Chapter 20

RELATIONS AND MAPPINGS

In Class VII, you learnt the basic ideas about relations and mappings. In this chapter, we shall strengthen these concepts and introduce an idea of equation form of functions.

RELATIONS

Let A and B be two (non-empty) sets, then a ***relation*** *R from A to B is a rule which associates elements of set A to elements of set B.*

If an element a of A is associated by R to an element b of B, we say that a is related to b and we write it as a R b.

For example :

(*i*) Let A = {2, 3, 5, 7}, B = {3, 6} and R be the relation 'is less than' from A to B, then we get

2 R 3, 2 R 6, 3 R 6, 5 R 6.

(*ii*) Let A = {–1, 0, 3, –4, –5, 5}, B = {0, –3, –12, –15, 15, 8, 10} and R be the relation 'is one-third of' from A to B, then we get

–1 R –3, 0 R 0, –4 R –12, –5 R –15, 5 R 15.

Representation of a relation

We already know two methods of representing a relation.

Roster form

In this form, a relation R from A to B is represented by the set of all ordered pairs (a, b) where $a \in$ A, $b \in$ B which satisfy the given relation R *i.e.*

$$R = \{(a, b) : a \in A, b \in B, a\ R\ b\}.$$

In the above example, relations R from A to B in the roster form can be written as

(*i*) R = {(2, 3), (2, 6), (3, 6), (5, 6)}

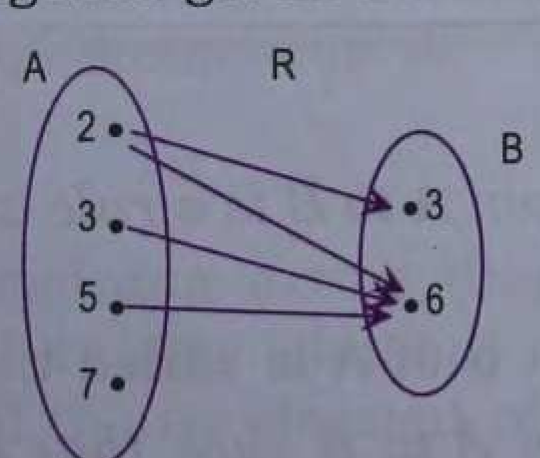
(*ii*) R = {(–1, –3), (0, 0), (–4, –12), (–5, –15), (5, 15)}.

Arrow diagram

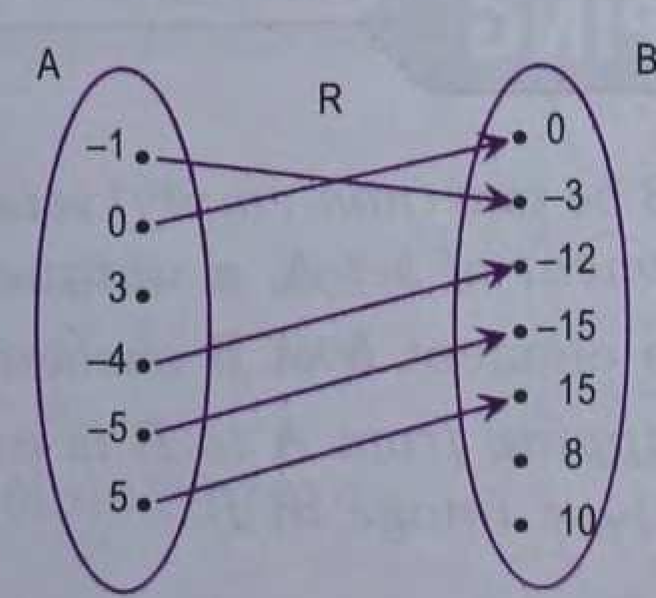
In this form, a relation R from A to B is represented by drawing arrows from first components to second components of all ordered pairs which satisfy the given relation R.

In the above example, relations R from A to B can be represented by arrow diagrams shown in the figures given below :

(*i*)



(*ii*)



Domain and range of a relation

Let A and B be two (non-empty) sets and R be a relation from A to B, then

domain of R = the set of first components of all ordered pairs which belong to R, and

range of R = the set of second components of all ordered pairs which belong to R.

In the above example,

(*i*) Domain of R = {2, 3, 5} and range of R = {3, 6}

(*ii*) Domain of R = {–1, 0, –4, –5, 5} and range of R = {–3, 0, –12, –15, 15}.

Example 1. Let A = {2, 3, 4, 5}, B = {4, 8, 10, 11, 15, 21} and R be the relation 'is a prime factor of' from A to B, then

(*i*) write R in the roster form.

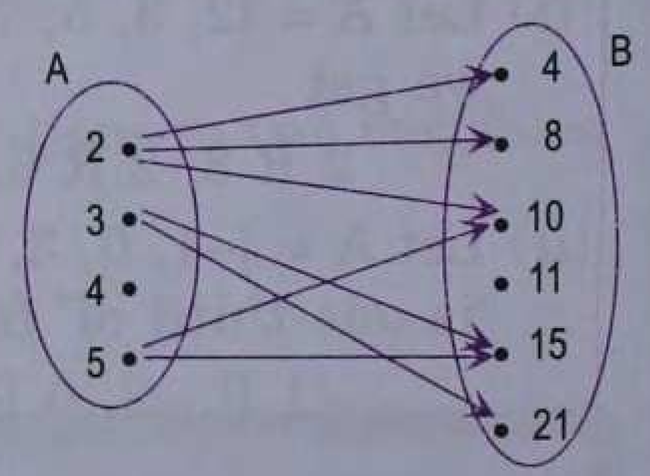
(*ii*) find domain and range of R.

(*iii*) represent R by an arrow diagram.

Solution. Given A = {2, 3, 4, 5}, B = {4, 8, 10, 11, 15, 21} and R is the relation 'is a prime factor of' from A to B.

(*i*) To write R in roster form, we form ordered pairs (a, b), $a \in A$, $b \in B$ such that a is a prime factor of b. We get

R = {(2, 4), (2, 8), (2, 10), (3, 15), (3, 21), (5, 10), (5, 15)}.



(*ii*) Domain of R = {2, 3, 5} and range of R = {4, 8, 10, 15, 21}.

(*iii*) The given relation R from A to B can be represented by an arrow diagram shown in the adjoining figure.

Example 2. Let A = {2, 3, 4, 5, 7, 9}, B = {0, 1, 2, 3, 4, 5, 6} and R = {$(a, b) : a \in A, b \in B$ and $a + b = 10$}, then

(*i*) find R in roster form. (*ii*) find domain and range of R.

Solution. Given A = {2, 3, 4, 5, 7, 9}, B = {0, 1, 2, 3, 4, 5, 6}

and R = {$(a, b) : a \in A, b \in B$ and $a + b = 10$}

(*i*) To find R in roster form, we note that

4 + 6 = 10, 5 + 5 = 10, 7 + 3 = 10, 9 + 1 = 10

∴ R = {(4, 6), (5, 5), (7, 3), (9, 1)}.

(*ii*) Domain of R = {4, 5, 7, 9} and range of R = {6, 5, 3, 1}.

MAPPING

Let A and B be two (non-empty) sets, then a ***mapping*** *from A to B is a rule which associates to every element of set A, a unique element of set B.*

The unique element b of B associated with an element a of A is called the **image** of a.

Thus, a mapping from A to B is a special relation from A to B such that each element of A has a unique image in B.

Conditions for a relation to be a mapping

(A) *When the relation is represented in roster form*

Let A and B be two (non-empty) sets and R be a relation from A to B, then R is a mapping if

(i) domain of R = A and

(ii) first components of different ordered pairs of R are not repeated.

Note that second components of different ordered pairs of R may be repeated, and some elements of B may not appear as a second component in ordered pairs of R.

For example :

(1) Let A = {0, –1, 2, 5}, B = $\left\{0, -1, \frac{1}{2}, \frac{1}{5}, 2, 5\right\}$ and R be the relation *'is reciprocal of'* from A to B, then

$R = \left\{(-1, -1), \left(2, \frac{1}{2}\right), \left(5, \frac{1}{5}\right)\right\}$.

Here, domain of R = {–1, 2, 5} ≠ A

Therefore, R is not a mapping from A to B.

(2) Let A = {1, –1, 2, 3, 4, 5}, B = {0, 1, 4, 9, 16, 25, 36} and R be the relation *'has as its square'* from A to B, then

R = {(1, 1), (–1, 1), (2, 4), (3, 9), (4, 16), (5, 25)}.

Here, domain of R = {1, –1, 2, 3, 4, 5} = A and the first components of different ordered pairs of R are not repeated.

Therefore, R is a mapping from A to B.

(3) Let A = {6, 8, 14, 20}, B = {3, 4, 5, 6, 7, 8, 9} and R be the relation *'is a multiple of'* from A to B, then

R = {(6, 3), (6, 6), (8, 4), (8, 8), (14, 7), (20, 4), (20, 5)}.

Here, domain of R = {6, 8, 14, 20} = A, but the first components of different ordered pairs (6, 3) and (6, 6) of R are repeated, therefore, R is not a mapping from A to B.

(B) *When the relation is represented by an arrow diagram*

Let A and B be two (non-empty) sets and R be a relation from A to B, then R is a mapping if

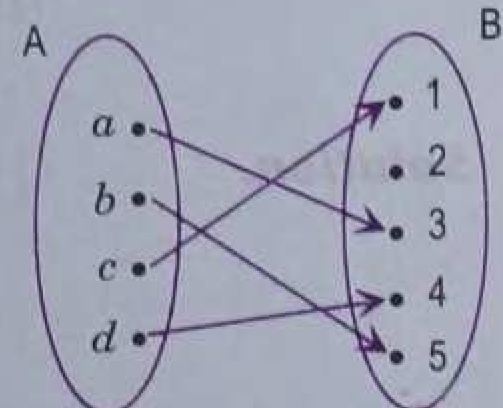
(i) each element of A is connected with some element of B i.e. each element of A must have an arrow emanating from it, and

(ii) no element of A should be connected with more than one element of B i.e. no element of A should have more than one arrow emanating from it.

Note that arrows from different elements of A may be pointing to the same element of B, and some elements of B may not have arrows pointing at them.

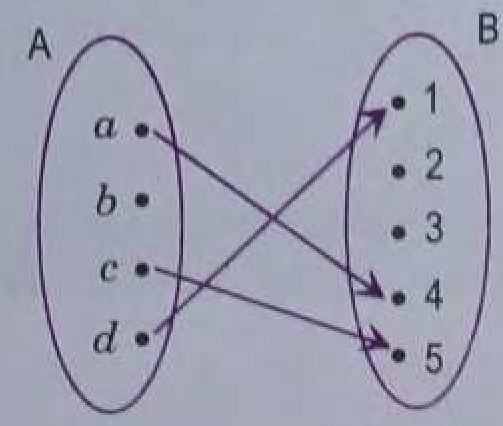
For example :

(1) The relation represented by the adjoining arrow diagram is a mapping from A to B because each element of A is connected to a unique element of B.

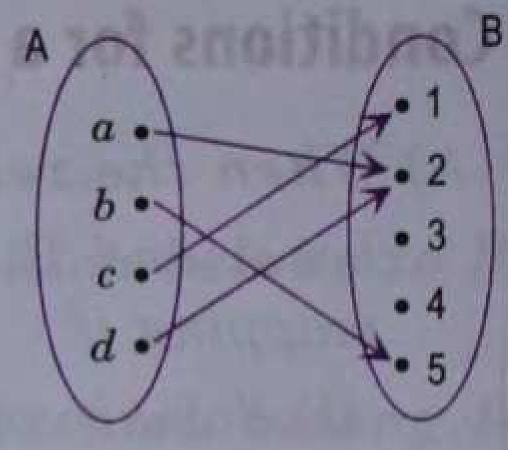


(2) In the adjoining arrow diagram, an element *b* of A is not connected to any element of B *i.e.* element *b* of A is not associated to any element of B.

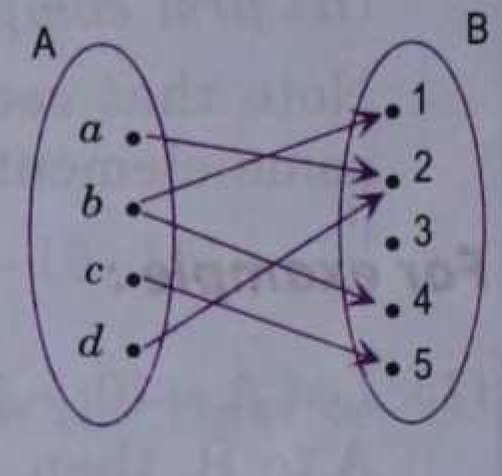
Therefore, the relation represented by the adjoining arrow diagram is not a mapping from A to B.



(3) The relation represented by the adjoining arrow diagram is a mapping from A to B because each element of A is connected to a unique element of B.
Note that the elements a and d of A are associated to the same element 2 of B.



(4) In the adjoining arrow diagram, an element b of A is connected to two elements 1 and 4 of B. So, each element of A is not associated to a unique element of B. Therefore, the relation represented by the adjoining arrow diagram is not a mapping from A to B.



Example 3. Let A = {1, 2, 3, 4}, B = {1, 3, 4, 5} and R be the relation 'is greater than or equal to' from A to B, then

(*i*) write R in roster form.

(*ii*) find domain and range of R.

(*iii*) represent R by an arrow diagram.

(*iv*) Is R a mapping from A to B? Give reason for your answer.

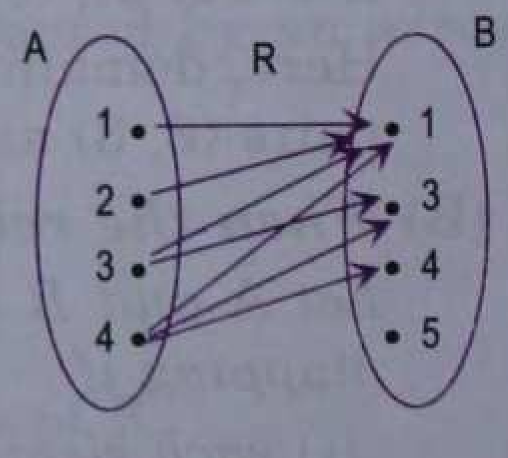
Solution. Given A = {1, 2, 3, 4}, B = {1, 3, 4, 5} and R is the relation 'is greater than or equal to' from A to B.

(*i*) To write R in roster form, we form ordered pairs (a, b), $a \in$ A, $b \in B$ such that $a \geq b$. We get

R = {(1, 1), (2, 1), (3, 1), (3, 3), (4, 1), (4, 3), (4, 4)}

(*ii*) Domain of R = {1, 2, 3, 4} and range of R = {1, 3, 4}

(*iii*) The given relation R from A to B can be represented by an arrow diagram shown in the adjoining figure.



(*iv*) The given relation R from A to B is not a mapping because element 3 of A is associated to two elements 1 and 3 of B.

Example 4. Let A = {4, 5, 6, 7, 8}, B = {0, 1, 2, 3} and R = {(a, b) : $a \in$ A, $b \in$ B and $a - b$ is divisible by 3}, then

(*i*) describe R in roster form.

(*ii*) find domain and range of R.

(*iii*) represent R by an arrow diagram.

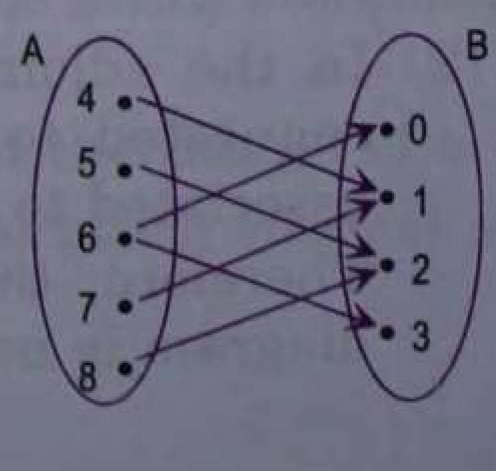
(*iv*) Is R a mapping from A to B? Give reason for your answer.

Solution. Given A = {4, 5, 6, 7, 8}, B = {0, 1, 2, 3} and R = {(a, b) : $a \in$ A, $b \in$ B and $a - b$ is divisible by 3}

(*i*) R = {(4, 1), (5, 2), (6, 0), (6, 3), (7, 1), (8, 2)}.

(*ii*) Domain of R = {4, 5, 6, 7, 8} and range of R = {0, 1, 2, 3}.

(*iii*) The given relation R from A to B can be represented by an arrow diagram shown in the adjoining figure.



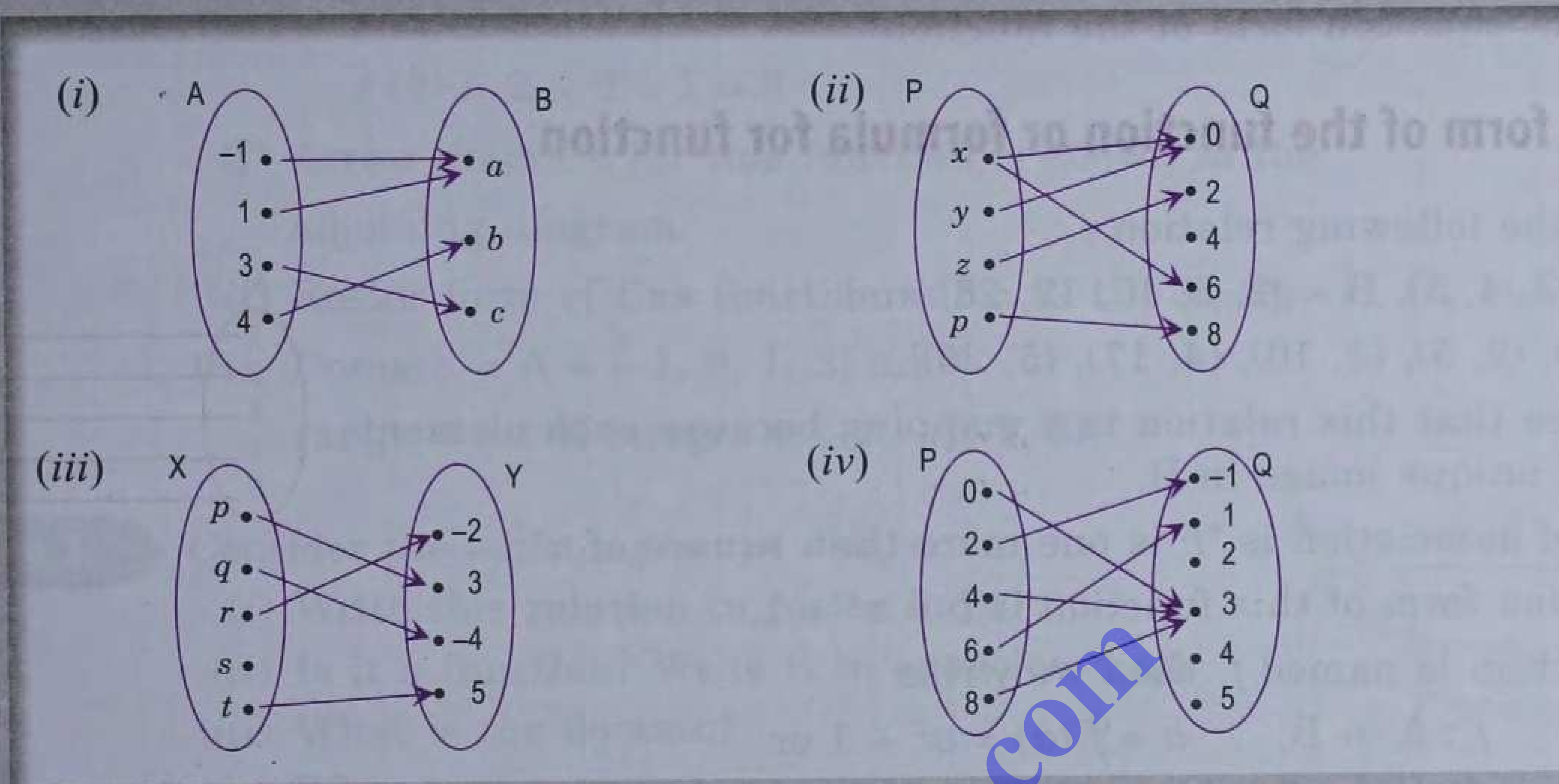
(*iv*) The given relation R from A to B is not a mapping because the element 6 of A is associated to two elements 0 and 3 of B.

Exercise 20.1

1. Let A = {–1, 3, 5, 7}, B = {0, 2, 4, 6, 7} and R = {(3, 0), (5, 6), (7, 6), (–1, 7)}.
State whether the relation R from A to B is a mapping or not. Justify your answer.

2. Let P = {2, 4, 6, 8, 10}, Q = {3, 5, 7, 9} and R = {(2, 3), (4, 5), (6, 7), (8, 9), (4, 7)}.
Is R a mapping from P to Q? Give reason.

3. Which of the arrow diagrams (given below) represent a mapping?



4. Let A = {–1, 2, 4, 5, 0}, B = {–2, 4, 3, 1, –1, 0} and R be the relation 'is one greater than' from A to B.
   (*i*) Write R in the roster form.
   (*ii*) Find domain and range of R.
   (*iii*) Is R a mapping from A to B? Give reason.

5. Let P = {4, 6, 8, 9}, Q = {2, 3, 5, 7, 9} and R be the relation 'is divisible by' from P to Q.
   (*i*) Write R as a set of ordered pairs.
   (*ii*) Find domain and range of R.
   (*iii*) Represent R by an arrow diagram.
   (*iv*) Is R a mapping from P to Q? Give reason.

6. Let A = {1, 3, 5, 6}, B = {3, 4, 5} and R = {(a, b) : $a \in$ A, $b \in$ B and $a + b$ is even}.
   (*i*) Describe R in roster form.
   (*ii*) Represent R by an arrow diagram.
   (*iii*) Is R a mapping from A to B? Give reason.

7. Let A = {6, 7, 8, 9, 10, 11}, B = {0, 1, 2, 3, 4, 5} and
R = {(a, b) : $a \in$ A, $b \in$ B and $a - b$ is divisible by 5}.
   (*i*) Describe R in roster form.
   (*ii*) Find domain and range of R.
   (*iii*) Represent R by an arrow diagram.
   (*iv*) Is R a mapping from A to B? Justify your answer.

8. Let P = {2, 3, 4}, Q = {0, 1, 2, 3, 4} and R be the relation 'is less than or equal to' from P to Q.
   (*i*) Write R in the roster form.
   (*ii*) Find domain and range of R.
   (*iii*) Represent R by an arrow diagram.
   (*iv*) Is R a mapping from P to Q? Give reason.

FUNCTIONS

A relation R from set A to set B is called a ***function****, if for each a in A, there is a unique b in B such that (a, b) ∈ R. The element b is called* ***image*** *of a and element a is called* ***pre-image*** *of b*. The set A is called the **domain** of function. Set of images is called **range** of function. Range may be same as set B or may be a proper subset of B.

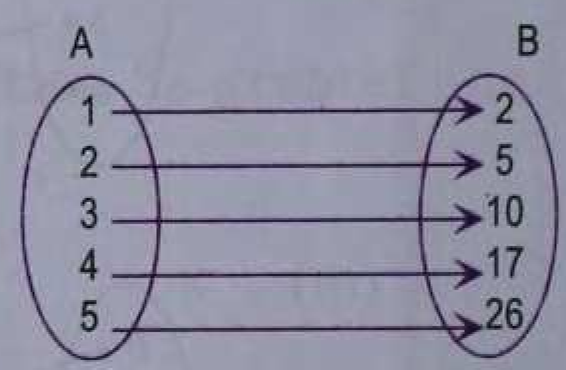
Thus, all mappings are examples of functions. You have already seen the roster form (set of ordered pairs) and the arrow diagram. There is a third way to represent the functions. It is called "equation form of the function".

Equation form of the function or formula for function

Consider the following relation :

A = {1, 2, 3, 4, 5}, B = {2, 5, 10, 17, 26} and

R = {(1, 2), (2, 5), (3, 10), (4, 17), (5, 26)}.



You can see that this relation is a mapping because each element of A has a unique image in B.

The rule of association is "b is one more than square of a".

The equation form of this function is $b = a^2 + 1$.

If the function is named f, then we write

$$f : A \to B, \quad b = f(a) = a^2 + 1 \text{ or}$$

$$f : a \to a^2 + 1 \quad \text{or} \quad f = \{(x, f(x)), x \in A\}.$$

Generally, symbols x and y are used in place of a and b.

We may write

$$y = f(x) = x^2 + 1 \text{ or } f : x \to x^2 + 1.$$

The **value** of function can be found as below :

$f(x) = x^2 + 1$ ...(*i*)

$f(1) = 1^2 + 1 = 2,$ (putting $x = 1$ in (*i*))

$f(2) = 2^2 + 1 = 5,$ (putting $x = 2$ in (*i*))

and so on.

Example 1. Given $f : x \to 2x + 3$ where A = {0, 1, 2, 3} and B is the set of images. List the elements of f and elements of B. Also draw the arrow diagram. What is the domain and range of this function?

Solution. Given $f(x) = 2x + 3$

Putting $x = 0, 1, 2, 3$, we get

$f(0) = 2 \times 0 + 3 = 0 + 3 = 3, \quad f(1) = 2 \times 1 + 3 = 5,$

$f(2) = 2 \times 2 + 3 = 7, \quad f(3) = 2 \times 3 + 3 = 9.$

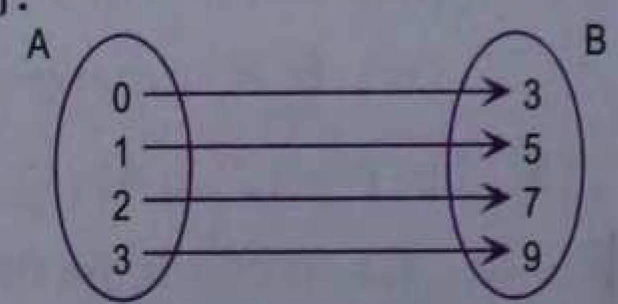
Since B is the set of images, we get B = {3, 5, 7, 9}.

Since $f = \{(x, f(x)), x \in A\}$, we get

$f = \{(0, 3), (1, 5), (2, 7), (3, 9)\}.$

Domain = A = {0, 1, 2, 3} and

range = set of images = {3, 5, 7, 9}.



Arrow diagram for this function is shown in the adjoining diagram.

Example 2. Let $f : x \to 2x - 1, x \in \{-1, 0, 1, 2\}$.

(*i*) Draw arrow diagram for this function.

(*ii*) Write roster form of this function.

(*iii*) What is the domain and range of this function?

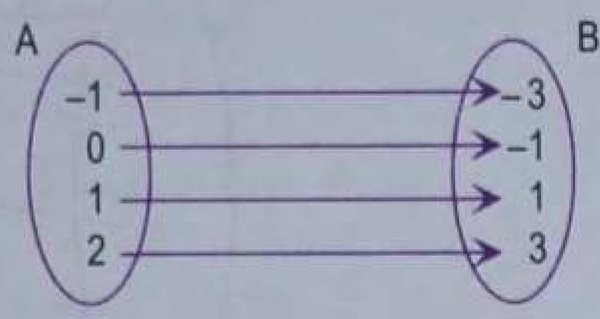
Solution. Since $f(x) = 2x - 1$, putting $x = -1, 0, 1, 2$, we get

$f(-1) = 2(-1) - 1 = -3,$

$f(0) = 2 \times 0 - 1 = -1,$

$f(1) = 2 \times 1 - 1 = 1,$

$f(2) = 2 \times 2 - 1 = 3$

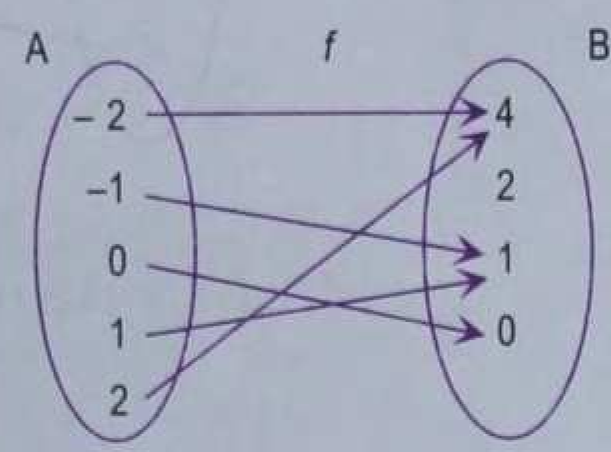


(*i*) Arrow diagram for this function is shown in the adjoining diagram.

(*ii*) Roster form of this function is $f = \{(-1, -3), (0, -1), (1, 1), (2, 3)\}$.

(*iii*) Domain = A = $\{-1, 0, 1, 2\}$ and range = set of images = $\{-3, -1, 1, 3\}$.

Example 3. Consider the adjoining arrow diagram :



(*i*) Write this relation in roster form.

(*ii*) Is it a function? Write it in equation form.

(*iii*) What is the domain?

(*iv*) What is the range?

Solution. (*i*) Relation f can be written in roster form as $f = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$.

(*ii*) Since each element of A has a unique image, f is a function. We see that each image of B is square of pre-image. Hence the formula for f is $f(x) = x^2$ or $f : x \to x^2$.

(*iii*) Domain of f = A = $\{-2, -1, 0, 1, 2\}$.

(*iv*) Range of f = set of images = $\{0, 1, 4\}$

Note that B = $\{0, 1, 2, 4\}$. Here range is a proper subset of B.

Example 4. Consider the following set of ordered pairs :

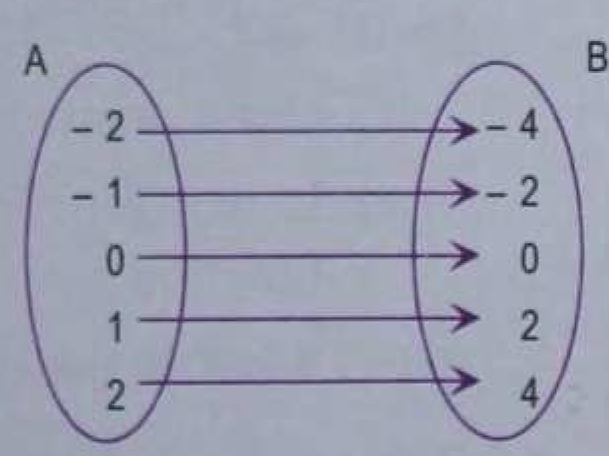
$g = \{(-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4)\}$

(*i*) Draw arrow diagram for this relation.

(*ii*) Is it a function? If yes, find the rule of association. Hence write the equation for this function.

(*iii*) What is the domain and range of this function?

Solution. (*i*) Arrow diagram is shown in the adjoining diagram.



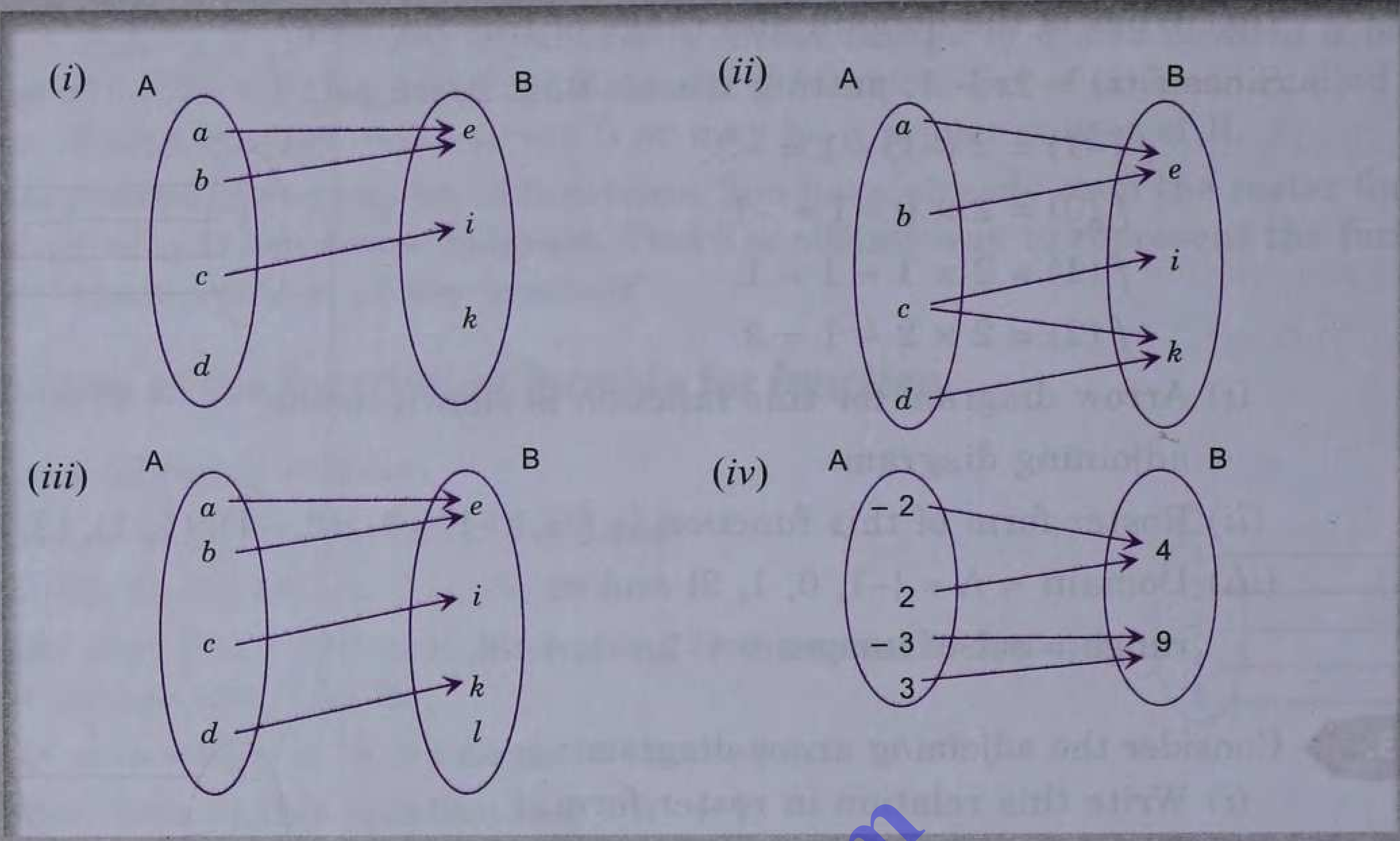
(*ii*) The arrow diagram clearly shows that each element of A has a unique image. So the given relation is a function. The rule of association is that "y is twice of x". Hence, the equation for this function is

$$g(x) = 2x.$$

(*iii*) Domain = set of pre-images = $\{-2, -1, 0, 1, 2\}$ and range = set of images = $\{-4, -2, 0, 2, 4\}$.

Exercise 20.2

1. Consider the following diagrams :



In each case, write the roster form of relation. Which of these are not functions and why not?

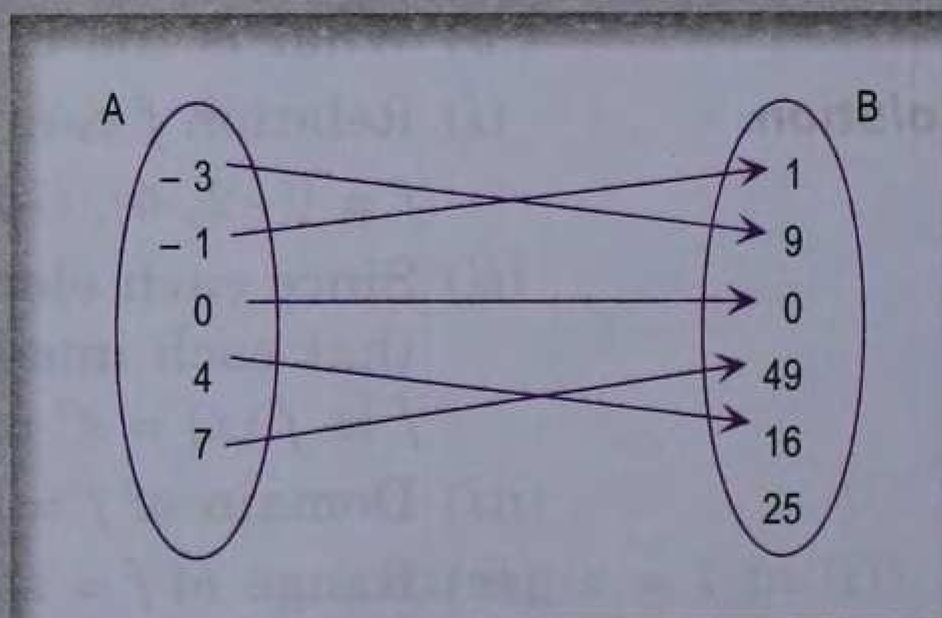
2. Consider the adjoining arrow diagram :

(*i*) Write this relation in roster form.

(*ii*) Is it a function? Write it in equation form.

(*iii*) What is the domain?

(*iv*) What is the range?



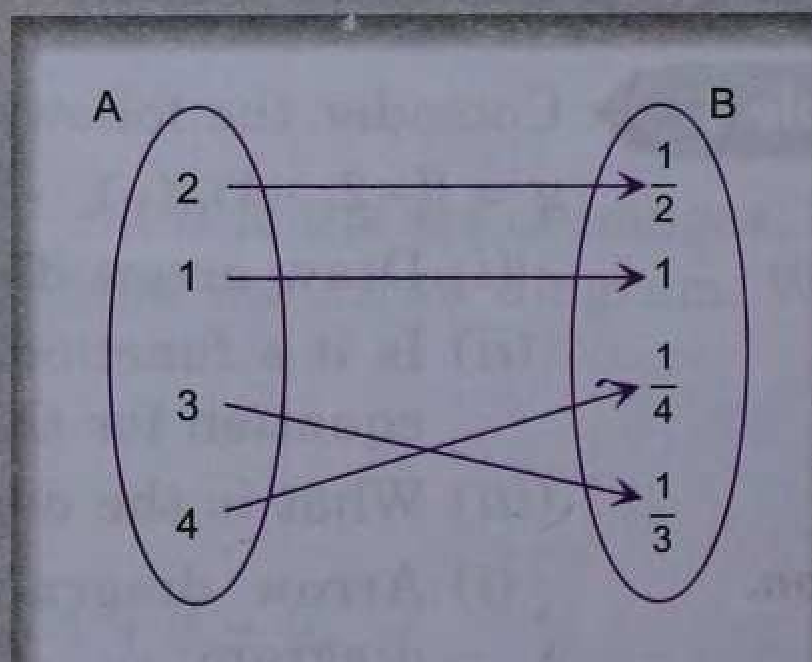
3. Consider the adjoining arrow diagram :

(*i*) Write this relation as a set of ordered pairs.

(*ii*) Is it a function? Write it in equation form.

(*iii*) What is the domain?

(*iv*) What is the range?



4. Consider the following set of ordered pairs :

$f = \{(-2, 1), (-1, 2), (0, 3), (5, 8)\}$

(*i*) Draw arrow diagram for this relation.

(*ii*) Is it a function? If yes, find the rule of association. Hence write the equation for this function.

(*iii*) What is the domain and range of this function?

5. Consider the relation $g = \{(1, 10), (2, 20), (3, 30), (4, 40), ...\}$.

(*i*) Is it a function? If yes, write the equation for this function.

(*ii*) What is the domain and range of this function?

6. Given $f : x \to 2x + 5$ where $x \in \{0, 2, 4, 6\}$.
   (*i*) List the elements of f. (*ii*) What is the range of f?

7. Let $f : x \to x^2 + 2$, where $x \in \mathbf{I}$, the set of integers.
   Write down an equation for $f(x)$ and hence evaluate
   (*i*) $f(-11)$ (*ii*) $f(-3)$ (*iii*) $f(0)$
   (*iv*) $f(3)$ (*v*) $f(11)$.

8. Given $f : x \to x + 3$ where $x \in \{-2, -1, 0, 1, 2\}$.
   (*i*) Write down an equation for $f(x)$. (*ii*) Evaluate $f(x)$ for all even values of x.
   (*iii*) If $f(x) = 2$, find x. (*iv*) If $f(x) = 4$, find x.

Summary

- Let A and B be two (non-empty) sets, then a relation from A to B is a rule which associates elements of set A to elements of set B.
- We learnt two methods of representing a relation from A to B — Roaster form and Arrow diagram.
- Let R be a relation from A to B, then
  domain of R = set of first components of all ordered pairs which belong to R and
  range of R = set of second components of all ordered pairs which belong to R.
- Let A and B be two (non-empty) sets, then a mapping from A to B is a rule which associates to every element of set A, a unique element of set B.
  The unique element b of B associated with an element a of A is called the image of a.
- A relation R from A to B is a mapping if each element of A has a unique image in B *i.e.* if
  (*i*) domain of R = A and
  (*ii*) first components of different ordered pairs of R are not repeated.
- Mappings are also called functions, usually denoted by letters f, g etc. The set of images is called range and the set of pre-images is called domain of a function.
- A function may be written in roster form (set of ordered pairs) or as arrow diagram or in equation form (formula).

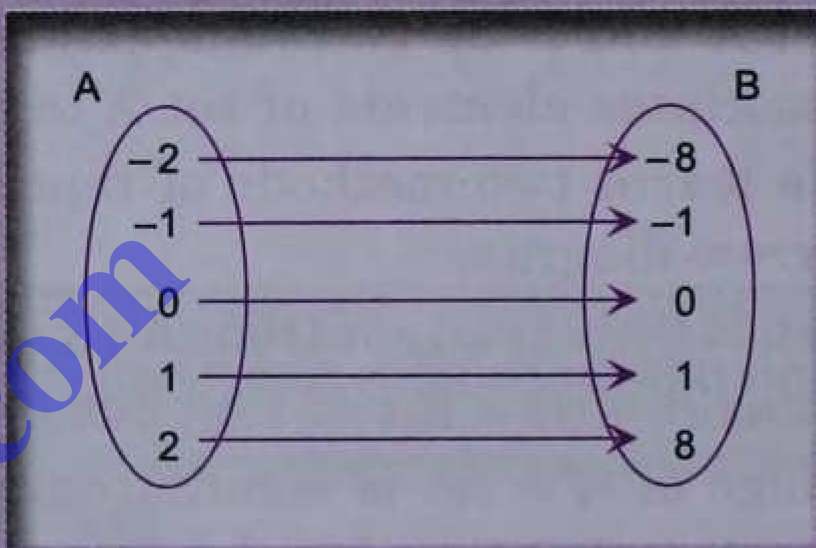
Check Your Progress

1. Write true or false :
   (*i*) An ordered pair is a pair of objects taken in any order.
   (*ii*) $(2, 3) = (3, 2)$.
   (*iii*) $\{2, 3\} = (2, 3)$.
   (*iv*) If $(x, 2) = (3, y)$ then $x = 3$, $y = 2$.
   (*v*) If $(x + 1, y + 1) = (5, 6)$ then $x = 4$, $y = 5$.
   (*vi*) The relation $\{(0, 0), (0, 1), (0, 2)\}$ is a mapping.

(*vii*) The relation {(0, 0), (1, 0), (2, 0)} is a mapping.
(*viii*) In a function each element has a unique image.
(*ix*) Domain is the set of pre-images.
(*x*) Range is the set of images.

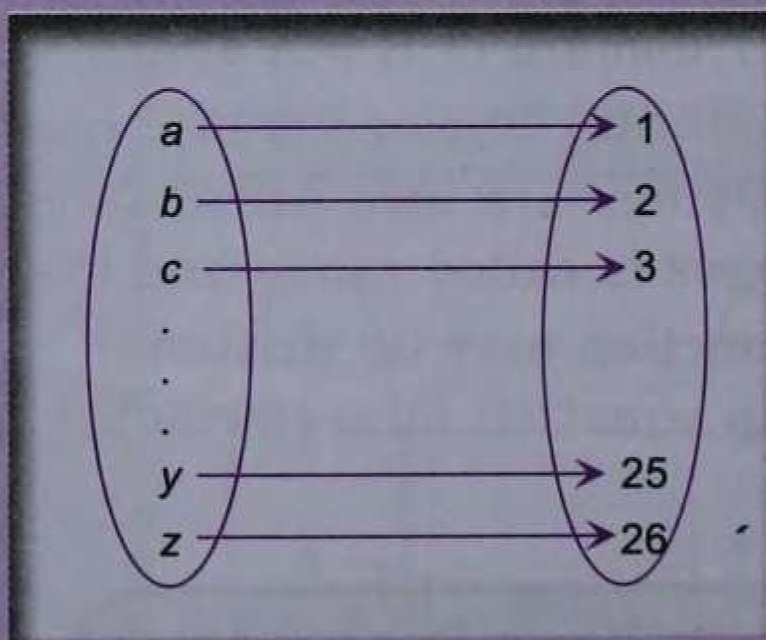
2. Let A = {3, 4, 5, 7}, B = {5, 6, 7, 9} and
R = {(a, b) : $a \in$ A, $b \in$ B and $a + b$ is odd}.
(*i*) Describe R in roster form. (*ii*) Find domain and range of R.
(*iii*) Represent R by an arrow diagram. (*iv*) Is R a mapping from A to B? Give reason.

3. Consider R = $\left\{(1, 2), \left(2, 2\frac{1}{2}\right), \left(3, 3\frac{1}{3}\right), ..., \left(10, 10\frac{1}{10}\right)\right\}$
(*i*) Is it a function? If yes, write it in equation form. Hence evaluate $f(5)$.
(*ii*) If $f(x) = \frac{37}{6}$, find x.
(*iii*) What is the domain?
(*iv*) What is the range?



4. Consider the adjoining arrow diagram :
(*i*) Is it a function? Write it in equation form.
(*ii*) List the elements of this function.

5. Given $f : x \rightarrow \frac{x}{3}$ where $x \in$ {multiples of 6}
(*i*) What is the range of this function?
(*ii*) Find $f(18)$.
(*iii*) If $f(x) = 18$, find x.



6. Consider the adjoining arrow diagram :
(*i*) Is it a mapping? Write in roster form.
(*ii*) Find the image of 'h'.
(*iii*) Find the pre-image of 24.