15 Locus

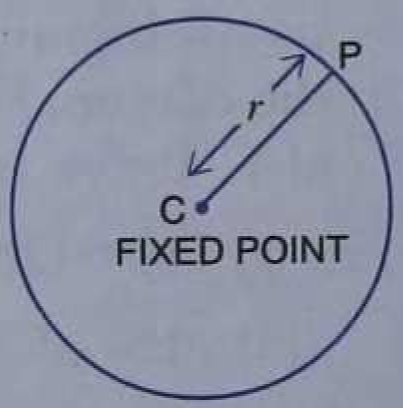
15.1 LOCUS

*The **locus** of a point is the path traced out by the point moving under given geometrical condition (or conditions).*

*Alternatively, the **locus** is the set of all those points which satisfy the given geometrical condition (or conditions).*

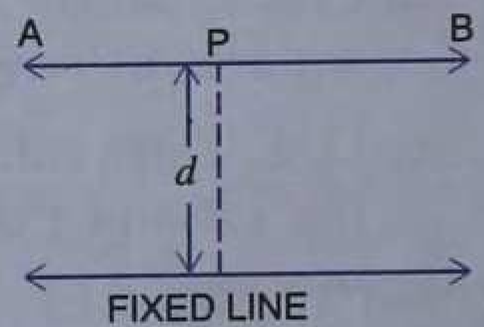
For example :

1. Let a point P move in a plane such that its distance from a fixed point (in the plane), say C, is always equal to r. The point P will trace out a circle with centre C (the fixed point) and radius 'r'.



   Thus, the locus of a point (in a plane) equidistant from a fixed point (in the plane) is a circle with the fixed point as centre.

2. Let a point P move such that its distance from a fixed line (on one side of the line) is always equal to d. The point P will trace out a straight line AB parallel to the fixed line.



   Thus, the locus of a point P is the straight line AB (shown in the adjoining diagram).

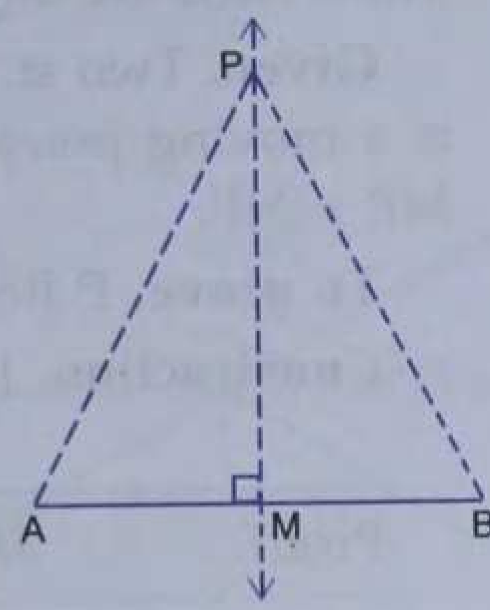
Remarks

- Every point which satisfies the given geometrical condition (or conditions) lies on the locus.
- A point which does not satisfy the given geometrical condition (or conditions) cannot lie on the locus.
- Every point which lies on the locus satisfies the given geometrical condition (or conditions).
- A point which does not lie on the locus cannot satisfy the given geometrical condition (or conditions).
- *The locus of a point moving in a plane under a given geometrical condition (or conditions) is always a straight or curved line (or lines).*

- ❒ *To find the locus of a moving point, plot some points satisfying the given geometrical condition (or conditions), and then join these points.*
- ❒ ***In a theorem on locus, we have to prove the theorem and its converse.***
- ❒ The plural of locus is 'loci' and is read as 'losai'.

15.2 THEOREMS ON LOCUS

Theorem 24. *The locus of a point, which is equidistant from two fixed points, is the perpendicular bisector of the line segment joining the two fixed points.*



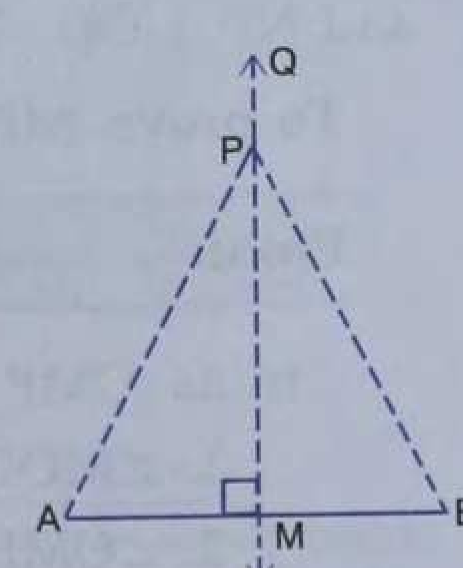
Given. Two fixed points A and B, P is a moving point such that AP = BP.

To prove. P lies on the perpendicular bisector of the line segment AB.

Construction. Join AB, let M be mid-point of AB and join MP.

Proof. *Statements*	*Reasons*
In Δs PAM and PBM	
1. AP = BP	1. Given.
2. AM = MB	2. M is mid-point of AB (construction).
3. MP = MP	3. Common.
4. Δ PAM ≅ Δ PBM	4. S.S.S. axiom of congruency.
5. ∠AMP = ∠PMB	5. 'c.p.c.t.'.
6. ∠AMP + ∠PMB = 180°	6. AMB is a st. line.
7. ∠AMP = 90°	7. From 5 and 6.
∴ MP is the perpendicular bisector of AB.	
Hence, P lies on the perpendicular bisector of AB.	

Conversely, *any point on the perpendicular bisector of a line segment joining two fixed points is equidistant from the fixed points.*



Given. Two fixed points A and B, MQ is perpendicular bisector of AB and P is any point on MQ.

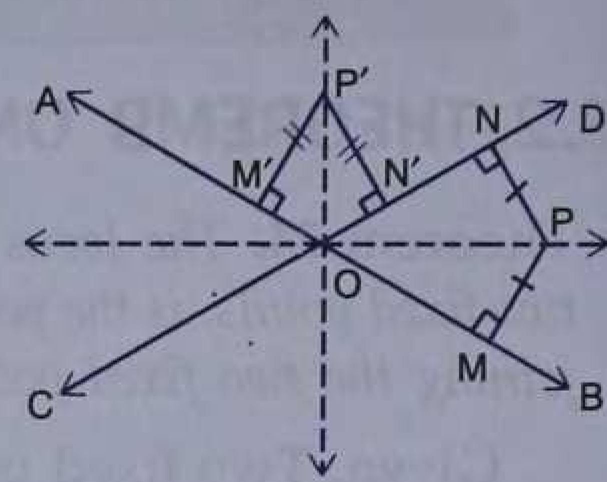
To prove. AP = BP.

Construction. Join AP and BP.

Proof. *Statements*	*Reasons*
In Δs PAM and PBM	
1. AM = MB	1. M is mid-point of AB (given).
2. ∠AMP = ∠PMB	2. MP ⊥ AB (given) ⇒ ∠AMP = 90° = ∠PMB.
3. MP = MP	3. Common.
4. Δ PAM ≅ Δ PBM	4. R.H.S. axiom of congruency.
5. AP = BP	5. 'c.p.c.t.'.

Conclusion : *From the above theorem and its converse, it follows that the locus of a point, which is equidistant from two fixed points, is the perpendicular bisector of the line segment joining the two fixed points.*

Theorem 25. *The locus of a point, which is equidistant from two intersecting straight lines, consists of a pair of straight lines which bisect the angles between the two given lines.*



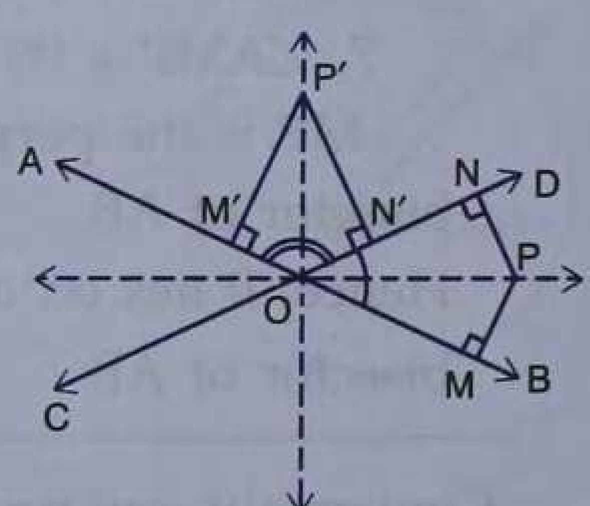
Given. Two st. lines AB and CD intersecting at O. P is a moving point such that MP ⊥ OB, NP ⊥ OD and MP = NP.

To prove. P lies on the bisector of ∠BOD.

Construction. Join OP.

Proof. *Statements*	*Reasons*
In Δs OMP and ONP	
1. MP = NP	1. Given.
2. ∠OMP = ∠ONP	2. MP ⊥ OB, NP ⊥ OD (given) ⇒ ∠OMP = 90°, ∠ONP = 90°.
3. OP = OP	3. Common.
4. Δ OMP ≅ Δ ONP	4. R.H.S. axiom of congruency.
5. ∠BOP = ∠POD	5. 'c.p.c.t.'.
∴ P lies on the bisector of ∠BOD.	

Similarly, if P′ is a moving point such that M′P′ ⊥ AO, N′P′ ⊥ OD and M′P′ = N′P′, then P′ lies on the bisector of ∠AOD.



Conversely, *any point on the bisector of an angle is equidistant from the arms of the angle.*

Given. Two st. lines AB and CD intersecting at O. P is a point on the bisector of ∠BOD, MP ⊥ OB and NP ⊥ OD.

To prove. MP = NP.

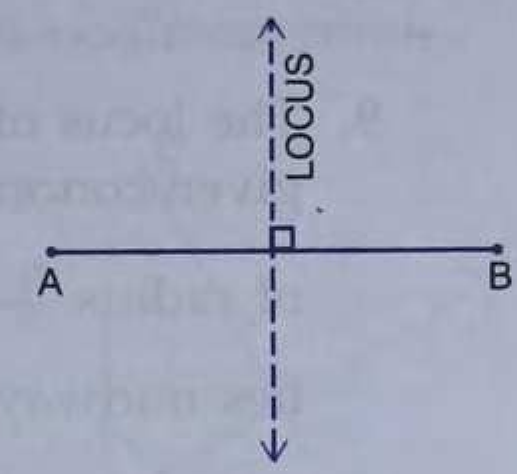
Proof. *Statements*	*Reasons*
In Δs OMP and ONP	
1. ∠MOP = ∠NOP	1. P lies on bisector of ∠BOD (given).
2. ∠OMP = ∠ONP	2. MP ⊥ OB, NP ⊥ OD (given) ⇒ ∠OMP = 90°, ∠ONP = 90°.
3. OP = OP	3. Common.
4. Δ OMP ≅ Δ ONP	4. A.A.S. axiom of congruency.
5. MP = NP	5. 'c.p.c.t.'.

Similarly, if P′ is a point on the bisector of ∠AOD and M′P′ ⊥ AO, N′P′ ⊥ OD, then M′P′ = N′P′.

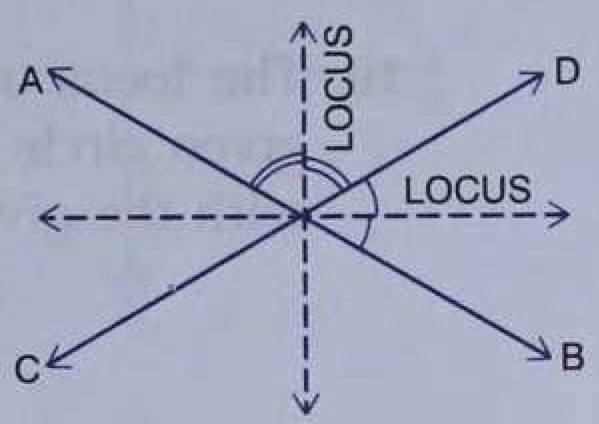
Conclusion : *From the above theorem and its converse, it follows that the locus of a point, which is equidistant from two intersecting straight lines, consists of a pair of straight lines which bisect the angles between the two given lines.*

15.3 LOCUS IN SOME STANDARD CASES

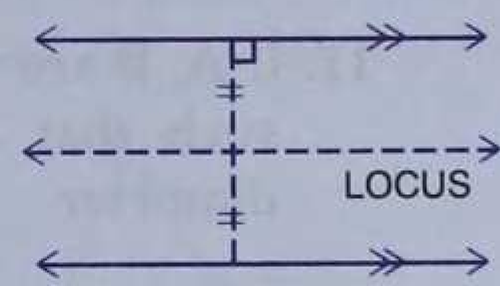
1. The locus of a point, which is equidistant from two fixed points, is the perpendicular bisector of the line segment joining the two fixed points.



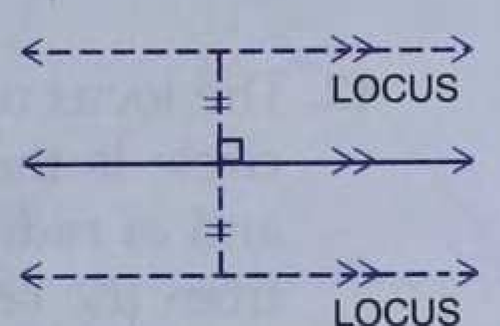
2. The locus of a point, which is equidistant from two intersecting straight lines, consists of a pair of straight lines which bisect the angles between the two given lines.



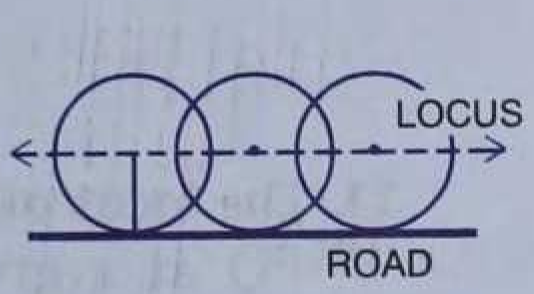
3. The locus of a point, which is equidistant from two parallel straight lines, is a straight line parallel to the given lines and midway between them.



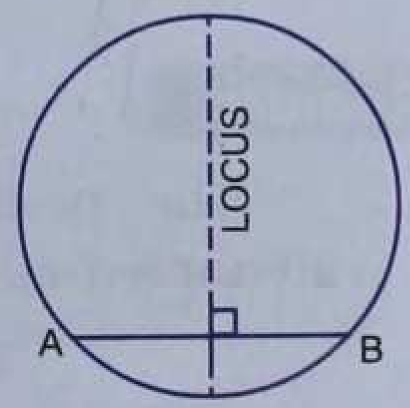
4. The locus of a point, which is at a given distance from a given straight line, consists of a pair of straight lines parallel to the given line and at a given distance from it.



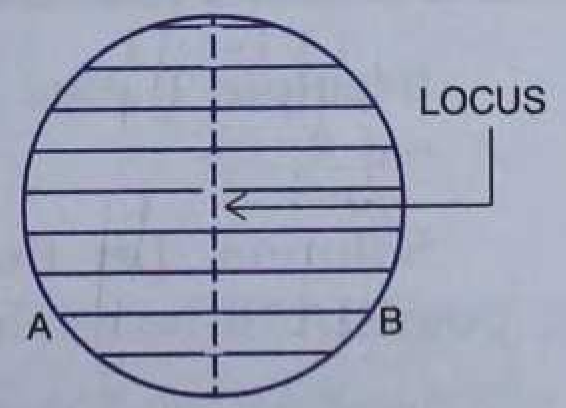
5. The locus of the centre of a wheel, which moves on a straight horizontal road, is a straight line parallel to the road and at a distance equal to the radius of the wheel.



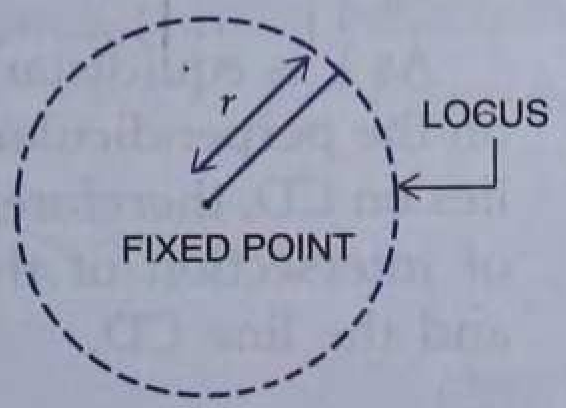
6. The locus of a point, which is inside a circle and is equidistant from two points on the circle, is the diameter of the circle which is perpendicular to the chord of the circle joining the given points.



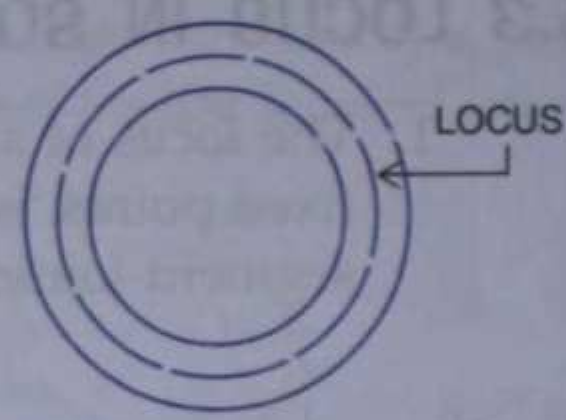
7. The locus of the mid-points of all parallel chords of a circle is the diameter of the circle which is perpendicular to the given parallel chords.



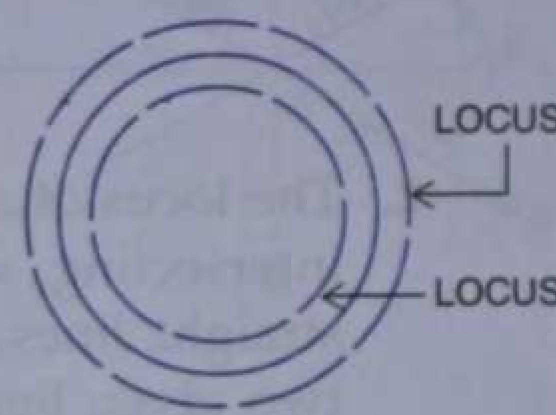
8. The locus of a point (in a plane), which is at a given distance r from a fixed point (in the plane), is a circle with the fixed point as its centre and radius r.



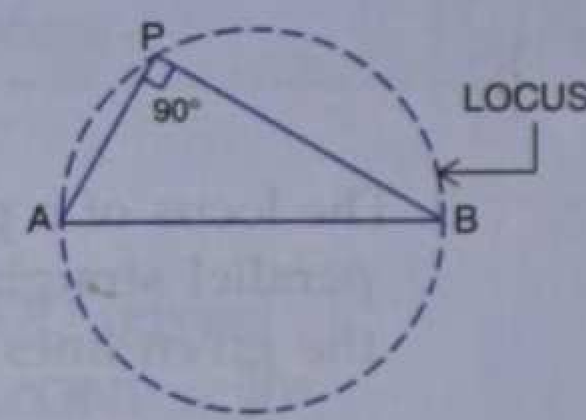
9. The locus of a point which is equidistant from two given concentric circles of radii r_1 and r_2 is the circle of radius $\frac{r_1 + r_2}{2}$ concentric with the given circles. It lies midway between them.



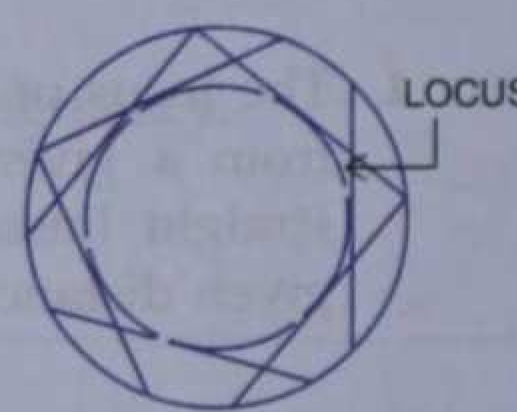
10. The locus of a point which is equidistant from a given circle consists of a pair of circles concentric with the given circle.



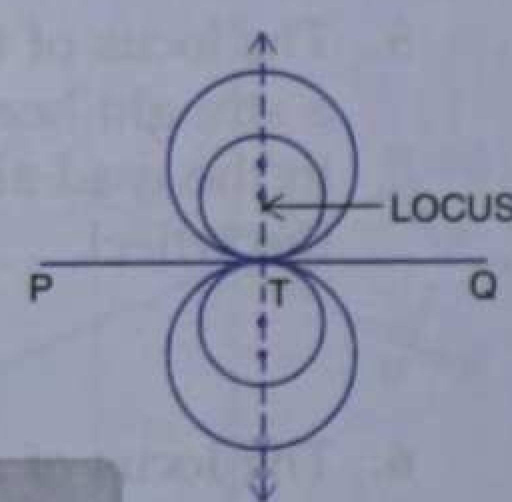
11. If A, B are fixed points, then the locus of a point P such that $\angle APB = 90°$ is the circle with AB as diameter.



12. The locus of the mid-points of all equal chords of a circle is the circle concentric with the given circle and of radius equal to the distance of equal chords from the centre of the given circle.



13. The locus of centres of circles touching a given line PQ at a given point T on it is the straight line perpendicular to PQ at T.



Remark

The problems on locus, concerning circle, should be attempted after learning chapter 16 on circles.

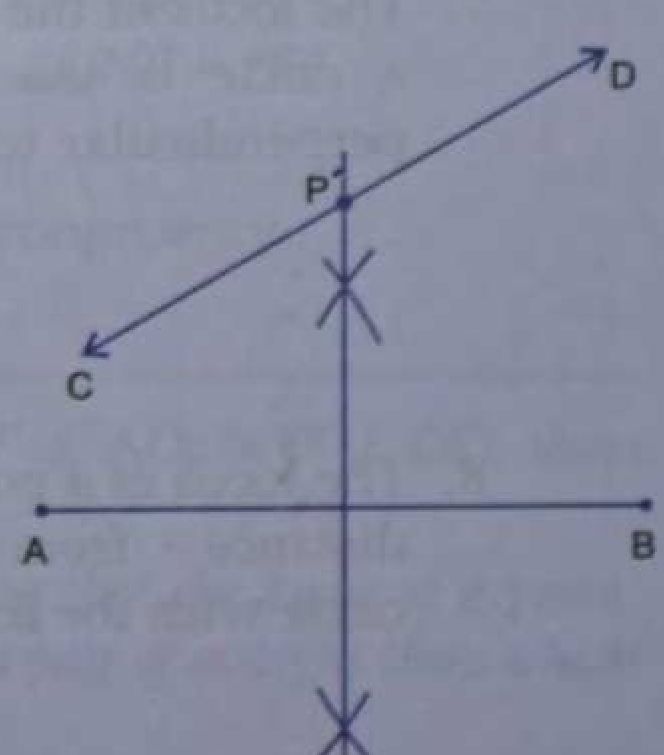
ILLUSTRATIVE EXAMPLES

Example 1. *Find a point P in a given line CD which is equidistant from two fixed points A and B.*

Solution. Let the given line CD and the fixed points A, B be as shown in the figure along side.

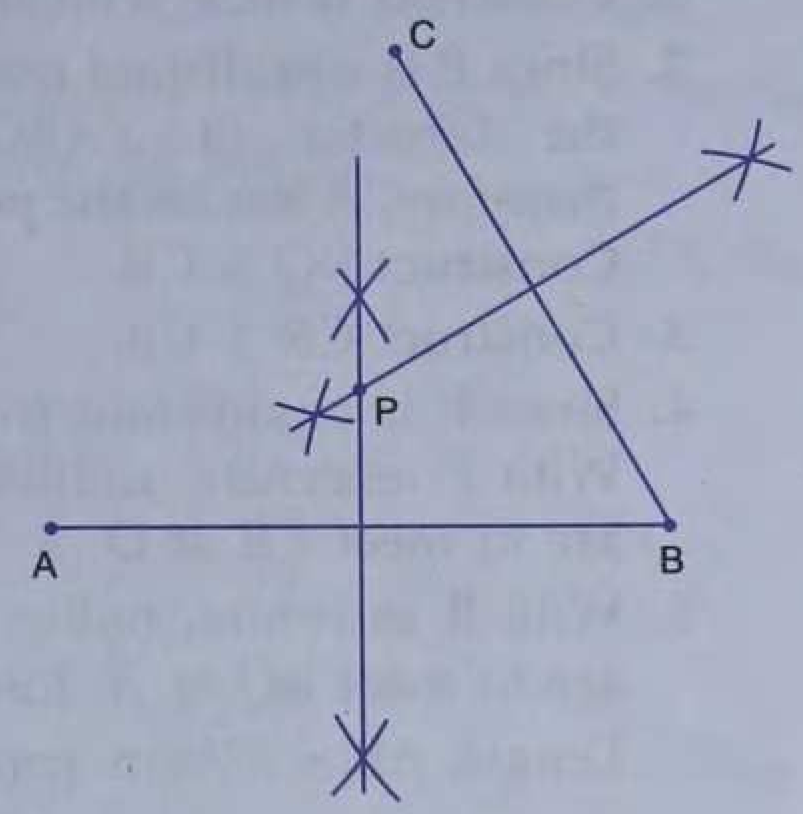
Join AB and construct perpendicular bisector of AB.

As P is equidistant from the points A and B, it lies on the perpendicular bisector of AB. Also the point P lies on CD, therefore, the required point P is the point of intersection of the perpendicular bisector of AB and the line CD.



Example 2. *Find a point P which is equidistant from three given non-collinear points.*

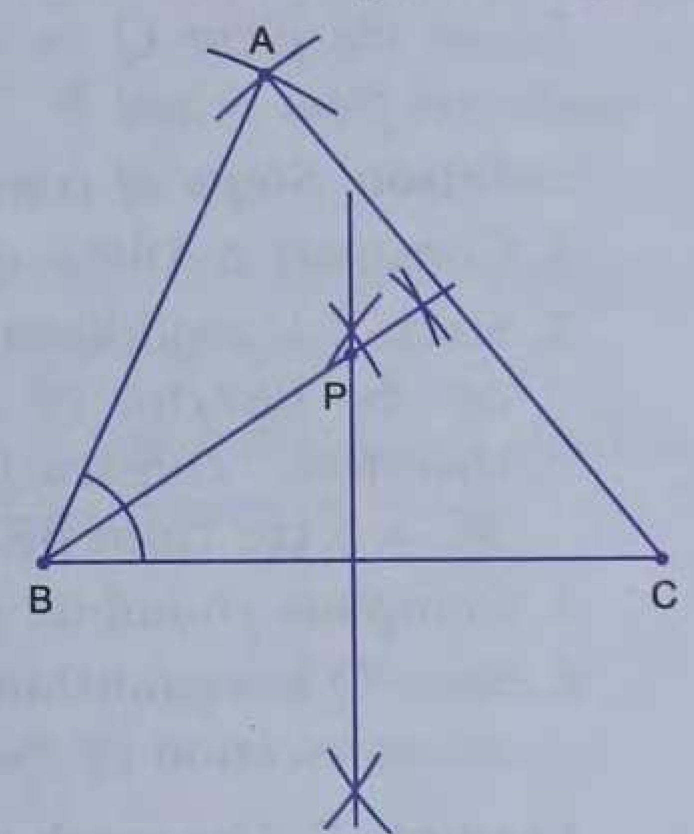
Solution. Let A, B and C be three given non-collinear points as shown in the figure alongside.



As the point P is equidistant from the points A and B, it lies on the perpendicular bisector of AB. Also as the point P is equidistant from the points B and C, it lies on the perpendicular bisector of BC. Construct perpendicular bisectors of AB and BC. Then the required point P is the point of intersection of the perpendicular bisectors of AB and BC.

Example 3. *Construct a triangle ABC in which BC = 5 cm, CA = 4·6 cm and AB = 3·8 cm. Find by construction a point P which is equidistant from BC and AB, and also equidistant from B and C.*

Solution. Construct Δ ABC with the given data.

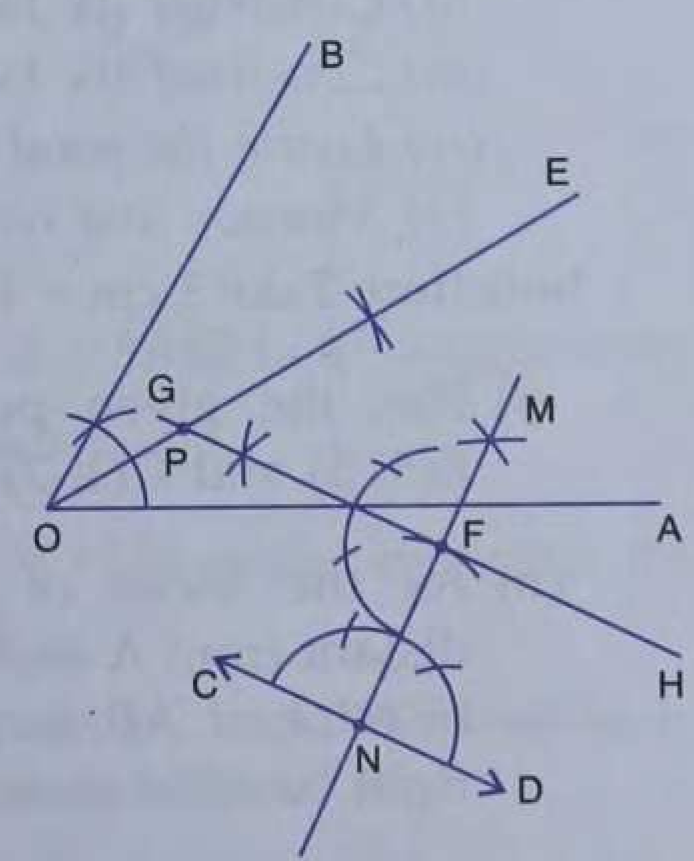


As the point P is equidistant from the intersecting lines BC and AB, it lies on the bisector of ∠ABC. Also as the point P is equidistant from the point B and C, it lies on the perpendicular bisector of the line segment BC.

Construct bisector of ∠ABC and the perpendicular bisector of BC. Then the required point P is the point of intersection of the bisector of ∠ABC and the perpendicular bisector of BC.

Example 4. *Construct ∠AOB = 60°. Mark a point P equidistant from OA and OB such that its distance from another given line CD is 2·5 cm.*

Solution. Construct ∠AOB = 60° as shown in the figure. As the point P is equidistant from the intersecting lines OA and OB, it lies on the bisector of ∠AOB. Construct OE, the bisector of ∠AOB.



Let CD be the other given line. Take any point N on CD and draw a perpendicular MN to CD. Cut off NF = 2·5 cm, and through F draw a st. line GH parallel to CD. Let GH meet OE at point P. Then P is a required point which is equidistant from the lines OA and OB, and is also at a distance 2·5 cm from another given line CD.

Remark

If we draw GH ∥ CD on the other side of CD, then GH will intersect OE at some other point, say Q. Thus, we get one more point satisfying the given conditions.

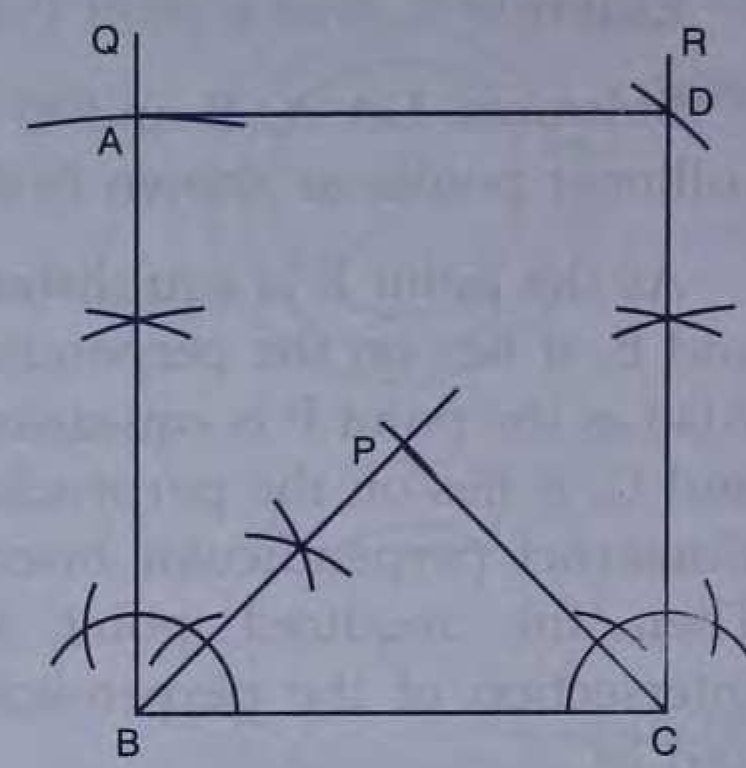
Example 5. *Construct triangle BCP, where BC = 5 cm, BP = 4 cm, ∠PBC = 45°. Complete the rectangle ABCD such that*

(i) P is equidistant from AB and BC; and

(ii) P is equidistant from C and D.

Measure and record the length of AB. (2007)

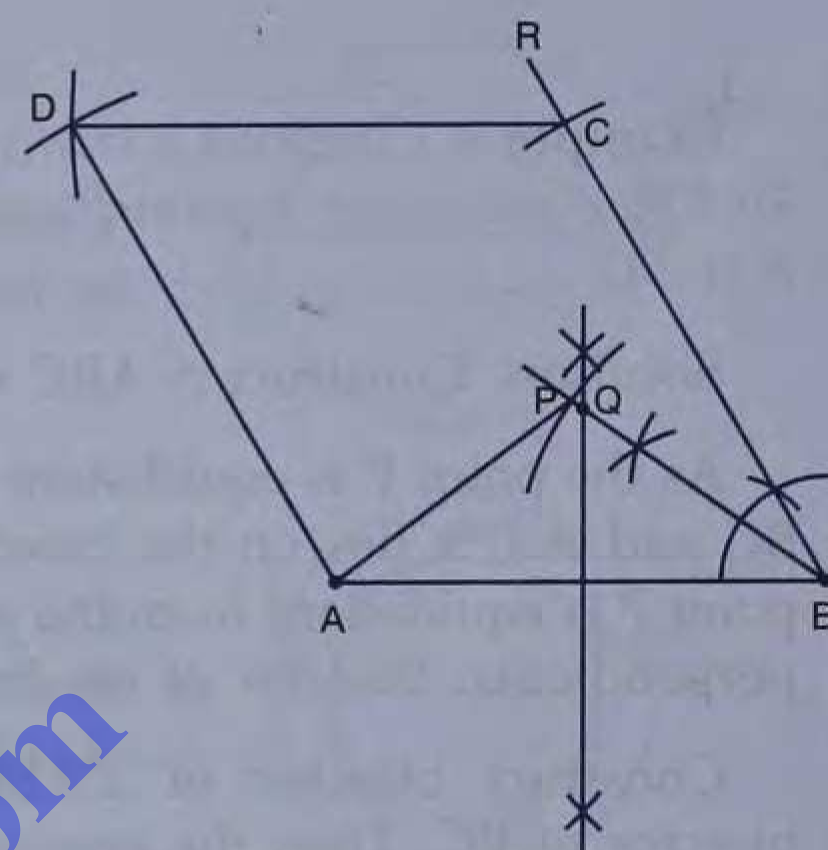
Solution. *Steps of construction.*



1. Construct Δ BCP with the given data.
2. Since P is equidistant from AB and BC, P lies on the bisector of ∠ABC. But ∠CBP = 45°, therefore, A lies on the perpendicular to CB at B. Construct BQ ⊥ CB.
3. Construct CR ⊥ CB.
4. Since P is equidistant from C and D, PD = PC. With P as centre, radius equal to CP, draw an arc to meet CR at D.
5. With B as centre, radius equal to CD, draw an arc to meet BQ at A. Join AD.
   Length AB = 5·7 cm approximately.

Example 6. *Construct a triangle ABP such that AB = 5 cm, BP = 3 cm and ∠ABP = 30°. Complete rhombus ABCD such that P is equidistant from AB and BC.*

Locate the point Q on the line BP such that Q is equdistant from A and B.



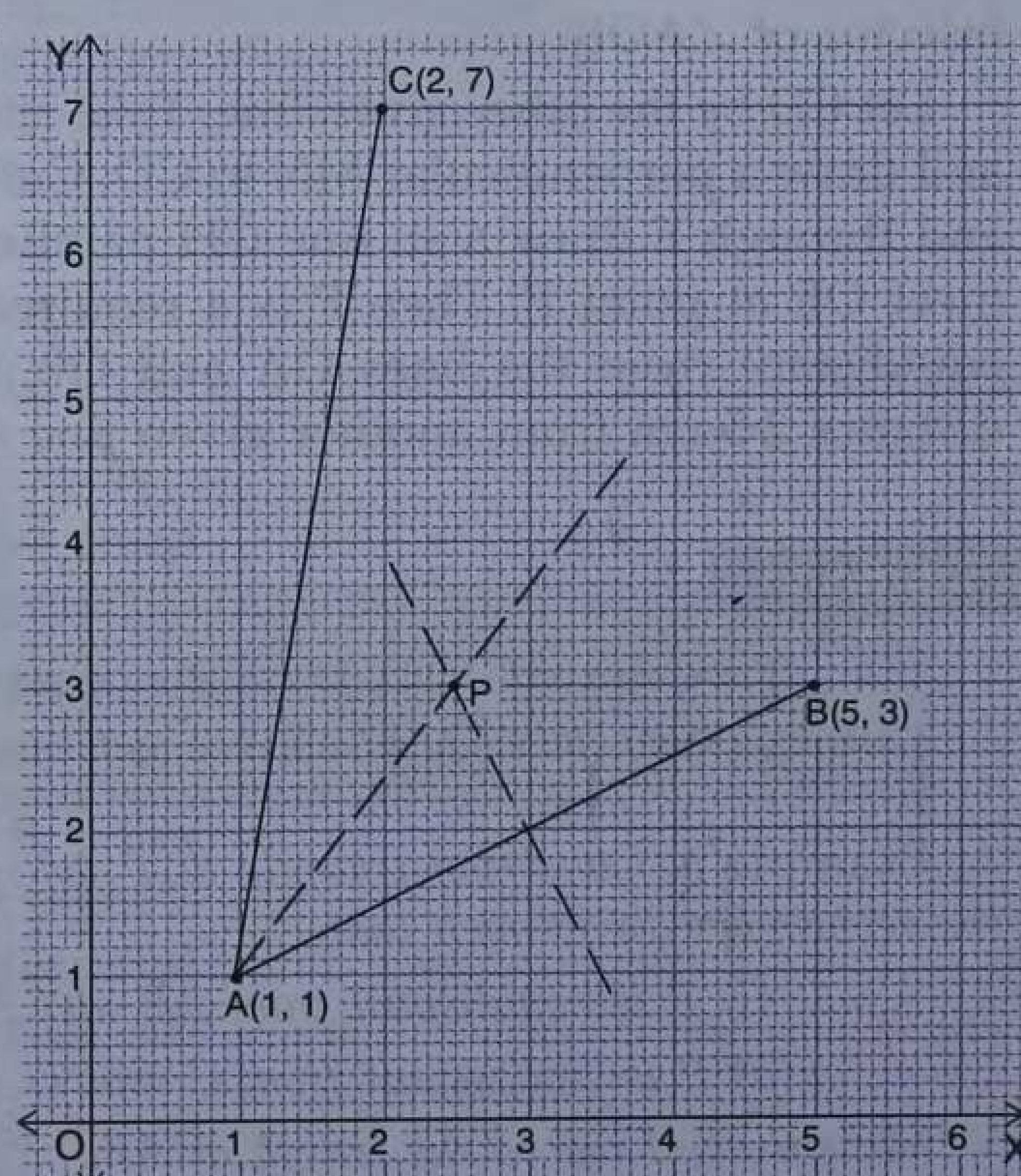
Solution. *Steps of construction.*

1. Construct Δ ABP with the given data.
2. Since P is equidistant from AB and BC, P lies on the bisector of ∠ABC. But ∠ABP = 30°, therefore, construct ∠ABR = 60°. Cut off BC = 5 cm from BR.
3. Complete rhombus ABCD.
4. Since Q is equidistant from A and B, draw perpendicular bisector of AB. The point of intersection of the right bisector of AB and the line BP is the required point Q.

Example 7. *Use graph paper for this question. Take 1 cm = 1 unit on both axes.*

(i) Plot the points A(1, 1), B(5, 3) and C(2, 7).
(ii) Construct the locus of points equidistant from A and B.
(iii) Construct the locus of points equidistant from AB and AC.
(iv) Locate the point P such that PA = PB and P is equidistant from AB and AC.
(v) Measure and record the length PA in cm.

Solution. Take 1 cm = 1 unit on both axes.



(*i*) Plot the given points A(1, 1), B(5, 3) and C(2, 7).

(*ii*) As the locus of points equidistant from A and B is the right bisector of AB, so construct the right bisector of segment AB.

(*iii*) As the locus of points equidistant from AB and AC is the bisector of ∠BAC, so construct the bisector of ∠BAC.

(*iv*) P is the point of intersection of the right bisector of the segment AB and the bisector of ∠BAC.

(*v*) Length PA = 2·5 cm approx.

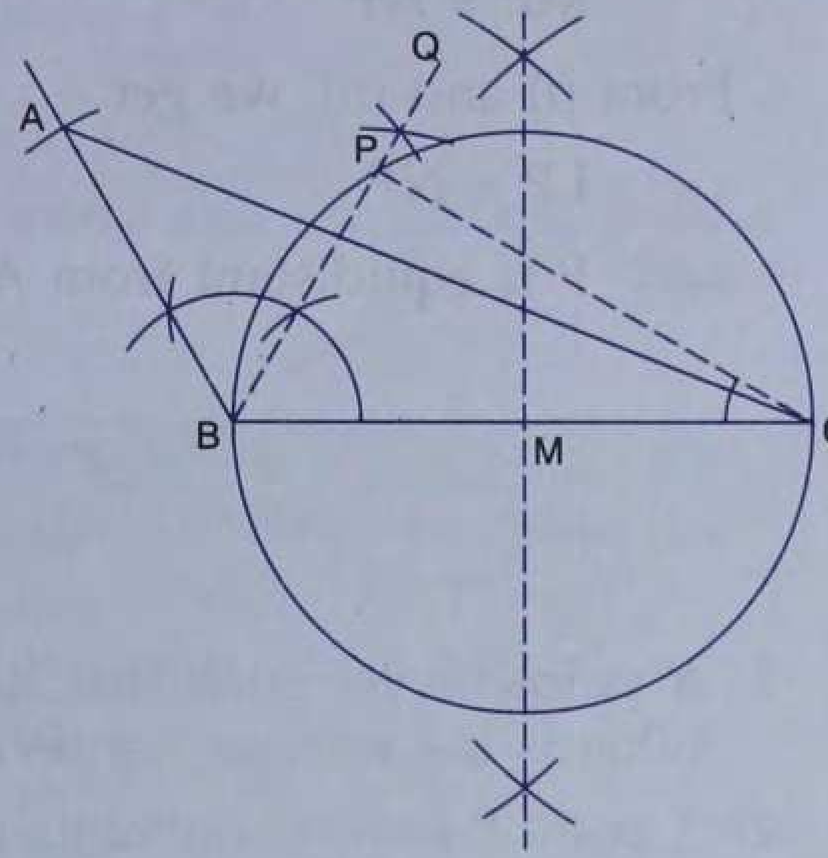
Example 8. *Using ruler and compasses only :*

(i) *Construct a triangle ABC with BC = 6 cm, ∠ABC = 120° and AB = 3·5 cm.*

(ii) *In the above figure, draw a circle with BC as diameter. Find a point 'P' on the circumference of the circle which is equidistant from AB and BC.*

(iii) *Measure ∠BCP.* (2013, 05)

Solution. ***Steps of construction.***



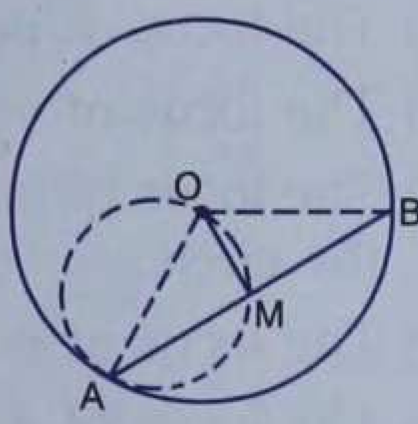
1. Construct Δ ABC with the given data.
2. Draw right bisector of BC to meet it at M. With M as centre and radius BM, draw the circle.
3. As P is equidistant from AB and BC. Draw bisector BQ of ∠ ABC.
4. Let BQ meet the circle at P, then P is the required point.

On measuring, we find that ∠BCP = 30°.

Example 9. *A is a fixed point on the circumference of a circle of radius 2·5 cm with centre O. M is mid-point of a variable chord AB. State the locus of M and justify your answer.*

Solution. Draw a circle of radius 2·5 cm with centre O. A is fixed point and AB is a chord. M is mid-point of AB. Locus of M is a circle with OA as diameter.

Justification



Join OA, OB and OM.

In Δs OAM and OBM,

AM = MB (∵ M is mid-point of AB)

OA = OB (radii of circle)

and OM is common

⇒ Δ OAM ≅ Δ OBM

⇒ ∠AMO = ∠OMB

but ∠AMO + ∠OMB = 180° (∵ AMB is a straight line)

⇒ ∠AMO = 90° ⇒ M lies on a circle with OA as diameter.

Example 10. *If the diagonals of a quadrilateral bisect each other at right angles, prove that the quadrilateral is a rhombus.*

Solution. Let ABCD be a quadrilateral in which the diagonals AC and BD bisect each other at right angles. Since A lies on the perpendicular bisector of BD,

AB = AD ...(*i*)

Similarly, BC = CD ...(*ii*)

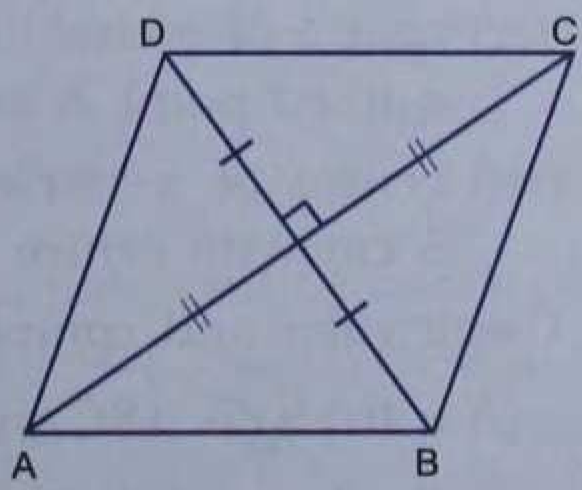
Also B lies on the perpendicular bisector of AC,

AB = BC ...(*iii*)

From (*i*), (*ii*) and (*iii*), we get

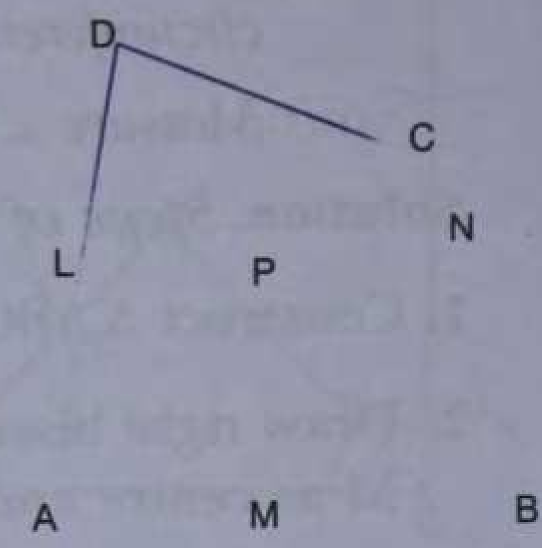
AB = BC = CD = AD

⇒ ABCD is a rhombus.



Example 11. *If the bisectors of ∠A and ∠B of a quadrilateral ABCD intersect each other at the point P, prove that P is equidistant from AD and BC.*

Solution. From P, draw PM ⊥ AB, PN ⊥ BC and PL ⊥ AD.



Since P lies on the bisector of ∠A, MP = LP ...(*i*)

Also P lies on the bisector of ∠B,

MP = NP ...(*ii*)

From (*i*) and (*ii*), we get

LP = NP

⇒ P is equidistant from AD and BC.

Exercise 15

1. A point moves such that its distance from a fixed line AB is always the same. What is the relation between AB and the path travelled by P ?
2. A point P moves so that its perpendicular distances from two given lines AB and CD are equal. State the locus of the point P.
3. P is a fixed point and a point Q moves such that the distance PQ is constant. What is the locus of the path traced out by the point Q ?
4. (*i*) AB is a fixed line. State the locus of the point P so that ∠APB = 90°.

   (*ii*) A, B are fixed points. State the locus of P so that ∠APB = 60°.
5. Draw and describe the locus in each of the following cases :

   (*i*) The locus of points at a distance 2·5 cm from a fixed line.

   (*ii*) The locus of vertices of all isosceles triangles having a common base.

   (*iii*) The locus of points inside a circle and equidistant from two fixed points on the circle.

   (*iv*) The locus of centres of all circles passing through two fixed points.

   (*v*) The locus of a point in rhombus ABCD which is equidistant from AB and AD.

   (*vi*) The locus of a point in the rhombus ABCD which is equidistant from points A and C.
6. Describe completely the locus of points in each of the following cases :

   (*i*) mid-point of radii of a circle.

   (*ii*) centre of a ball, rolling along a straight line on a level floor.

   (*iii*) point in a plane equidistant from a given line.

   (*iv*) point in a plane, at a constant distance of 5 cm from a fixed point (in the plane).

   (*v*) centre of a circle of varying radius and touching two arms of ∠ABC.

   (*vi*) centre of a circle of varying radius and touching a fixed circle, centre O, at a fixed point A on it.

   (*vii*) centre of a circle of radius 2 cm and touching a fixed circle of radius 3 cm with centre O.
7. Using ruler and compasses, construct

   (*i*) a triangle ABC in which AB = 5·5 cm, BC = 3·4 cm and CA = 4·9 cm.

   (*ii*) the locus of points equidistant from A and C. *(2009)*

8. Construct triangle ABC, with AB = 7 cm, BC = 8 cm and $\angle ABC = 60°$. Locate by construction the point P such that :

(*i*) P is equidistant from B and C and

(*ii*) P is equidistant from AB and BC.

(*iii*) Measure and record the length of PB. (*2000*)

9. A straight line AB is 8 cm long. Locate by construction the locus of a point which is :

(*i*) Equidistant from A and B.

(*ii*) Always 4 cm from the line AB.

(*iii*) Mark two points X and Y, which are 4 cm from AB and equidistant from A and B. Name the figure AXBY. (*2008*)

10. Use ruler and compasses only for this question.

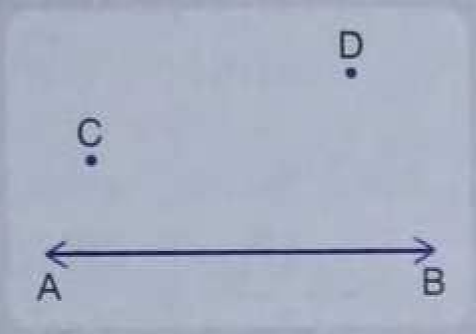
(*i*) Construct ΔABC, where AB = 3·5 cm, BC = 6 cm and $\angle ABC = 60°$.

(*ii*) Construct the locus of points inside the triangle which are equidistant from BA and BC.

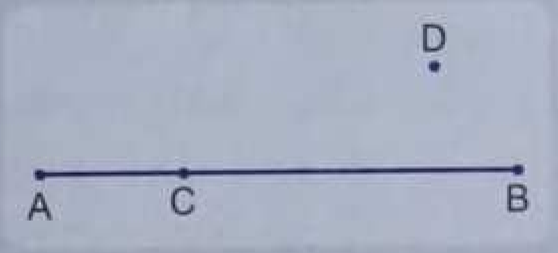
(*iii*) Construct the locus of points inside the triangle which are equidistant from B and C.

(*iv*) Mark the point P which is equidistant from AB, BC and also equidistant from B and C. Measure and record the length of PB. (*2010*)

11. In the adjoining diagram, AB is a fixed line and C, D are fixed points. Locate the point P on the line AB such that CP = PD.



12. In the adjoining diagram, A, B and C are fixed collinear points ; D is a fixed point outside the line. Locate



(*i*) the point P on AB such that CP = DP.

(*ii*) the points Q such that CQ = DQ = 3 cm. How many such points are possible ?

(*iii*) the points R on AB such that DR = 4 cm. How many such points are possible ?

(*iv*) the points S such that CS = DS and S is 4 cm away from the line CD. How many such points are possible ?

(*v*) Are the points P, Q, R collinear ?

(*vi*) Are the points P, Q, S collinear ?

13. Points A, B and C represent position of three towers such that AB = 60 m, BC = 73 m and CA = 52 m. Taking a scale of 10 m to 1 cm, make an accurate drawing of ΔABC. Find by drawing, the location of a point which is equidistant from A, B and C, and its actual distance from any of the towers.

14. Draw two intersecting lines to include an angle of 30°. Use ruler and compasses to locate points which are equidistant from these lines and also 2 cm away from their point of intersection. How many such points exist ?

Hint

To find points 2 cm away, draw a circle with centre as the point of intersection of the lines including an angle of 30° and of radius 2 cm. This circle meets bisectors (internal and external) of angle of 30° at four points.

15. Without using set square or protractor, construct the quadrilateral ABCD in which $\angle BAD = 45°$, AD = AB = 6 cm, BC = 3·6 cm and CD = 5 cm.

(*i*) Measure $\angle BCD$.

(*ii*) Locate the point P on BD which is equidistant from BC and CD.

16. Without using set square or protractor, construct rhombus ABCD with sides of length 4 cm and diagonal AC of length 5 cm. Measure $\angle ABC$. Find the point R on AD such that RB = RC. Measure the length of AR.

17. Without using set square or protractor construct :

(*i*) Triangle ABC, in which AB = 5·5 cm, BC = 3·2 cm and CA = 4·8 cm.

(*ii*) Draw the locus of a point which moves so that it is always 2·5 cm from B.

(*iii*) Draw the locus of a point which moves so that it is equidistant from the sides BC and CA.

(*iv*) Mark the point of intersection of the loci with the letter P and measure PC.

18. By using ruler and compasses only, construct an isosceles triangle ABC in which BC = 5 cm, AB = AC and $\angle BAC = 90°$. Locate the point P such that

(*i*) P is equidistant from the sides BC and AC.

(*ii*) P is equidistant from the points B and C.

Hint

Draw a semicircle with BC as diameter. Draw the right bisector of BC to meet the semicircle at A.

19. Using ruler and compasses only, construct a quadrilateral ABCD in which AB = 6 cm, BC = 5 cm, $\angle B = 60°$, AD = 5 cm and D is equidistant from AB and BC. Measure CD.

20. Construct an isosceles triangle ABC such that AB = 6 cm, BC = AC = 4 cm. Bisect $\angle C$ internally and mark a point P on this bisector such that CP = 5 cm. Find the points Q and R which are 5 cm from P and also 5 cm from the line AB. (*2001*)

21. Use ruler and compasses only for this question. Draw a circle of radius 4 cm and mark two chords AB and AC of the circle of length 6 cm and 5 cm respectively.

(*i*) Construct the locus of points, inside the circle, that are equidistant from A and C. Prove your construction.

(*ii*) Construct the locus of points, inside the circle, that are equidistant from AB and AC.

22. Ruler and compasses only may be used in this question. All construction lines and arcs must be clearly shown, and be of sufficient length and clarity to permit assessment.

(*i*) Construct a triangle ABC, in which BC = 6 cm, AB = 9 cm, and $\angle ABC = 60°$.

(*ii*) Construct the locus of all points, inside Δ ABC, which are equidistant from B and C.

(*iii*) Construct the locus of the vertices of the triangles with BC as base, which are equal in area to Δ ABC.

(*iv*) Mark the point Q, in your construction, which would make Δ QBC equal in area to Δ ABC, and isosceles.

(*v*) Measure and record the length of CQ.

CHAPTER TEST

1. Draw a straight line AB of length 8 cm. Draw the locus of all points which are equidistant from A and B. Prove your statement.

2. A point P is allowed to travel in space. State the locus of P so that it always remains at a constant distance from a fixed point C.

3. Draw a line segment AB of length 7 cm. Construct the locus of a point P such that area of triangle PAB is 14 cm^2.

Hint

Let h be height of Δ PAB, then $\frac{1}{2} \times 7 \times h = 14$.

4. Draw a line segment AB of length 12 cm. Mark M, the mid-point of AB. Draw and describe the locus of a point which is

   (*i*) at a distance of 3 cm from AB.

   (*ii*) at a distance of 5 cm from the point M.

   Mark the points P, Q, R, S which satisfy both the above conditions. What kind of quadrilateral is PQRS ? Compute the area of the quadrilateral PQRS.

5. AB and CD are two intersecting lines. Find the position of a point which is at a distance of 2 cm from AB and 1·6 cm from CD.

Hint

Draw EF ∥ AB at a distance 2 cm, and draw GH ∥ CD at a distance 1·6 cm. Point of intersection of EF and GH is the required point.

6. Two straight lines PQ and PK cross each other at P at an angle of 75°. S is a stone on the road PQ, 800 m from P towards Q. By drawing a figure to scale 1 cm = 100 m, locate the position of a flag staff X, which is equidistant from P and S, and is also equidistant from the roads.

7. Construct a rhombus PQRS whose diagonals PR, QS are 8 cm and 6 cm respectively. Find by construction a point X equidistant from PQ, PS and equidistant from R, S. Measure XR.

8. Without using set square or protractor, construct the parallelogram ABCD in which AB = 5·1 cm, the diagonal AC = 5·6 cm and the diagonal BD = 7 cm. Locate the point P on DC, which is equidistant from AB and BC.

9. By using ruler and compass only, construct a quadrilateral ABCD in which AB = 6·5 cm, AD = 4 cm and ∠DAB = 75°. C is equidistant from the sides AB and AD, also C is equidistant from the points A and B.