Chapter 13

SPECIAL PRODUCTS AND EXPANSIONS

Products of algebraic expressions can be obtained by using distributive laws :

$a(b + c) = ab + ac$

and $(a + b)c = ac + bc$

In this chapter, we shall study some special products and expansions.

SOME SPECIAL PRODUCTS

1. $(x + a)(x + b) = x^2 + (a + b)x + ab$

Simplifying the left hand side, we get

$(x + a)(x + b) = x(x + b) + a(x + b) = x^2 + bx + ax + ab$

$= x^2 + (ax + bx) + ab = x^2 + (a + b)x + ab$

2. $(x + a)(x - b) = x^2 + (a - b)x - ab$

Simplifying the left hand side, we get

$(x + a)(x - b) = x(x - b) + a(x - b) = x^2 - bx + ax - ab$

$= x^2 + (ax - bx) - ab = x^2 + (a - b)x - ab$

Aliter : Using the product 1, we get

$(x + a)(x - b) = (x + a)[x + (-b)] = x^2 + [a + (-b)]x + a \times (-b)$

$= x^2 + (a - b)x - ab$

3. $(x - a)(x + b) = x^2 - (a - b)x - ab$

Using the product 1, we get

$(x - a)(x + b) = [x + (-a)](x + b) = x^2 + [(-a) + b]x + (-a) \times b$

$= x^2 - (a - b)x - ab$

4. $(x - a)(x - b) = x^2 - (a + b)x + ab$

Using the product 1, we get

$(x - a)(x - b) = [x + (-a)][x + (-b)] = x^2 + [(-a) + (-b)]x + (-a) \times (-b)$

$= x^2 - (a + b)x + ab$

Example 1. Using the product $(x + a)(x + b) = x^2 + (a + b)x + ab$, simplify the following:

(*i*) $(x + 7)(x + 12)$ (*ii*) $(y + 3)(y - 8)$

(*iii*) $(s - 5)(s + 9)$ (*iv*) $(p - 6)(p - 13)$.

Solution. (*i*) $(x + 7)(x + 12) = x^2 + (7 + 12)x + 7 \times 12$

$= x^2 + 19x + 84.$

(*ii*) $(y + 3)(y - 8) = (y + 3)[y + (-8)]$

$= y^2 + [3 + (-8)]y + 3 \times (-8)$

$= y^2 - 5y - 24.$

(*iii*) $(s - 4)(s + 9) = [s + (-4)](s + 9)$

$= s^2 + [(-4) + 9]s + (-4) \times 9$

$= s^2 + 5s - 36.$

(iv) $(p-6)(p-13) = [p+(-6)][p+(-13)]$
$= p^2 + [(-6)+(-13)]p + (-6) \times (-13)$
$= p^2 - 19p + 78.$

Example 2. Find the following products :

(i) $(3x+5)(4x-7)$ (ii) $(4x-3y)(5x+6y)$
(iii) $(5a^2-4b^2)(3a^2-7b^2)$ (iv) $(x+2)(2x-3)(3x+2)$.

Solution. (i) $(3x+5)(4x-7) = 3x(4x-7) + 5(4x-7)$
$= 12x^2 - 21x + 20x - 35$
$= 12x^2 - x - 35.$

Use distributive laws

(ii) $(4x-3y)(5x+6y) = 4x(5x+6y) - 3y(5x+6y)$
$= 20x^2 + 24xy - 15xy - 18y^2$
$= 20x^2 + 9xy - 18y^2.$

(iii) $(5a^2-4b^2)(3a^2-7b^2) = 5a^2(3a^2-7b^2) - 4b^2(3a^2-7b^2)$
$= 15a^4 - 35a^2b^2 - 12a^2b^2 + 28b^4$
$= 15a^4 - 47a^2b^2 + 28b^4.$

(iv) $(x+2)(2x-3)(3x+2) = (x+2)[2x(3x+2) - 3(3x+2)]$
$= (x+2)(6x^2 + 4x - 9x - 6)$
$= (x+2)(6x^2 - 5x - 6)$
$= x(6x^2 - 5x - 6) + 2(6x^2 - 5x - 6)$
$= 6x^3 - 5x^2 - 6x + 12x^2 - 10x - 12$
$= 6x^3 + 7x^2 - 16x - 12.$

Exercise 13.1

Find the following (1 to 12) products :

1. (i) $(x+3)(x+5)$ (ii) $(y+2)(y-5)$
2. (i) $(a-3)(a+8)$ (ii) $(t-11)(t-6)$
3. (i) $\left(a+\frac{1}{2}\right)\left(a+\frac{1}{3}\right)$ (ii) $\left(b+\frac{2}{5}\right)\left(b-\frac{2}{3}\right)$
4. (i) $(x-3)\left(x+\frac{2}{7}\right)$ (ii) $(x+0{\cdot}4)(x-0{\cdot}7)$
5. (i) $(8-x)(5+x)$ (ii) $(3-z)(11-z)$
6. (i) $(2x+3)(2x+7)$ (ii) $(5y-2)(5y+9)$
7. (i) $(7c-11)(7c-3)$ (ii) $(p^2+3)(p^2-5)$
8. (i) $(3x^2-7)(3x^2+5)$ (ii) $(3+xy)(7-xy)$
9. (i) $\left(\frac{y}{3}-2\right)\left(\frac{y}{3}-7\right)$ (ii) $(5x+2y)(2x+5y)$
10. (i) $(3a-5b)(7a+2b)$ (ii) $(3mn-5)(4mn+6)$
11. (i) $(3x^2+2y^2)(4x^2-5y^2)$ (ii) $(2c^2-3d^2)(7c^2-2d^2)$
12. (i) $(ab-2c)(3ab+5c)$ (ii) $(x+1)(2x+5)(3x-1)$.

5. Product of sum and difference of two terms

$$(a + b)(a - b) = a^2 - b^2$$

Simplifying the left hand side, we get

$(a + b)(a - b) = a(a - b) + b(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$

In words, this result can be stated as :

> ***(1st term + 2nd term) × (1st term – 2nd term)***
> ***= (1st term)² – (2nd term)²***

Example 1. Using the product $(a + b)(a - b) = a^2 - b^2$, simplify the following :

(i) $(5x + 7y)(5x - 7y)$ (ii) $\left(\frac{2}{3}a + \frac{5}{4}b\right)\left(\frac{2}{3}a - \frac{5}{4}b\right)$

(iii) $(7pq + 11)(7pq - 11)$ (iv) $\left(6c^2 - \frac{5}{7}d^2\right)\left(6c^2 + \frac{5}{7}d^2\right)$.

Solution. (i) $(5x + 7y)(5x - 7y) = (5x)^2 - (7y)^2 = 25x^2 - 49y^2$.

(ii) $\left(\frac{2}{3}a + \frac{5}{4}b\right)\left(\frac{2}{3}a - \frac{5}{4}b\right) = \left(\frac{2}{3}a\right)^2 - \left(\frac{5}{4}b\right)^2 = \frac{4}{9}a^2 - \frac{25}{16}b^2$.

(iii) $(7pq + 11)(7pq - 11) = (7pq)^2 - (11)^2 = 49p^2q^2 - 121$.

(iv) $\left(6c^2 - \frac{5}{7}d^2\right)\left(6c^2 + \frac{5}{7}d^2\right) = (6c^2)^2 - \left(\frac{5}{7}d^2\right)^2 = 36c^4 - \frac{25}{49}d^4$.

Example 2. Find the product of :

(i) $(x + 3)(x - 3)(x^2 + 9)$ (ii) $(3p - 2q)(3p + 2q)(9p^2 + 4q^2)$.

Solution. (i) $(x + 3)(x - 3)(x^2 + 9) = [(x + 3)(x - 3)](x^2 + 9)$

$= (x^2 - 3^2)(x^2 + 9) = (x^2 - 9)(x^2 + 9)$

$= (x^2)^2 - (9)^2 = x^4 - 81$.

(ii) $(3p - 2q)(3p + 2q)(9p^2 + 4q^2) = [(3p - 2q)(3p + 2q)](9p^2 + 4q^2)$

$= [(3p)^2 - (2q)^2](9p^2 + 4q^2)$

$= (9p^2 - 4q^2)(9p^2 + 4q^2)$

$= (9p^2)^2 - (4q^2)^2 = 81p^4 - 16q^4$.

Example 3. Using the product $(a + b)(a - b) = a^2 - b^2$, find the value of :

(i) 507×493 (ii) $25{\cdot}3 \times 24{\cdot}7$.

Solution. (i) $507 \times 493 = (500 + 7)(500 - 7)$

$= (500)^2 - 7^2 = 250000 - 49 = 249951$.

(ii) $25{\cdot}3 \times 24{\cdot}7 = (25 + 0{\cdot}3)(25 - 0{\cdot}3)$

$= (25)^2 - (0{\cdot}3)^2 = 625 - 0{\cdot}09 = 624{\cdot}91$.

Exercise 13.2

Find the following (1 to 8) products :

1. (i) $(x + 7)(x - 7)$ (ii) $(5x + 9)(5x - 9)$

2. (i) $\left(y + \frac{2}{3}\right)\left(y - \frac{2}{3}\right)$ (ii) $(4 + 3x)(4 - 3x)$

3. (i) $(4x + 11y)(4x - 11y)$ (ii) $\left(\frac{2}{3}p - \frac{4}{5}q\right)\left(\frac{2}{3}p + \frac{4}{5}q\right)$

4. (i) $(3 - ab)(3 + ab)$ (ii) $\left(p + \frac{1}{q}\right)\left(p - \frac{1}{q}\right)$

5. (i) $\left(\frac{2}{a} + \frac{5}{b}\right)\left(\frac{2}{a} - \frac{5}{b}\right)$ (ii) $\left(\frac{1}{5x} + \frac{3}{2y}\right)\left(\frac{1}{5x} - \frac{3}{2y}\right)$

6. (i) $\left(3x^2 - \frac{2}{5}y^2\right)\left(3x^2 + \frac{2}{5}y^2\right)$ (ii) $(1{\cdot}4a - 0{\cdot}3b)(1{\cdot}4a + 0{\cdot}3b)$

7. (i) $(y + 2)(y - 2)(y^2 + 4)$ (ii) $(2p + 3)(2p - 3)(4p^2 + 9)$

8. (i) $(x + a)(x - a)(x^2 + a^2)$ (ii) $(x + yz)(x - yz)(x^2 + y^2z^2)$.

9. Using the product $(a + b)(a - b) = a^2 - b^2$, find the value of :
   (i) 108×92 (ii) 306×294
   (iii) $10{\cdot}4 \times 9{\cdot}6$ (iv) $14{\cdot}7 \times 15{\cdot}3$

EXPANSIONS

Module 2

The result of multiplication of any algebraic expression with itself any number of times is called the **expansion**.

For example :

$$(3x + 5)(3x + 5) = 3x(3x + 5) + 5(3x + 5) = 9x^2 + 15x + 15x + 25$$

$$\Rightarrow \quad (3x + 5)^2 = 9x^2 + 30x + 25$$

Thus, $9x^2 + 30x + 25$ is the expansion of $(3x + 5)^2$.

Some special expansions

1. $\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2}$

   Simplifying the left hand side, we get

   $$(a + b)^2 = (a + b)(a + b) = a(a + b) + b(a + b)$$
   $$= a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

   In words, this result can be stated as :

   (sum of two terms)² = (1st term)² + 2 × 1st term × 2nd term + (2nd term)²

   In particular, $\left(a + \frac{1}{a}\right)^2 = a^2 + 2 \times a \times \frac{1}{a} + \left(\frac{1}{a}\right)^2$

   $$\Rightarrow \quad \left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2.$$

2. $\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2}$

   Simplifying the left hand side, we get

   $$(a - b)^2 = (a - b)(a - b) = a(a - b) - b(a - b)$$
   $$= a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

   In words, this result can be stated as :

   (difference of two terms)² = (1st term)² – 2 × 1st term × 2nd term + (2nd term)²

In particular, $\left(a - \frac{1}{a}\right)^2 = a^2 - 2 \times a \times \frac{1}{a} + \left(\frac{1}{a}\right)^2$

$\Rightarrow \quad \left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2.$

Example 1. Expand the following :

(i) $\left(3x + \frac{2}{5}y\right)^2$ (ii) $\left(5a + \frac{3}{b}\right)^2$ (iii) $\left(\sqrt{2}\left(\frac{p^2}{2} + \frac{2}{q^2}\right)\right)^2$.

Solution. (i) $\left(3x + \frac{2}{5}y\right)^2 = (3x)^2 + 2 \times 3x \times \frac{2}{5}y + \left(\frac{2}{5}y\right)^2 = 9x^2 + \frac{12}{5}xy + \frac{4}{25}y^2.$

(ii) $\left(5a + \frac{3}{b}\right)^2 = (5a)^2 + 2 \times 5a \times \frac{3}{b} + \left(\frac{3}{b}\right)^2 = 25a^2 + 30\frac{a}{b} + \frac{9}{b^2}.$

(iii) $\left(\sqrt{2}\left(\frac{p^2}{2} + \frac{2}{q^2}\right)\right)^2 = (\sqrt{2})^2\left(\frac{p^2}{2} + \frac{2}{q^2}\right) = 2\left[\left(\frac{p^2}{2}\right)^2 + 2 \times \frac{p^2}{2} \times \frac{2}{q^2} + \left(\frac{2}{q^2}\right)^2\right]$

$= 2\left[\frac{p^4}{4} + 2\frac{p^2}{q^2} + \frac{4}{q^4}\right] = \frac{p^4}{2} + 4\frac{p^2}{q^2} + \frac{8}{q^4}.$

Example 2. Expand the following :

(i) $\left(\frac{3}{5}p - 2q\right)^2$ (ii) $(\sqrt{2}a - \sqrt{3}b)^2$ (iii) $\left(\frac{2x}{3y} - \frac{3y}{2x}\right)^2$.

Solution. (i) $\left(\frac{3}{5}p - 2q\right)^2 = \left(\frac{3}{5}p\right)^2 - 2 \times \frac{3}{5}p \times 2q + (2q)^2$

$= \frac{9}{25}p^2 - \frac{12}{5}pq + 4q^2.$

(ii) $(\sqrt{2}a - \sqrt{3}b)^2 = (\sqrt{2}a)^2 - 2 \times \sqrt{2}a \times \sqrt{3}b + (\sqrt{3}b)^2$

$= 2a^2 - 2\sqrt{6}ab + 3b^2.$

(iii) $\left(\frac{2x}{3y} - \frac{3y}{2x}\right)^2 = \left(\frac{2x}{3y}\right)^2 - 2 \times \frac{2x}{3y} \times \frac{3y}{2x} + \left(\frac{3y}{2x}\right)^2 = \frac{4x^2}{9y^2} - 2 + \frac{9y^2}{4x^2}.$

Example 3. Using special expansions, find the values of :

(i) $(1003)^2$ (ii) $(10{\cdot}2)^2$ (iii) $(998)^2$.

Solution. (i) $(1003)^2 = (1000 + 3)^2$

$= (1000)^2 + 2 \times 1000 \times 3 + (3)^2$

$= 1000000 + 6000 + 9 = 1006009.$

(ii) $(10{\cdot}2)^2 = (10 + 0{\cdot}2)^2$

$= (10)^2 + 2 \times 10 \times 0{\cdot}2 + (0{\cdot}2)^2$

$= 100 + 4 + 0{\cdot}04 = 104{\cdot}04.$

(iii) $(998)^2 = (1000 - 2)^2$

$= (1000)^2 - 2 \times 1000 \times 2 + (2)^2$

$= 1000000 - 4000 + 4 = 996004.$

Perfect square trinomial

Since $(a + b)^2 = a^2 + 2ab + b^2$, $a^2 + 2ab + b^2$ is square of $a + b$.

We say that $a^2 + 2ab + b^2$ is a *perfect square trinomial.*

Similarly, $(a - b)^2 = a^2 - 2ab + b^2$, so $a^2 - 2ab + b^2$ is square of $a - b$.

We say that $a^2 - 2ab + b^2$ is a *perfect square trinomial*.

Thus, if the given trinomial can be expressed as $a^2 + 2ab + b^2$ or as $a^2 - 2ab + b^2$, then it is a perfect square trinomial, otherwise, it is not a perfect square trinomial.

For example :

(*i*) In $9x^2 + 30xy + 25y^2$, $9x^2 = (3x)^2$, $25y^2 = (5y)^2$ and

$2 \times 3x \times 5y = 30xy$, so $9x^2 + 30xy + 25y^2 = (3x + 5y)^2$.

$\therefore$ The given trinomial is a perfect square.

(*ii*) In $16x^2 - 56xy + 49y^2$, $16x^2 = (4x)^2$, $49y^2 = (7y)^2$ and

$2 \times 4x \times 7y = 56xy$, so $16x^2 - 56xy + 49y^2 = (4x - 7y)^2$.

$\therefore$ The given trinomial is a perfect square.

(*iii*) In $36x^2 + 30xy + 25y^2$, $36x^2 = (6x)^2$, $25y^2 = (5y)^2$ and

$2 \times 6x \times 5y = 60xy$, which is not equal to the middle term.

$\therefore$ $36x^2 + 30xy + 25y^2$ is not a perfect square trinomial.

Example 4. Write each of the following trinomials as a perfect square :

(*i*) $81x^2 + 90xy + 25y^2$ (*ii*) $9a^2 - \frac{12}{5}a + \frac{4}{25}$.

Solution. (*i*) $81x^2 + 90xy + 25y^2 = (9x)^2 + 2 \times 9x \times 5y + (5y)^2 = (9x + 5y)^2$.

(*ii*) $9a^2 - \frac{12}{5}a + \frac{4}{25} = (3a)^2 - 2 \times 3a \times \frac{2}{5} + \left(\frac{2}{5}\right)^2 = \left(3a - \frac{2}{5}\right)^2$.

Example 5. If $x - \frac{1}{x} = 3$, evaluate : (*i*) $x^2 + \frac{1}{x^2}$ (*ii*) $x^4 + \frac{1}{x^4}$.

Solution. (*i*) Given $x - \frac{1}{x} = 3 \Rightarrow \left(x - \frac{1}{x}\right)^2 = 3^2$

$\Rightarrow x^2 - 2 \times x \times \frac{1}{x} + \left(\frac{1}{x}\right)^2 = 9$

$\Rightarrow x^2 - 2 + \frac{1}{x^2} = 9 \Rightarrow x^2 + \frac{1}{x^2} = 9 + 2 = 11.$

(*ii*) From (*i*), $x^2 + \frac{1}{x^2} = 11 \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = 11^2$

$\Rightarrow (x^2)^2 + 2 \times x^2 \times \frac{1}{x^2} + \left(\frac{1}{x^2}\right)^2 = 121$

$\Rightarrow x^4 + 2 + \frac{1}{x^4} = 121 \Rightarrow x^4 + \frac{1}{x^4} = 121 - 2 = 119.$

Example 6. (*i*) If $a + b = 7$ and $ab = 10$, find the value of $a^2 + b^2$.

(*ii*) If $a - b = 5$ and $a^2 + b^2 = 37$, find the value of ab.

Solution. (*i*) We know that $(a + b)^2 = a^2 + b^2 + 2ab$

$\Rightarrow 7^2 = a^2 + b^2 + 2 \times 10$

$\Rightarrow 49 - 20 = a^2 + b^2$

$\Rightarrow a^2 + b^2 = 29.$

(*ii*) We know that $(a - b)^2 = a^2 + b^2 - 2ab$

$\Rightarrow 5^2 = 37 - 2ab$

$\Rightarrow \quad 2ab = 37 - 25 = 12$

$\Rightarrow \quad ab = 6.$

Example 7. If $a^2 + b^2 = 34$ and $ab = 15$, find the values of : (*i*) $a + b$ (*ii*) $a - b$.

Solution. (*i*) We know that $(a + b)^2 = a^2 + b^2 + 2ab$

$\Rightarrow \quad (a + b)^2 = 34 + 2 \times 15 = 34 + 30 = 64$

$\Rightarrow \quad a + b = \pm\sqrt{64} = \pm 8.$

(*ii*) We know that $(a - b)^2 = a^2 + b^2 - 2ab$

$\Rightarrow \quad (a - b)^2 = 34 - 2 \times 15 = 34 - 30 = 4$

$\Rightarrow \quad a - b = \pm\sqrt{4} = \pm 2.$

Example 8. If $x^2 + \frac{1}{x^2} = 18$, find the values of : (*i*) $x + \frac{1}{x}$ (*ii*) $x - \frac{1}{x}$.

Solution. (*i*) $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 = 18 + 2 = 20$

$\Rightarrow \quad x + \frac{1}{x} = \pm\sqrt{20} = \pm 2\sqrt{5}.$

(*ii*) $\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2 = 18 - 2 = 16$

$\Rightarrow \quad x - \frac{1}{x} = \pm\sqrt{16} = \pm 4.$

Exercise 13.3

Expand the following (1 to 7) :

1. (*i*) $(3a + 7b)^2$ (*ii*) $(8 + 5p)^2$
2. (*i*) $\left(2x + \frac{3}{y}\right)^2$ (*ii*) $\left(\sqrt{3}p + \frac{2}{5}q\right)^2$
3. (*i*) $\left(\frac{2x}{3y} + \frac{3y}{2x}\right)^2$ (*ii*) $\left[\sqrt{3}\left(\frac{a}{3} + \frac{3}{a}\right)\right]^2$
4. (*i*) $\left(2m^2 + \frac{3}{7}n^2\right)^2$ (*ii*) $\left(3ab + \frac{1}{2}c\right)^2$
5. (*i*) $(3a - 7)^2$ (*ii*) $(3p - 5q)^2$
6. (*i*) $\left(\frac{x}{2} - \frac{y}{3}\right)^2$ (*ii*) $\left(\frac{2}{m} - \frac{3}{n}\right)^2$
7. (*i*) $\left(3x - \frac{1}{3x}\right)^2$ (*ii*) $\left[\sqrt{2}(\sqrt{3}c - 2d)\right]^2$.
8. Write down the squares of the following :

   (*i*) $2a + 5$ (*ii*) $3b - 2$ (*iii*) $4p + \frac{2}{3}$ (*iv*) $\frac{2}{3}z - \frac{5}{7}$

   (*v*) $3x + \frac{5}{2}y$ (*vi*) $5c^2 - 2d$ (*vii*) $\sqrt{3}\left(\sqrt{2}a - \frac{1}{\sqrt{2}}\right)$ (*viii*) $2p - \frac{1}{2p}$.

9. Use $(a+b)^2 = a^2 + 2ab + b^2$ to find the values of :

(*i*) $(501)^2$ (*ii*) $(1005)^2$ (*iii*) $(10{\cdot}3)^2$.

10. Use $(a-b)^2 = a^2 - 2ab + b^2$ to find the values of :

(*i*) $(199)^2$ (*ii*) $(997)^2$ (*iii*) $(9{\cdot}8)^2$.

11. Write each of the following trinomials as a perfect square :

(*i*) $16x^2 + 40xy + 25y^2$ (*ii*) $4p^2 + 44p + 121$

(*iii*) $9a^2 - 42ab + 49b^2$ (*iv*) $25m^2 - \frac{10}{3}mn + \frac{n^2}{9}$.

12. If $x + \frac{1}{x} = 4$, evaluate :

(*i*) $x^2 + \frac{1}{x^2}$ (*ii*) $x^4 + \frac{1}{x^4}$.

13. If $p - \frac{1}{p} = 7$, evaluate :

(*i*) $p^2 + \frac{1}{p^2}$ (*ii*) $p^4 + \frac{1}{p^4}$.

14. If $a + b = 7$ and $ab = 10$, find the value of $a^2 + b^2$.

15. If $a + b = 9$ and $a^2 + b^2 = 51$, find the value of ab.

16. If $a - b = 5$ and $ab = 4$, find the value of $a^2 + b^2$.

17. If $a - b = 6$ and $a^2 + b^2 = 42$, find the value of ab.

18. If $a^2 + b^2 = 41$ and $ab = 4$, find the values of : (*i*) $a + b$ (*ii*) $a - b$.

19. If $a^2 + b^2 = 67$ and $ab = 9$, find the values of : (*i*) $a + b$ (*ii*) $a - b$.

20. If $2a + \frac{1}{2a} = 5$, find the values of :

(*i*) $4a^2 + \frac{1}{4a^2}$ (*ii*) $16a^4 + \frac{1}{16a^4}$.

21. If $3p - \frac{1}{3p} = 1$, find the values of :

(*i*) $9p^2 + \frac{1}{9p^2}$ (*ii*) $81p^4 + \frac{1}{81p^4}$.

SOME MORE SPECIAL EXPANSIONS

1. $\boldsymbol{(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)}$

Simplifying the left hand side, we get

$$\begin{aligned}(a + b + c)^2 &= [(a + b) + c]^2 \\ &= (a + b)^2 + 2(a + b)c + c^2 \\ &= (a^2 + 2ab + b^2) + 2(ca + bc) + c^2 \\ &= a^2 + b^2 + c^2 + 2(ab + bc + ca).\end{aligned}$$

2. $\boldsymbol{(a + b)^3 = a^3 + b^3 + 3ab(a + b)}$

Simplifying the left hand side, we get

$$\begin{aligned}(a + b)^3 &= (a + b) \times (a + b)^2 \\ &= (a + b)(a^2 + 2ab + b^2)\end{aligned}$$

$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$
$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$
$= a^3 + b^3 + 3a^2b + 3ab^2$
$= a^3 + b^3 + 3ab(a + b).$

In particular, $\left(x + \frac{1}{x}\right)^3 = x^3 + \left(\frac{1}{x}\right)^3 + 3 \times x \times \frac{1}{x} \times \left(x + \frac{1}{x}\right)$

$\Rightarrow \quad \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right).$

3. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

Simplifying the left hand side, we get

$(a - b)^3 = (a - b) \times (a - b)^2$
$= (a - b)(a^2 - 2ab + b^2)$
$= a(a^2 - 2ab + b^2) - b(a^2 - 2ab + b^2)$
$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$
$= a^3 - b^3 - 3a^2b + 3ab^2$
$= a^3 - b^3 - 3ab(a - b).$

In particular, $\left(x - \frac{1}{x}\right)^3 = x^3 - \left(\frac{1}{x}\right)^3 - 3 \times x \times \frac{1}{x} \times \left(x - \frac{1}{x}\right)$

$\Rightarrow \quad \left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right).$

Example 1. Expand the following :

(*i*) $(a + b - c)^2$ (*ii*) $(x + 2y - 3z)^2$.

Solution. (*i*) $(a + b - c)^2 = [a + b + (-c)]^2$
$= a^2 + b^2 + (-c)^2 + 2[ab + b(-c) + (-c)a]$
$= a^2 + b^2 + c^2 + 2ab - 2bc - 2ca.$

(*ii*) $(x + 2y - 3z)^2 = [x + 2y + (-3z)]^2$
$= x^2 + (2y)^2 + (-3z)^2 + 2[x \times 2y + 2y \times (-3z) + (-3z) \times x]$
$= x^2 + 4y^2 + 9z^2 + 4xy - 12yz - 6zx.$

Example 2. Expand the following :

(*i*) $(3x + 2y)^3$ (*ii*) $(2p - 3q)^3$.

Solution. (*i*) $(3x + 2y)^3 = (3x)^3 + (2y)^3 + 3 \times 3x \times 2y \times (3x + 2y)$
$= 27x^3 + 8y^3 + 18xy(3x + 2y)$
$= 27x^3 + 8y^3 + 54x^2y + 36xy^2.$

(*ii*) $(2p - 3q)^3 = (2p)^3 - (3q)^3 - 3 \times 2p \times 3q \times (2p - 3q)$
$= 8p^3 - 27q^3 - 18pq(2p - 3q)$
$= 8p^3 - 27q^3 - 36p^2q + 54pq^2.$

Example 3. (*i*) If $a + b + c = 9$ and $ab + bc + ca = 15$, find $a^2 + b^2 + c^2$.

(*ii*) If $a + b + c = 11$ and $a^2 + b^2 + c^2 = 81$, find $ab + bc + ca$.

Solution. (*i*) We know that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow \quad 9^2 = a^2 + b^2 + c^2 + 2 \times 15$

$\Rightarrow \quad 81 - 30 = a^2 + b^2 + c^2$

$\Rightarrow \quad a^2 + b^2 + c^2 = 51.$

(*ii*) We know that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow \quad 11^2 = 81 + 2(ab + bc + ca)$

$\Rightarrow \quad 121 - 81 = 2(ab + bc + ca)$

$\Rightarrow \quad 2(ab + bc + ca) = 40$

$\Rightarrow \quad ab + bc + ca = 20.$

Example 4. If $a^2 + b^2 + c^2 = 44$ and $ab + bc + ca = 10$, find $a + b + c$.

Solution. We know that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$= 44 + 2 \times 10 = 44 + 20 = 64$

$\Rightarrow \quad a + b + c = \pm\sqrt{64} = \pm 8.$

Example 5. If $a + b = 5$ and $ab = 6$, find $a^3 + b^3$.

Solution. We know that $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

$\Rightarrow \quad 5^3 = a^3 + b^3 + 3 \times 6 \times 5$

$\Rightarrow \quad 125 = a^3 + b^3 + 90$

$\Rightarrow \quad a^3 + b^3 = 125 - 90 = 35.$

Example 6. If $x - \frac{1}{x} = 6$, find the value of $x^3 - \frac{1}{x^3}$.

Solution. Given $x - \frac{1}{x} = 6 \quad \Rightarrow \left(x - \frac{1}{x}\right)^3 = 6^3$

$\Rightarrow \quad x^3 - \left(\frac{1}{x}\right)^3 - 3 \times x \times \frac{1}{x} \times \left(x - \frac{1}{x}\right) = 216$

$\Rightarrow \quad x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) = 216$

$\Rightarrow \quad x^3 - \frac{1}{x^3} - 3 \times 6 = 216$

$\Rightarrow \quad x^3 - \frac{1}{x^3} = 216 + 18 = 234.$

Example 7. If $2p + \frac{1}{2p} = 3$, find the value of $8p^3 + \frac{1}{8p^3}$.

Solution. Given $2p + \frac{1}{2p} = 3 \quad \Rightarrow \left(2p + \frac{1}{2p}\right)^3 = 3^3$

$\Rightarrow \quad (2p)^3 + \left(\frac{1}{2p}\right)^3 + 3 \times 2p \times \frac{1}{2p} \times \left(2p + \frac{1}{2p}\right) = 27$

$\Rightarrow \quad 8p^3 + \frac{1}{8p^3} + 3\left(2p + \frac{1}{2p}\right) = 27$

$\Rightarrow \quad 8p^3 + \frac{1}{8p^3} + 3 \times 3 = 27$

$\Rightarrow \quad 8p^3 + \frac{1}{8p^3} = 27 - 9 = 18.$

Example 8. If $3p - 4q = 5$ and $pq = 3$, find the value of $27p^3 - 64q^3$.

Solution. Given $3p - 4q = 5 \quad \Rightarrow \quad (3p - 4q)^3 = 5^3$

$\Rightarrow (3p)^3 - (4q)^3 - 3 \times 3p \times 4q \times (3p - 4q) = 125$

$\Rightarrow 27p^3 - 64q^3 - 36pq\,(3p - 4q) = 125$

$\Rightarrow 27p^3 - 64q^3 - 36 \times 3 \times 5 = 125$

$\Rightarrow 27p^3 - 64q^3 = 125 + 540 = 665.$

Exercise 13.4

Expand the following (1 to 5) :

1. (i) $(a - b - c)^2$ (ii) $(2x + 3y + 5z)^2$.
2. (i) $(2p - 3q + 1)^2$ (ii) $\left(x + \frac{1}{x} - 1\right)^2$.
3. (i) $(2a + b)^3$ (ii) $(7c + 4d)^3$.
4. (i) $(2x - 3)^3$ (ii) $(a - 5b)^3$.
5. (i) $\left(2x + \frac{1}{x}\right)^3$ (ii) $\left(3a - \frac{1}{3a}\right)^3$.
6. If $a + b + c = 10$ and $a^2 + b^2 + c^2 = 42$, find the value of $ab + bc + ca$.
7. If $a + b + c = 11$ and $ab + bc + ca = 31$, find the value of $a^2 + b^2 + c^2$.
8. If $a^2 + b^2 + c^2 = 49$ and $ab + bc + ca = 36$, find the value of $a + b + c$.
9. If $a + b - c = 9$ and $a^2 + b^2 + c^2 = 29$, find the value of $ab - bc - ca$.
10. If $a + b = 8$ and $ab = 15$, find the value of $a^3 + b^3$.
11. If $p - q = 5$ and $pq = 14$, find the value of $p^3 - q^3$.
12. If $x + \frac{1}{x} = 4$, find the value of $x^3 + \frac{1}{x^3}$.
13. If $x - \frac{1}{x} = 5$, find the value of $x^3 - \frac{1}{x^3}$.
14. If $3p + \frac{1}{3p} = 6$, find the value of $27p^3 + \frac{1}{27p^3}$.
15. If $2x - \frac{1}{2x} = 3$, find the value of $8x^3 - \frac{1}{8x^3}$.
16. If $2a + 3b = 9$ and $ab = 3$, find the value of $8a^3 + 27b^3$.
17. If $4p - 5q = 2$ and $pq = 6$, find the value of $64p^3 - 125q^3$.

Summary

- $(x + a)(x + b) = x^2 + (a + b)x + ab$
- $(x + a)(x - b) = x^2 + (a - b)x - ab$
- $(x - a)(x + b) = x^2 - (a - b)x - ab$
- $(x - a)(x - b) = x^2 - (a + b)x + ab$
- $(a + b)(a - b) = a^2 - b^2$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$
- $\left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2$

➡ $\left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2$

➡ $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

➡ $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

➡ $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

➡ $\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$

➡ $\left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)$

Check Your Progress

1. Find the following products :

(*i*) $(2x - 3y)(5x - 8y)$ (*ii*) $(4p^2 + 7q^2)(3p^2 - 5q^2)$

(*iii*) $\left(\frac{x}{3} - \frac{y}{4}\right)\left(\frac{x}{3} + \frac{y}{4}\right)$ (*iv*) $\left(\frac{2}{a} + \frac{3}{b}\right)\left(\frac{2}{a} - \frac{3}{b}\right)$.

2. Prove the following identities :

(*i*) $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$ (*ii*) $(a + b)^2 - (a - b)^2 = 4ab$

(*iii*) $(a + b)(a^2 - ab + b^2) = a^3 + b^3$ (*iv*) $(a - b)(a^2 + ab + b^2) = a^3 - b^3$

(*v*) $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc$.

3. Expand the following :

(*i*) $(5a + 2bc)^2$ (*ii*) $(3mn - p)^2$ (*iii*) $(2x - 3y + z)^2$

(*iv*) $(3x - 2y - 1)^2$ (*v*) $(3x + 2)^3$ (*vi*) $(2 - 3p)^3$.

4. Simplify : $\left(2x - \frac{1}{2x}\right)^2 - \left(2x + \frac{1}{2x}\right)\left(2x - \frac{1}{2x}\right)$.

5. Simplify : $(x + y + 1)(x + y - 1)$.

6. If $x - \frac{1}{x} = \sqrt{5}$, find the values of :

(*i*) $x^2 + \frac{1}{x^2}$ (*ii*) $x^4 + \frac{1}{x^4}$.

7. If $a^2 + b^2 = 20$ and $ab = 8$, find the values of :

(*i*) $a + b$ (*ii*) $a - b$.

8. If $x^2 + \frac{1}{x^2} = 47$, find the values of :

(*i*) $x + \frac{1}{x}$ (*ii*) $x - \frac{1}{x}$.

9. If $a + b + c = 11$ and $a^2 + b^2 + c^2 = 49$, find the value of $ab + bc + ca$.

10. If $x + \frac{1}{x} = \sqrt{3}$, find the value of $x^3 + \frac{1}{x^3}$.

11. If $x - \frac{1}{x} = 4$, find the value of $x^3 - \frac{1}{x^3}$.

12. If $3p + \frac{1}{3p} = 3$, find the values of :

(*i*) $9p^2 + \frac{1}{9p^2}$ (*ii*) $81p^4 + \frac{1}{81p^4}$ (*iii*) $27p^3 + \frac{1}{27p^3}$.