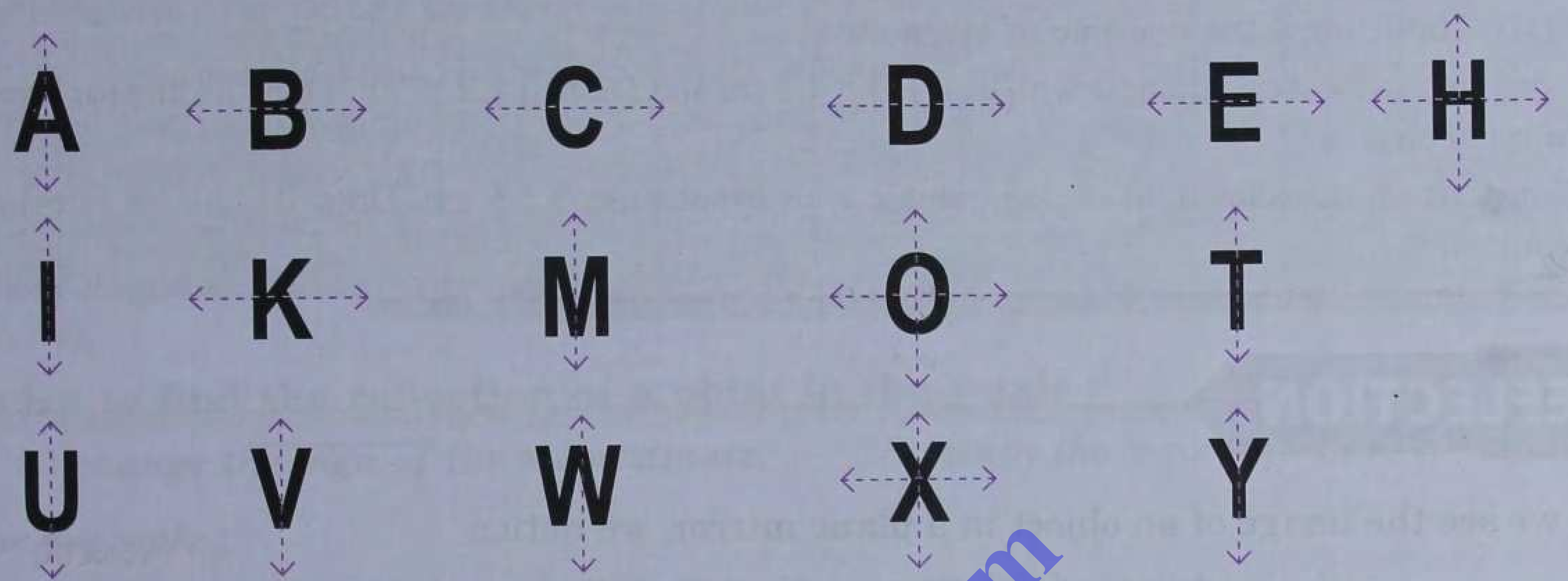
(i) *an infinite number of lines of symmetry* — all lines along diameters.

(ii) *one point of symmetry* — the centre of the circle.

(iii) *rotational symmetry of infinite order.*

Letters

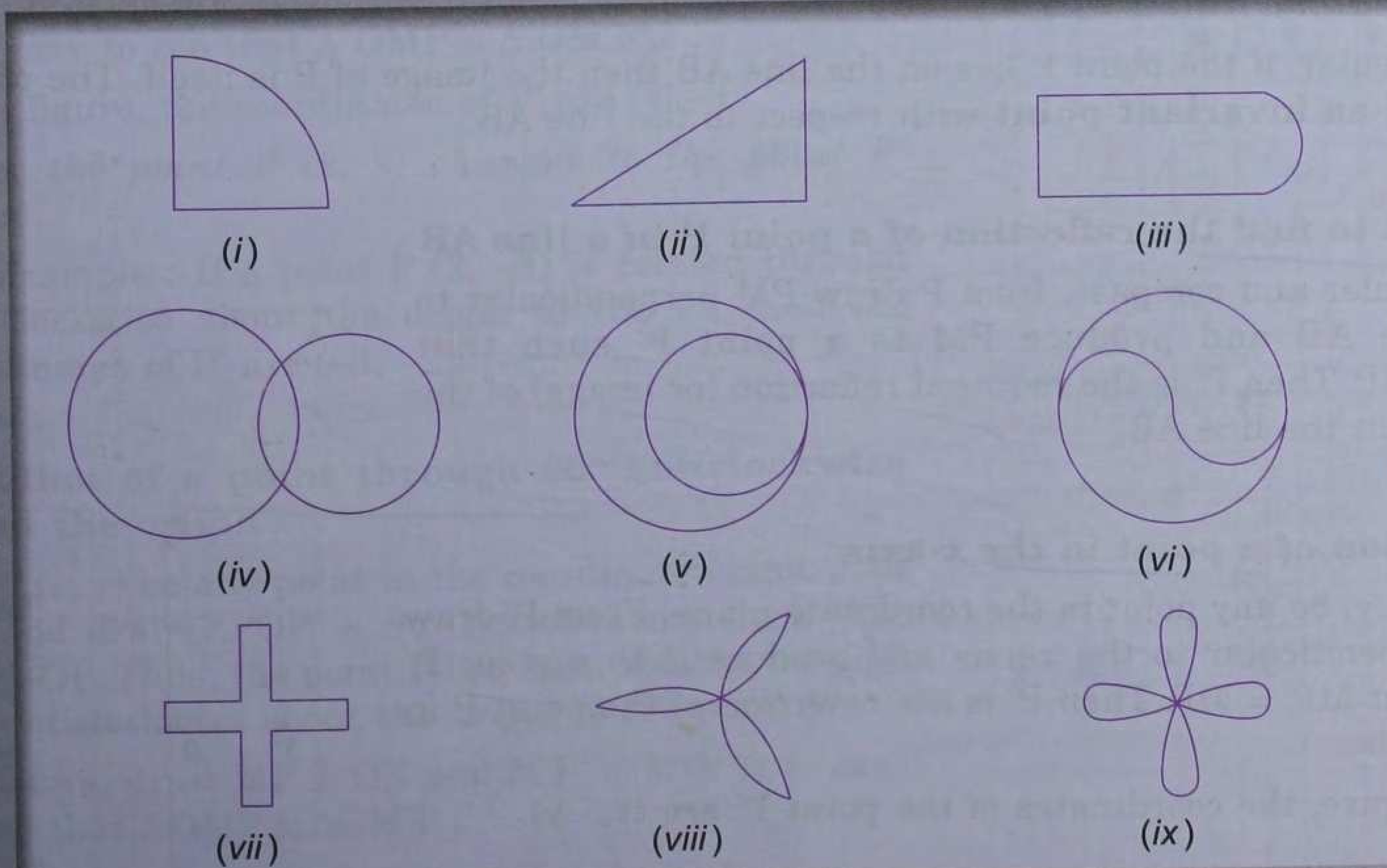
The following letters are symmetrical about the dotted line (or lines) :



The following letters have a point symmetry about a point marked by a dot and also have rotational symmetry of order 2 :

Exercise 27.1

1. Draw the line or lines of symmetry, if any, of the following shapes and count their number :



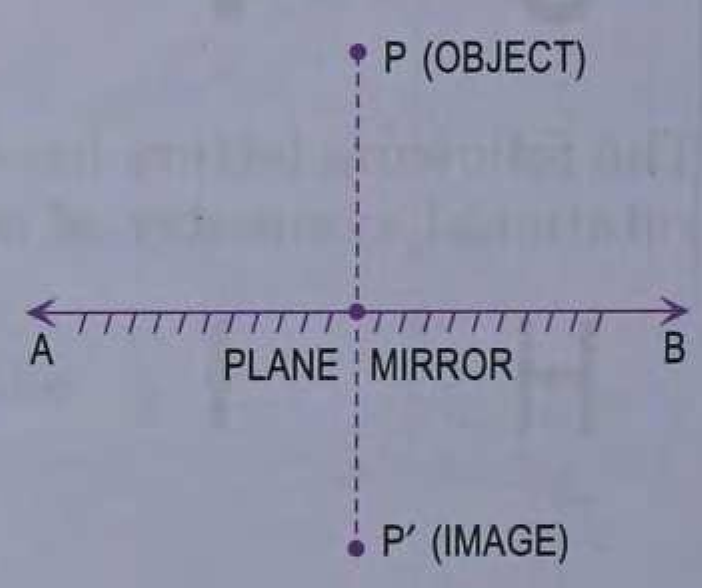
2. For each of the given shape in question 1, find the order of the rotational symmetry (if any).
3. State which of the following statements are true and which are false :
   (*i*) A parallelogram has diagonals as its lines of symmetry.
   (*ii*) A regular triangle has three lines of symmetry, one point of symmetry and has rotational symmetry of order 3.
   (*iii*) A regular quadrilateral has four lines of symmetry, one point of symmetry and has rotational symmetry of order 4.
   (*iv*) A parallelogram has no rotational symmetry.
   (*v*) A regular pentagon has one point of symmetry.
   (*vi*) The letter Z has one line of symmetry.
4. Construct an isosceles triangle with base PQ = 4·2 cm and vertical $\angle R = 30°$. Draw its line (or lines) of symmetry.
5. Construct an isosceles right angled triangle with hypotenuse = 5·6 cm. Draw its line (or lines) of symmetry.

REFLECTION

When we see the image of an object in a plane mirror, we notice that :

(*i*) the distance of the **image** behind the mirror is the same as the distance of the object in front of it.

(*ii*) the **mirror line** is perpendicular to the line joining the object and its image.



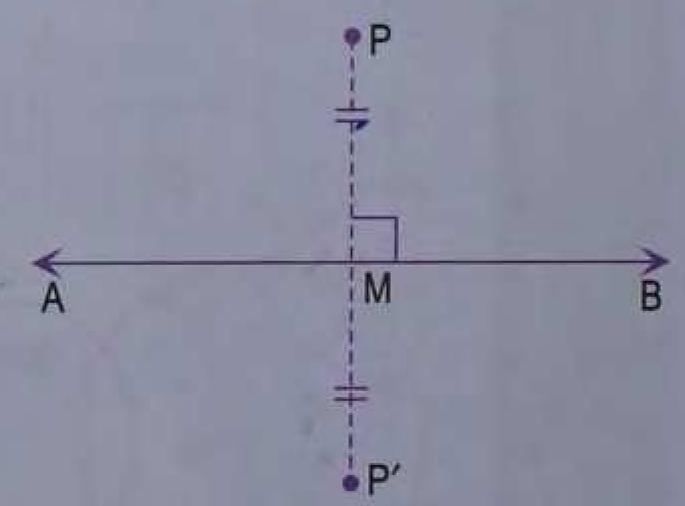
Thus, if P is the object and P′ its image in a plane mirror, then the mirror line, say AB, is the perpendicular bisector of the line segment PP′. This leads to :

*The **reflection** (or **image**) of a point P in a line AB is a point P′ such that AB is the perpendicular bisector of the line segment PP′, and the line AB is called **axis of reflection** (or **mediator**).*

In particular, if the point P lies on the line AB then the image of P is itself. The point P is called an **invariant point** with respect to the line AB.

Method to find the reflection of a point P in a line AB

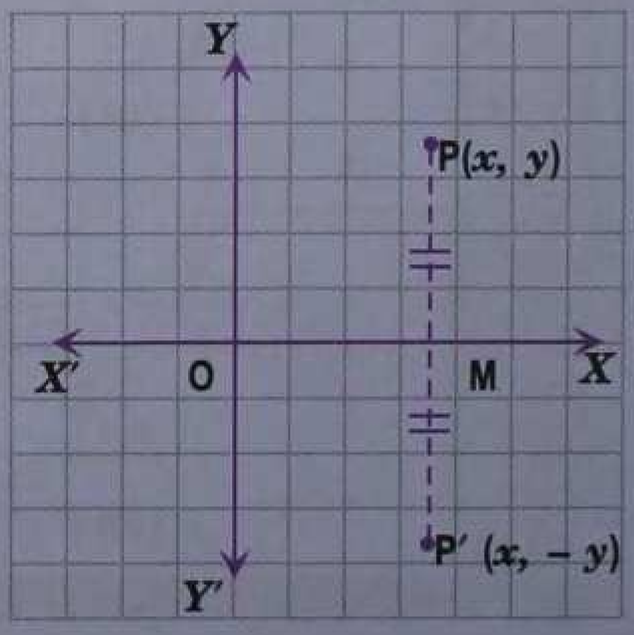
Using ruler and compass, from P draw PM perpendicular to the line AB and produce PM to a point P′ such that MP′ = MP. Then P′ is the required reflection (or image) of the point P in the line AB.



Reflection of a point in the x-axis

Let P (x, y) be any point in the coordinate plane. From P, draw PM perpendicular to the x-axis and produce it to a point P′ such that MP′ = MP. Then P′ is the *reflection* of the point P in the x-axis.

From figure, the coordinates of the point P′ are $(x, -y)$.



Rules to find the reflection of a point in the x-axis :

(*i*) *retain the x-coordinate.* (*ii*) *change the sign of the y-coordinate.*

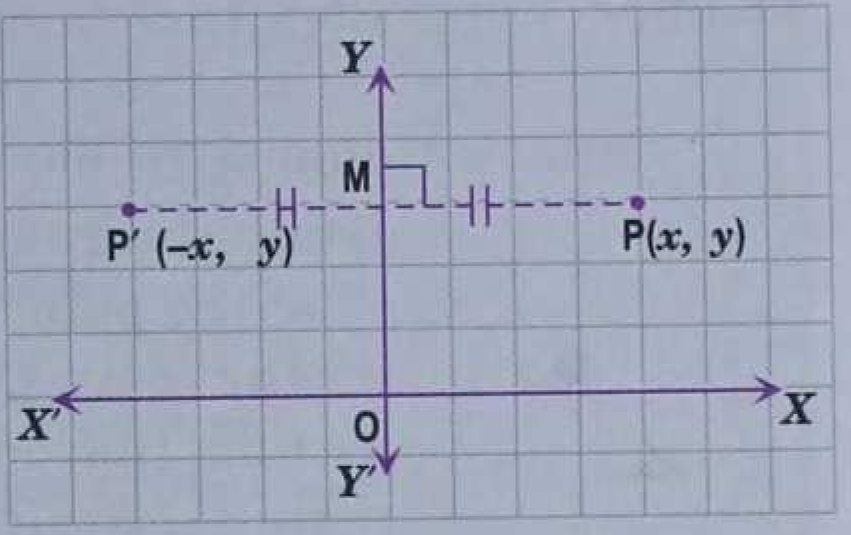
For example :

(*i*) The reflection of the point (2, 3) in the x-axis is the point (2, −3).

(*ii*) The reflection of the point (−4, −1) in the x-axis is the point (−4, 1).

(*iii*) The reflection of the point (5, 0) in the x-axis is the point itself, therefore, the point (5, 0) is invariant point with respect to x-axis.

Reflection of a point in the y-axis



Let P (x, y) be any point in the coordinate plane. From P, draw PM perpendicular to the y-axis and produce it to a point P′ such that MP′ = MP. Then P′ is the *reflection* of the point P in the y-axis.

From figure, the coordinates of the point P′ are ($-x$, y).

Rules to find the reflection of a point in the y-axis :

(*i*) *change the sign of the x-coordinate.* (*ii*) *retain the y-coordinate.*

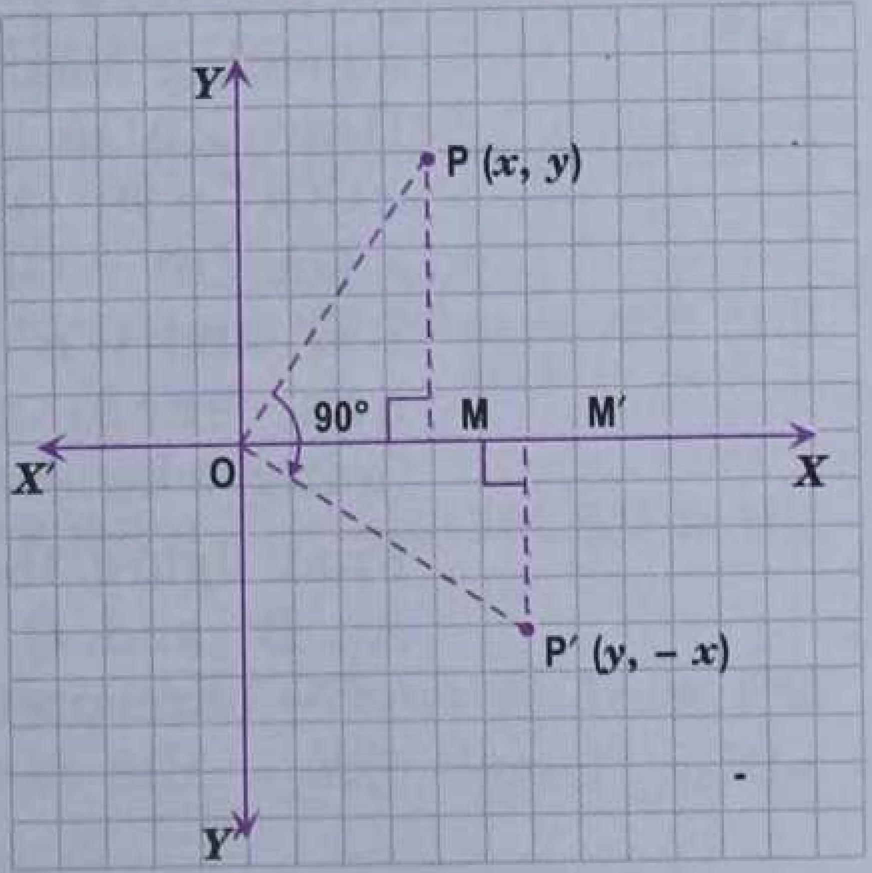
For example :

(*i*) The reflection of the point (2, 3) in the y-axis is the point (−2, 3).

(*ii*) The reflection of the point (−4, −1) in the y-axis is the point (4, −1).

(*iii*) The reflection of the point (0, −5) in the y-axis is the point itself, therefore, the point (0, −5) is invariant point with respect to y-axis.

Rotation of a point through 90° clockwise about the origin



Let P (x, y) be any point in the coordinate plane. Join OP and draw ∠POP′ = 90° clockwise such that OP′ = OP. Thus, the point P has been rotated through 90° clockwise about the origin to the point P′.

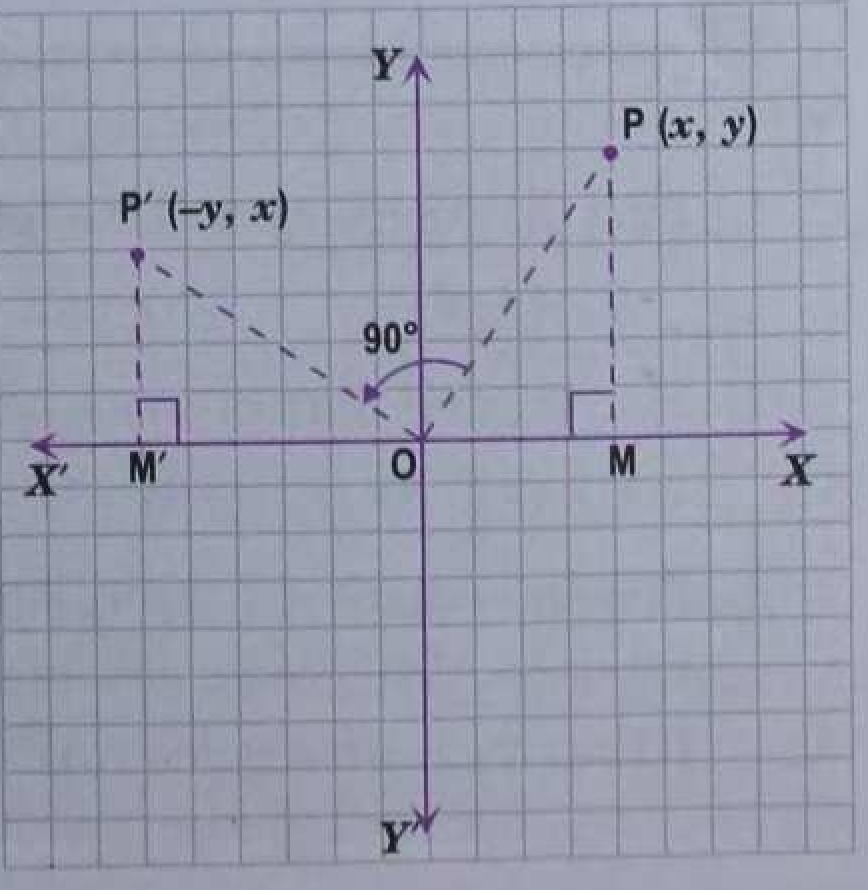
From P draw MP ⊥ OX and from P′ draw M′P′ ⊥ OX. It is easy to see that Δ OMP ≅ Δ OM′P′.

From figure, the coordinates of P′ are (y, $-x$).

Thus, the point P (x, y) changes to the point P′ (y, −x).

For example : If a point P (2, −3) is rotated through 90° clockwise about the origin to the P′, then the coordinates of P′ are (−3, −2).

Rotation of a point through 90° anticlockwise about the origin



Let P (x, y) be any point in the coordinate plane. Join OP and draw ∠POP′ = 90° anticlockwise such that OP′ = OP. Thus, the point P has been rotated through 90° anticlockwise about the origin to the point P′.

As before, draw MP ⊥ OX and M′P′ ⊥ X′O. It is easy to see that ΔOMP ≅ ΔOM′P′.

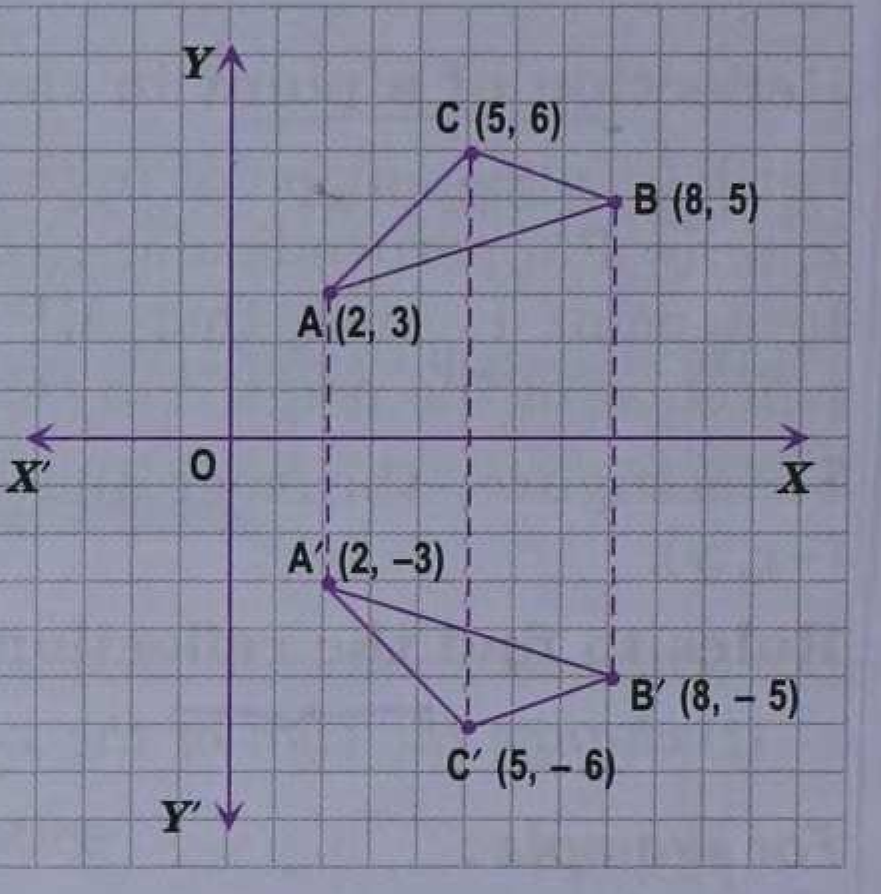
From figure, the coordinates of P′ are ($-y$, x).

Thus the point P (x, y) changes to the point P′ (–y, x).

For example, if a point P (2, –3) is rotated through 90° anticlockwise about the origin to the point P′, then the coordinates of P′ are (3, 2).

Example 1. The triangle ABC whose vertices are A (2, 3), B (8, 5) and C (5, 6) is reflected in the x-axis to the triangle A′B′C′. Draw the triangle A′B′C′ (on the graph paper). Also write down the coordinates of its vertices. Are the two triangles congruent?

Solution. Plot the points A (2, 3), B (8, 5) and C (5, 6) on a graph paper as shown in the adjoining diagram.

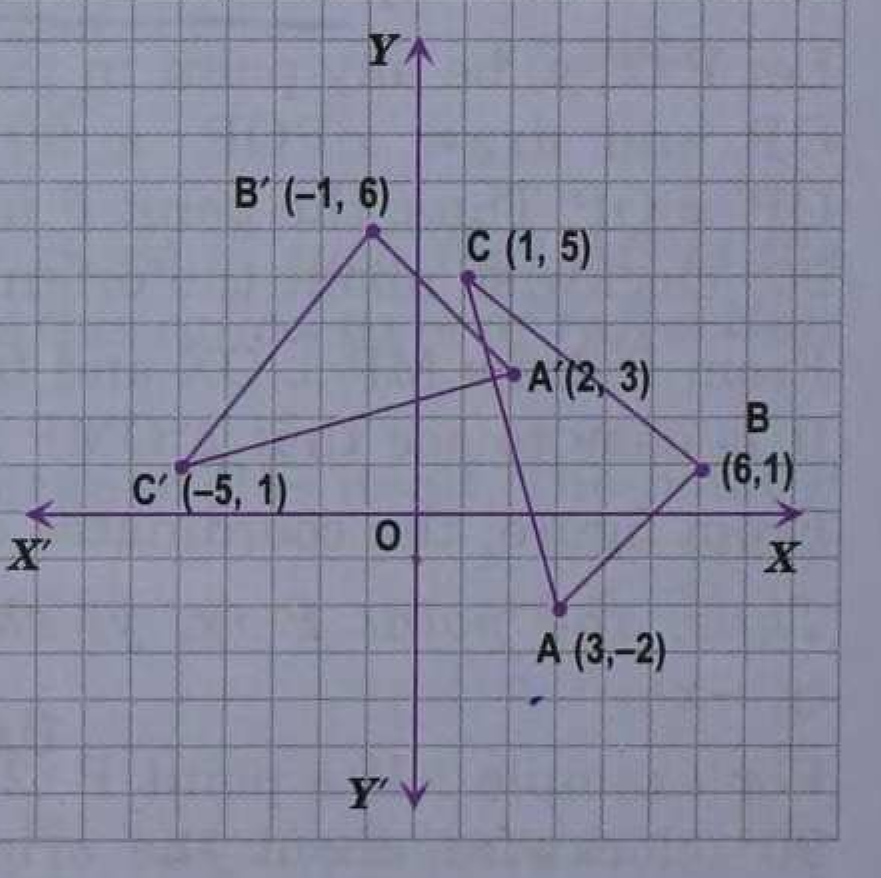


Draw the reflection (images) A′, B′ and C′ of the points A, B and C (respectively) in the x-axis. The triangle formed by joining the points A′, B′ and C′ is the required triangle A′B′C′. The coordinates of its vertices are A′ (2, –3), B′ (8, –5) and C′ (5, –6).

The two triangles ABC and A′B′C′ are congruent (measure the distances and check it).

Example 2. Plot the points A (3, –2), B (6, 1) and C (1, 5) on the graph paper. Rotate the triangle ABC through 90° anticlockwise about the origin to the position A′B′C′. Write down the coordinates of A′, B′ and C′.

Solution. Plot the point A (3, –2), B (6, 1) and C (1, 5) on a graph paper as shown in the adjoining diagram. Join AB, BC and CA. Draw ∠AOA′ = 90°, ∠BOB′ = 90° and ∠COC′ = 90° (anticlockwise) such that OA′ = OA, OB′ = OB and OC′ = OC. Join A′B′, B′C′ and C′A′. Thus, the triangle ABC has been rotated through 90° (anticlockwise) about the origin.



From figure, the coordinates of the points A′, B′ and C′ are (2, 3), (–1, 6) and (– 5, 1) respectively.

Exercise 27.2

1. Find the coordinates of the images of the following points under reflection in the x-axis :
   (*i*) (3, 5) (*ii*) (–3, 4) (*iii*) (–2, –6) (*iv*) (0, 3) (*v*) (–3, 0)
2. Find the coordinates of the images of the following points under reflection in the y-axis :
   (*i*) (–2, 5) (*ii*) (3, –4) (*iii*) (–2, –6) (*iv*) (–3, 0) (*v*) (0, –2)
3. Plot the points A (–3, 4) and B (2, 5) on the graph paper. Reflect the line segment AB in the x-axis to A′ B′. Write down the coordinates of A′ and B′. Are AB and A′B′ equal?
   [Hint. Measure AB and A′B′.**]**

4. Plot the points P (2, –5) and Q (3, 7) on the graph paper. Reflect the line segment PQ in the y-axis to P′Q′. Write down the coordinates of P′ and Q′. Are PQ and P′Q′ equal?

5. The triangle ABC whose vertices are A (2, –3), B (3, 4) and C (0, 5) is reflected in the y-axis to the triangle A′B′C′. Write down the coordinates of the vertices of triangle A′B′C′. Are the two triangles congruent?

6. The following points are rotated through 90° clockwise about the origin. Write the coordinates of their corresponding points :
   (*i*) (3, 4) (*ii*) (–3, 5) (*iii*) (5, –2) (*iv*) (0, 3) (*v*) (–3, 0)

7. The following points are rotated through 90° anticlockwise about the origin. Write down the coordinates of their corresponding points.
   (*i*) (3, 4) (*ii*) (–3, 5) (*iii*) (5, –2) (*iv*) (0, 3) (*v*) (–3, 0)

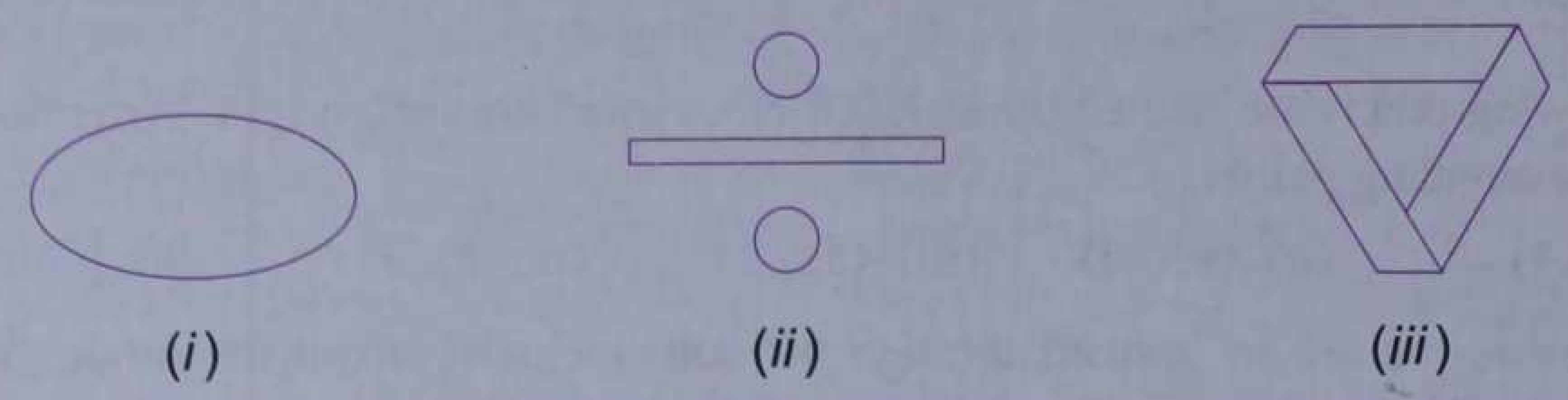
8. Plot the points A (3, –4) and B (–5, –2) on the graph paper. Rotate the line segment AB through 90° clockwise about the origin to the position A′B′. Write down the coordinates of A′ and B′. Are AB and A′B′ equal?

Summary

- A plane figure is symmetrical about a line if it is divided into two coincident parts by that line. The line is called its line (or axis) of symmetry.
- Point symmetry exists when a figure is built around a single point called the centre of the figure. For every point in the figure, there is another point found directly opposite it at the same distance on the other side of the centre.
- A plane figure has a rotational symmetry if on rotation through some angle (≤ 180°) about a point, it looks the same as it did in its starting position.
- If A° (≤ 180°) is the smallest angle through which a figure can be rotated and still look the same, then it has a rotational symmetry of order $\frac{360}{A}$.
- The reflection (or image) of a point P in a line AB is a point P′ such that AB is the perpendicular bisector of the line segment PP′.
- To find the reflection (or image) of a point P in a line AB — from P, draw PM perpendicular to AB and produce PM to P′ such that MP′ = MP, then P′ is the reflection (or image) of P in the line AB.
- The reflection of the point P (x, y) in the x-axis is the point P′ $(x, -y)$.
- The reflection of the point P (x, y) in the y-axis is the point P′ $(-x, y)$.
- If a point P (x, y) is rotated through 180° (clockwise or anticlockwise) about the origin to the point P′, then coordinates of P′ are $(-x, -y)$.
- If a point P (x, y) is rotated through 90° clockwise about the origin to the point P′, then coordinates of P′ are $(y, -x)$.
- If a point P (x, y) is rotated through 90° anticlockwise about the origin to the point P′, then coordinates of P′ are $(-y, x)$.

Check Your Progress

1. Draw the line (or lines) of symmetry, if any, of the following shapes and count their number:



2. For each of the given shape in question 1, find the order of the rotational symmetry (if any).
3. Plot the points A (2, –3), B (–1, 2) and C (0, –2) on the graph paper. Reflect the triangle ABC in the x-axis to the triangle A′B′C′. Write down the coordinates of the vertices of Δ A′B′C′. Are the two triangles congruent?
4. A (4, – 1), B (0, 7) and C (–2, 5) are the vertices of a triangle ABC. This triangle is reflected in the y-axis. Find the coordinates of the images of the vertices.
5. Plot the points P (–2, 3) and Q (4, –7) on the graph paper. Rotate the line segment PQ through 90° anticlockwise about the origin to the position P′Q′. Write down the coordinates of P′ and Q′.
6. Plot the points A (3, 5), B (–2, 4) and C (5, –6) on the graph paper. Rotate the triangle ABC through 90° clockwise about the origin to the position A′B′C′. Write down the coordinates of the vertices of Δ A′B′C′.
7. Plot the points A (3, 0), B (1, 3), C (–4, 2), D (–3, –2) and E (1, –4) on the graph paper. Rotate the pentagon ABCDE through 90° anticlockwise about the origin to take the position A′B′C′D′E′. Write down the coordinates of the vertices of the pentagon in new position.