

QUADRATIC EQUATIONS

QUADRATIC EQUATION

An equation of the form $ax^2 + bx + c = 0$, where a , b and c are real numbers and $a \neq 0$, is called a **quadratic equation** in the variable x .

For example :

$x^2 - 3x + 2 = 0$ and $6x^2 + x - 1 = 0$ are quadratic equations in the variable x .

A number α is a **root** (or **solution**) of the quadratic equation $ax^2 + bx + c = 0$ if it satisfies the equation *i.e.* if $a\alpha^2 + b\alpha + c = 0$.

For example :

When we substitute $x = 2$ in the quadratic equation $x^2 - 3x + 2 = 0$, we get $2^2 - 3 \times 2 + 2 = 0$ *i.e.* $4 - 6 + 2 = 0$ *i.e.* $0 = 0$, which is true. Therefore, 2 is a root of the quadratic equation $x^2 - 3x + 2 = 0$.

When we substitute $x = 3$ in the quadratic equation $x^2 - 3x + 2 = 0$, we get $3^2 - 3 \times 3 + 2 = 0$ *i.e.* $9 - 9 + 2 = 0$ *i.e.* $2 = 0$, which is wrong.

Therefore, 3 is not a root of the quadratic equation $x^2 - 3x + 2 = 0$.

SOLVING QUADRATIC EQUATIONS

Factorisation can be used to solve quadratic equations.

The equation $x^2 - 3x + 2 = 0$ can be written as $(x - 1)(x - 2) = 0$.

This equation can be solved by using a property of real numbers called **zero-product rule**.

Zero-product rule

If a and b are two numbers or expressions and if $ab = 0$, then either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$

Using this rule, the solutions of the equation $(x - 1)(x - 2) = 0$ can be obtained by putting each factor equal to zero and then solving for x . Thus, we get

$$x - 1 = 0 \text{ or } x - 2 = 0$$

$$\Rightarrow x = 1 \text{ or } x = 2.$$

Hence, the solutions of the equation $x^2 - 3x + 2 = 0$ are 1 and 2.

Method to Solve a Quadratic Equation by Factorisation

Proceed as under :

- * Clear all fractions and write the equation in the form $ax^2 + bx + c = 0$.
- * Factorise the left hand side into product of two linear factors.
- * Put each linear factor equal to zero and solve the resulting linear equations.

Remark

The solutions (roots) may be checked by substituting in the original equation.

Example 1. Solve : $x^2 - 2x - 15 = 0$.

Solution.

$$\begin{aligned} \text{Given } x^2 - 2x - 15 &= 0 \\ \Rightarrow x^2 - 5x + 3x - 15 &= 0 \\ \Rightarrow x(x - 5) + 3(x - 5) &= 0 \\ \Rightarrow (x - 5)(x + 3) &= 0 \\ \Rightarrow x - 5 = 0 \text{ or } x + 3 &= 0 \\ \Rightarrow x = 5 \text{ or } x = -3. \end{aligned}$$

Hence, the roots of the given equation are 5, -3.

Example 2. Solve : $6x^2 + x - 1 = 0$.

Solution.

$$\begin{aligned} \text{Given } 6x^2 + x - 1 &= 0 \\ \Rightarrow 6x^2 + 3x - 2x - 1 &= 0 \\ \Rightarrow 3x(2x + 1) - 1(2x + 1) &= 0 \\ \Rightarrow (2x + 1)(3x - 1) &= 0 \\ \Rightarrow 2x + 1 = 0 \text{ or } 3x - 1 &= 0 \\ \Rightarrow x = -\frac{1}{2} \text{ or } x = \frac{1}{3}. \end{aligned}$$

Hence, the roots of the given equation are $-\frac{1}{2}, \frac{1}{3}$.

Example 3. Solve : $9x^2 + 6x + 1 = 0$.

Solution.

$$\begin{aligned} \text{Given } 9x^2 + 6x + 1 &= 0 \\ \Rightarrow 9x^2 + 3x + 3x + 1 &= 0 \\ \Rightarrow 3x(3x + 1) + 1(3x + 1) &= 0 \\ \Rightarrow (3x + 1)(3x + 1) &= 0 \\ \Rightarrow 3x + 1 = 0 \text{ or } 3x + 1 &= 0 \\ \Rightarrow x = -\frac{1}{3} \text{ or } x = -\frac{1}{3}. \end{aligned}$$

Hence, the roots of the given equation are $-\frac{1}{3}, -\frac{1}{3}$.

Note that the given quadratic equation has equal roots.

Example 4. Solve : $2x - \frac{1}{x} = 1$.

Solution.

To clear the fractions, multiply both sides of the given equation by x . We get

$$\begin{aligned} 2x^2 - 1 &= x \\ \Rightarrow 2x^2 - x - 1 &= 0 \\ \Rightarrow 2x^2 - 2x + x - 1 &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow 2x(x-1) + 1(x-1) &= 0 \\ \Rightarrow (2x+1)(x-1) &= 0 \\ \Rightarrow 2x+1 = 0 \text{ or } x-1 &= 0 \\ \Rightarrow x = -\frac{1}{2} \text{ or } x &= 1. \end{aligned}$$

Hence, the roots of the given equation are $-\frac{1}{2}, 1$.

Example 5.

Solve : $\frac{x+3}{x-1} = \frac{2x+1}{3x-5}$.

Solution.

Given $\frac{x+3}{x-1} = \frac{2x+1}{3x-5}$

$$\begin{aligned} \Rightarrow (x+3)(3x-5) &= (x-1)(2x+1) && \text{(By cross-multiplication)} \\ \Rightarrow 3x^2 - 5x + 9x - 15 &= 2x^2 + x - 2x - 1 \\ \Rightarrow 3x^2 + 4x - 15 &= 2x^2 - x - 1 \\ \Rightarrow 3x^2 - 2x^2 + 4x + x - 15 + 1 &= 0 \\ \Rightarrow x^2 + 5x - 14 &= 0 \\ \Rightarrow x^2 + 7x - 2x - 14 &= 0 \\ \Rightarrow x(x+7) - 2(x+7) &= 0 \\ \Rightarrow (x+7)(x-2) &= 0 \\ \Rightarrow x+7 = 0 \text{ or } x-2 &= 0 \\ \Rightarrow x = -7 \text{ or } x &= 2. \end{aligned}$$

Hence, the roots of the given equation are $-7, 2$.

Example 6.

Solve $\frac{x}{x-1} + \frac{x-1}{x} = 2\frac{1}{2}$.

Solution.

Given $\frac{x}{x-1} + \frac{x-1}{x} = \frac{5}{2}$

To clear the fractions, multiply both sides of the given equation by L.C.M. of denominators *i.e.* by $2x(x-1)$. We get

$$\begin{aligned} 2x \times x + 2(x-1)(x-1) &= 5x(x-1) \\ \Rightarrow 2x^2 + 2x^2 - 4x + 2 &= 5x^2 - 5x \\ \Rightarrow 4x^2 - 4x + 2 &= 5x^2 - 5x \\ \Rightarrow 4x^2 - 5x^2 - 4x + 5x + 2 &= 0 \\ \Rightarrow -x^2 + x + 2 &= 0 \\ \Rightarrow x^2 - x - 2 &= 0 \\ \Rightarrow x^2 - 2x + x - 2 &= 0 \\ \Rightarrow x(x-2) + 1(x-2) &= 0 \\ \Rightarrow (x+1)(x-2) &= 0 \\ \Rightarrow x+1 = 0 \text{ or } x-2 &= 0 \\ \Rightarrow x = -1 \text{ or } x &= 2. \end{aligned}$$

Hence, the roots of the given equation are $-1, 2$.

Exercise 18

Solve the following (1 to 12) equations :

- | | |
|---|---|
| 1. (i) $x^2 - 11x + 30 = 0$ | (ii) $4x^2 - 25 = 0$ |
| 2. (i) $2x^2 - 5x = 0$ | (ii) $x^2 - 2x = 48$ |
| 3. (i) $6 + x = x^2$ | (ii) $2x^2 - 3x + 1 = 0$ |
| 4. (i) $3x^2 = 2x + 8$ | (ii) $4x^2 + 15 = 16x$ |
| 5. (i) $x(2x + 5) = 25$ | (ii) $(x + 3)(x - 3) = 40$ |
| 6. (i) $(2x + 3)(x - 4) = 6$ | (ii) $(3x + 1)(2x + 3) = 3$ |
| 7. (i) $4x^2 + 4x + 1 = 0$ | (ii) $(x - 4)^2 + 5^2 = 13^2$ |
| 8. (i) $21x^2 = 4(2x + 1)$ | (ii) $\frac{2}{3}x^2 - \frac{1}{3}x - 1 = 0$ |
| 9. (i) $6x + 29 = \frac{5}{x}$ | (ii) $x + \frac{1}{x} = 2\frac{1}{2}$ |
| 10. (i) $3x - \frac{8}{x} = 2$ | (ii) $\frac{x}{3} + \frac{9}{x} = 4$ |
| 11. (i) $\frac{x-1}{x+1} = \frac{2x-5}{3x-7}$ | (ii) $\frac{1}{x+2} + \frac{1}{x} = \frac{3}{4}$ |
| 12. (i) $\frac{8}{x+3} - \frac{3}{2-x} = 2$ | (ii) $\frac{x}{x+1} + \frac{x+1}{x} = 2\frac{1}{6}$ |

Summary

- An equation of the form $ax^2 + bx + c = 0$, where a , b and c are real numbers and $a \neq 0$, is called a quadratic equation in the variable x .
- A number α is a root (or solution) of the equation $ax^2 + bx + c = 0$ if it satisfies it i.e. if $a\alpha^2 + b\alpha + c = 0$.
- A quadratic equation can be solved by factorisation and then using zero-product rule.
- To solve a quadratic equation, proceed as under :
 - (i) Clear all fractions and write the equation in the form $ax^2 + bx + c = 0$.
 - (ii) Factorise left hand side into product of two linear factors.
 - (iii) Put each linear factor equal to zero and solve the resulting linear equations.

Check Your Progress

Solve the following (1 to 3) equations :

- | | |
|---|--|
| 1. (i) $x(2x + 5) = 3$ | (ii) $3x^2 - 4x - 4 = 0$ |
| 2. (i) $4x^2 - 2x + \frac{1}{4} = 0$ | (ii) $2x^2 + 7x + 6 = 0$ |
| 3. (i) $\frac{x-1}{x-2} + \frac{x-3}{x-4} = 3\frac{1}{3}$ | (ii) $\frac{6}{x} - \frac{2}{x-1} = \frac{1}{x-2}$ |