

Chapter 12

EXPONENTS

You already know that $3 \times 3 \times 3 \times 3 \times 3$ can be written as 3^5 . Here 3 is the base and 5 is the exponent or index. 3^5 is read as '3 raised to the power 5' or '3 to the power 5' or simply '3 power 5'. In general, we have :

If a is any real number and n is a natural number, then $a^n = a \times a \times a \dots n$ times where a is called the base, n is called the exponent or index and a^n is the exponential form. a^n is read as 'a raised to the power n' or 'a to the power n' or simply 'a power n'.

For zero power, we have : $a^0 = 1$ (where $a \neq 0$).

For example :

$$(i) 7^0 = 1 \qquad (ii) \left(-\frac{2}{3}\right)^0 = 1 \qquad (iii) (\sqrt{7})^0 = 1$$

For negative powers, we have :

$$a^{-n} = \frac{1}{a^n} \text{ and } \frac{1}{a^{-n}} = a^n \text{ (where } a \neq 0\text{)}.$$

For example :

$$(i) 5^{-3} = \frac{1}{5^3} = \frac{1}{125} \qquad (ii) (-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}$$
$$(iii) \frac{1}{2^{-5}} = 2^5 = 32 \qquad (iv) \frac{1}{(-7)^{-3}} = (-7)^3 = -243$$

For fractional indices, remember that :

$$\sqrt[n]{a} = a^{\frac{1}{n}} \text{ and } \sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$$

For example :

$$(i) \sqrt{3} = 3^{\frac{1}{2}} \qquad (ii) \sqrt[3]{8} = 8^{\frac{1}{3}} \qquad (iii) \sqrt[4]{5^3} = 5^{\frac{3}{4}}$$

LAWS OF EXPONENTS

If a and b are any two real numbers and m and n are any two integers, then the following results hold :

1. $a^m \times a^n = a^{m+n}$

2. $a^m \div a^n = a^{m-n}$ ($a \neq 0$)

3. $(a^m)^n = a^{mn}$

4. $(ab)^m = a^m b^m$

5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ ($b \neq 0$)

6. $a^n = a^m \Rightarrow n = m$, provided $a > 0$ and $a \neq 1$



Remark

If a is any real number and n is a natural number, then

$$(-a)^n = (-1 \times a)^n = (-1)^n a^n = \begin{cases} a^n & \text{if } n \text{ is even} \\ -a^n & \text{if } n \text{ is odd} \end{cases}$$

Example 1. Use the laws of exponents to simplify the following :

$$(i) [(2^3)^4]^5 \quad (ii) [3^6 \div 3^4]^3 \quad (iii) (2^4)^3 \times 2 \times 3^0$$

$$(iv) (81)^{-1} \times 3^5 \quad (v) \left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^{-2} \quad (vi) (3^{-2} \times 5^3)^4.$$

Solution.

$$(i) [(2^3)^4]^5 = [2^{3 \times 4}]^5 = [2^{12}]^5 = 2^{12 \times 5} = 2^{60}.$$

$$(ii) [3^6 \div 3^4]^3 = [3^{6-4}]^3 = [3^2]^3 = 3^{2 \times 3} = 3^6.$$

$$(iii) (2^4)^3 \times 2 \times 3^0 = 2^{12} \times 2^1 \times 1 = 2^{12+1} = 2^{13}.$$

$$(iv) (81)^{-1} \times 3^5 = (3^4)^{-1} \times 3^5 = 3^{-4} \times 3^5 = 3^{-4+5} = 3^1 = 3.$$

$$(v) \left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^{-2} = 1 + \frac{1}{\left(\frac{2}{3}\right)^2} = 1 + \frac{1}{\frac{2^2}{3^2}} = 1 + \frac{1}{\frac{4}{9}} = 1 + \frac{9}{4} = \frac{13}{4}.$$

$$(vi) (3^{-2} \times 5^3)^4 = (3^{-2})^4 \times (5^3)^4 = 3^{-2 \times 4} \times 5^{3 \times 4} = 3^{-8} \times 5^{12}.$$

Example 2. Simplify the following :

$$(i) \frac{(-2)^5 \times (3^5)^2}{12 \times 3^7} \quad (ii) \frac{(2^3 \times 3^4)^3 \times (-5)^3}{60 \times (-2)^5} \quad (iii) \frac{(2^{-4})^2 \times 2^{-5}}{2^{-6}}.$$

Solution.

$$(i) \frac{(-2)^5 \times (3^5)^2}{12 \times 3^7} = \frac{(-2^5) \times 3^{5 \times 2}}{2 \times 2 \times 3 \times 3^7} = \frac{2^5 \times 3^{10}}{2^2 \times 3^{7+1}}$$

$$= -2^{5-2} \times 3^{10-8} = -2^3 \times 3^2 = -8 \times 9 = -72.$$

$$(ii) \frac{(2^3 \times 3^4)^3 \times (-5)^3}{60 \times (-2)^5} = \frac{(2^3)^3 \times (3^4)^3 \times (-5^3)}{2 \times 2 \times 3 \times 5 \times (-2^5)} = \frac{-2^9 \times 3^{12} \times 5^3}{-2^2 \times 3 \times 5 \times 2^5}$$

$$= 2^{9-2-5} \times 3^{12-1} \times 5^{3-1} = 2^2 \times 3^{11} \times 5^2.$$

$$(iii) \frac{(2^{-4})^2 \times 2^{-5}}{2^{-6}} = \frac{2^{-8} \times 2^{-5}}{2^{-6}} = 2^{-8+(-5)-(-6)} = 2^{-7} = \frac{1}{2^7} = \frac{1}{128}.$$

Example 3. Simplify and write in the exponential form :

$$2^3 \times 3^2 + (-11)^2 + 2^{-5} \div 2^{-8} - \left(-\frac{2}{5}\right)^0$$

Solution.

$$2^3 \times 3^2 + (-11)^2 + 2^{-5} \div 2^{-8} - \left(-\frac{2}{5}\right)^0$$

$$= 8 \times 9 + 11^2 + 2^{-5-(-8)} - 1 = 72 + 121 + 2^3 - 1$$

$$= 72 + 121 + 8 - 1 = 200$$

$$= 2 \times 2 \times 2 \times 5 \times 5 = 2^3 \times 5^2.$$

Example 4. Simplify the following :

$$(i) \frac{3x^4y^3}{18x^3y^5} \quad (ii) \left(\frac{-2x^2}{y^3}\right)^3 \quad (iii) \frac{5^{n+2} - 5^{n+1}}{5^{n+3}}.$$

Solution.

$$(i) \frac{3x^4y^3}{18x^3y^5} = \frac{3}{18} \cdot \frac{x^4}{x^3} \cdot \frac{y^3}{y^5} = \frac{1}{6} x^{4-3} \cdot y^{3-5} = \frac{1}{6} xy^{-2} = \frac{x}{6y^2}.$$

$$(ii) \left(\frac{-2x^2}{y^3}\right)^3 = (-2)^3 \cdot \frac{(x^2)^3}{(y^3)^3} = -\frac{8x^6}{y^9}.$$

$$(iii) \frac{5^{n+2} - 5^{n+1}}{5^{n+3}} = \frac{5^n \cdot 5^2 - 5^n \cdot 5^1}{5^n \cdot 5^3} = \frac{5^n(5^2 - 5^1)}{5^n \cdot 5^3} = \frac{25 - 5}{125} = \frac{20}{125} = \frac{4}{25}.$$

Example 5. Simplify and express the following in positive indices only :

$$(i) \frac{3^{-5} \times 5^{-7} \times (-2)^3}{3^4 \times 5^{-2} \times (-2)^{-3}} \quad (ii) (4^{-2} \times 3^{-3})^2 \div 6^{-4} \quad (iii) \frac{a^{-4} \times b^{-7} \times c^{-3} \times d^3}{a^{-7} \times b^{-9} \times c^3 \times d^3}.$$

Solution.

$$(i) \frac{3^{-5} \times 5^{-7} \times (-2)^3}{3^4 \times 5^{-2} \times (-2)^{-3}} = 3^{-5-4} \times 5^{-7-(-2)} \times (-2)^{3-(-3)}$$

$$= 3^{-9} \times 5^{-5} \times (-2)^6 = \frac{2^6}{3^9 \times 5^5}.$$

$$(ii) (4^{-2} \times 3^{-3})^2 \div 6^{-4} = \frac{(4^{-2})^2 \times (3^{-3})^2}{6^{-4}} = \frac{4^{-4} \times 3^{-6}}{(2 \times 3)^{-4}} = \frac{(2^2)^{-4} \times 3^{-6}}{2^{-4} \times 3^{-4}}$$

$$= 2^{-8-(-4)} \times 3^{-6-(-4)} = 2^{-4} \times 3^{-2} = \frac{1}{2^4 \times 3^2}.$$

$$(iii) \frac{a^{-4} \times b^{-7} \times c^{-3} \times d^3}{a^{-7} \times b^{-9} \times c^3 \times d^3} = a^{-4-(-7)} \times b^{-7-(-9)} \times c^{-3-3} \times d^{3-3}$$

$$= a^3 \times b^2 \times c^{-6} \times d^0 = \frac{a^3 \times b^2 \times 1}{c^6} = \frac{a^3 b^2}{c^6}.$$

Example 6. Simplify : $\frac{5^{n+2} - 6 \times 5^{n+1}}{13 \times 5^n - 2 \times 5^{n+1}}$

Solution.

$$\frac{5^{n+2} - 6 \times 5^{n+1}}{13 \times 5^n - 2 \times 5^{n+1}} = \frac{5^n \times 5^2 - 6 \times 5^n \times 5^1}{13 \times 5^n - 2 \times 5^n \times 5^1} = \frac{5^n(5^2 - 6 \times 5)}{5^n(13 - 2 \times 5)} = \frac{25 - 30}{13 - 10} = -\frac{5}{3}.$$

Example 7. Show that $\left(\frac{x^m}{x^n}\right)^{m+n} \cdot \left(\frac{x^n}{x^l}\right)^{n+l} \cdot \left(\frac{x^l}{x^m}\right)^{l+m} = 1$

Solution.

$$\left(\frac{x^m}{x^n}\right)^{m+n} \cdot \left(\frac{x^n}{x^l}\right)^{n+l} \cdot \left(\frac{x^l}{x^m}\right)^{l+m} = (x^{m-n})^{m+n} \cdot (x^{n-l})^{n+l} \cdot (x^{l-m})^{l+m}$$

$$= x^{m^2-n^2} \cdot x^{n^2-l^2} \cdot x^{l^2-m^2} = x^{m^2-n^2+n^2-l^2+l^2-m^2}$$

$$= x^0 = 1.$$

Example 8. Simplify the following :

$$(i) (27)^{2/3} \quad (ii) (125)^{-1/3} \quad (iii) \left(\frac{16}{81}\right)^{-3/4}.$$

Solution.

$$(i) (27)^{2/3} = (3^3)^{2/3} = 3^{3 \times \frac{2}{3}} = 3^2 = 9.$$

$$(ii) (125)^{-1/3} = (5^3)^{-1/3} = 5^{3 \times (-\frac{1}{3})} = 5^{-1} = \frac{1}{5^1} = \frac{1}{5}.$$

$$(iii) \left(\frac{16}{81}\right)^{-3/4} = \left(\frac{2^4}{3^4}\right)^{-3/4} = \left(\left(\frac{2}{3}\right)^4\right)^{-3/4} = \left(\frac{2}{3}\right)^{4 \times (-\frac{3}{4})} = \left(\frac{2}{3}\right)^{-3} = \frac{2^{-3}}{3^{-3}} = \frac{3^3}{2^3} = \frac{27}{8}.$$

Example 9. Simplify : $7^0 + \sqrt[5]{32} + (27)^{-2/3} + \sqrt[3]{3^6} - \left(\frac{16}{25}\right)^{-1/2}$

Solution.

$$7^0 + \sqrt[5]{32} + (27)^{-2/3} + \sqrt[3]{3^6} - \left(\frac{16}{25}\right)^{-1/2}$$

$$= 1 + (2^5)^{\frac{1}{5}} + (3^3)^{-\frac{2}{3}} + (3^6)^{\frac{1}{3}} - \left(\left(\frac{4}{5}\right)^2\right)^{-\frac{1}{2}}$$

$$\begin{aligned}
 &= 1 + 2^{5 \times \frac{1}{5}} + 3^{3 \times \left(-\frac{2}{3}\right)} + 3^{6 \times \frac{1}{3}} - \left(\frac{4}{5}\right)^{2 \times \left(-\frac{1}{2}\right)} \\
 &= 1 + 2^1 + 3^{-2} + 3^2 - \left(\frac{4}{5}\right)^{-1} \\
 &= 1 + 2 + \frac{1}{3^2} + 9 - \frac{1}{\left(\frac{4}{5}\right)^1} = 1 + 2 + \frac{1}{9} + 9 - \frac{1}{\frac{4}{5}} \\
 &= 12 + \frac{1}{9} - \frac{5}{4} = \frac{432 + 4 - 45}{36} = \frac{391}{36} = 10\frac{31}{36}.
 \end{aligned}$$

Example 10. Find n so that $2^{11} \div 2^5 = 2^{-3} \times 2^{2n-1}$.

Solution.

$$\text{Given } 2^{11} \div 2^5 = 2^{-3} \times 2^{2n-1}$$

$$\Rightarrow 2^{11-5} = 2^{-3+2n-1}$$

$$\Rightarrow 2^6 = 2^{2n-4} \Rightarrow 6 = 2n - 4$$

$$\Rightarrow 6 + 4 = 2n \Rightarrow 2n = 10 \Rightarrow n = 5$$

Hence, $n = 5$.

$$| a^n = a^m \Rightarrow n = m$$

Example 11. Determine $(8x)^x$ if $9^{x+2} = 240 + 9^x$.

Solution.

$$\text{Given } 9^{x+2} = 240 + 9^x$$

$$\Rightarrow 9^x \times 9^2 - 9^x = 240 \Rightarrow 9^x (81 - 1) = 240$$

$$\Rightarrow 80 \times 9^x = 240 \Rightarrow 9^x = \frac{240}{80} = 3$$

$$\Rightarrow (3^2)^x = 3 \Rightarrow 3^{2x} = 3^1 \Rightarrow 2x = 1$$

$$\Rightarrow x = \frac{1}{2}$$

$$| a^n = a^m \Rightarrow n = m$$

$$\therefore (8x)^x = \left(8 \times \frac{1}{2}\right)^{\frac{1}{2}} = 4^{1/2} = (2^2)^{1/2} = 2^{2 \times \frac{1}{2}} = 2^1 = 2.$$

Exercise 12

1. Find the value of the following :

(i) $2^3 \times (-3)^2$

(ii) $(2^3)^2$

(iii) $(3^5 \div 3^3)^2$

(iv) $(2^2 \times 3^{-1})^3$

(v) $\left(1\frac{1}{4}\right)^{-1}$

(vi) $8^{-2} \times 2^7$

(vii) $5^{-1} + 5^0 + 5^1$

(viii) $\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{2}\right)^{-2}$

2. Find the value of :

(i) $7^4 \times 7^8 \div (7^5)^2$

(ii) $5^0 \times 8^3 \times 4^{-2}$

(iii) $((-2)^3)^{-1}$

(iv) $\frac{2^{-3}}{5^{-3}}$

(v) $\frac{3^{-5} \times 3^7}{3^{-2}}$

(vi) $(3^{-4} \div 3^{-3})^3$

3. Find the value of :

(i) $(3^{-2} - 3^{-3}) \times (3^0 + 3^{-1})$

(ii) $\frac{(2^3 \times 3^2)^3 \times 6^{-2}}{(-2)^3 \times 3^3}$

(iii) $\frac{3^4 \times 3^{-2} \times 5^2}{120 \times (-6)^2}$

4. Simplify and write the following in exponential form :

(i) $((-2)^3)^2 + 5^{-3} \div 5^{-5} - \left(-\frac{1}{2}\right)^0$

(ii) $3^{-5} \times 3^2 \div 3^{-6} + (2^2 \times 3)^2 + \left(\frac{2}{3}\right)^{-1} + 2^{-1} + \left(\frac{1}{19}\right)^{-1}$

5. Simplify the following and write each with positive exponent :

(i) $\frac{5^{-3} \times 7^4 \times 11^{-4}}{5^{-5} \times 7^6 \times 11^3}$

(ii) $\frac{3^{-4} \times 3^2 \times 5^{-3}}{3^{-6} \times 5^{-4} \times 5^7}$

6. Simplify the following :

(i) $5x^9y^4 \div x^5y^{-2}$

(ii) $(a^3b^{-2})^{-5}$

(iii) $\left(\frac{-3xz^2}{2y^2}\right)^{-3}$

(iv) $2g^2\left(g^3 - g + \frac{1}{g} - \frac{1}{g^3}\right)$

7. Find the value of :

(i) $(125)^{2/3}$

(ii) $(32)^{-2/5}$

(iii) $\left(\frac{8}{27}\right)^{-2/3}$

(iv) $(16)^{-3/4}$

(v) $\left(\frac{81}{16}\right)^{-1/4}$

(vi) $\left(\frac{27}{8}\right)^{-4/3}$

8. Simplify the following :

(i) $(8x^3)^{1/3}$

(ii) $(27p^{-3})^{2/3}$

(iii) $\left(-3x^4 y^{-3/4}\right)^4$

(iv) $(32p^{10}q^{-15})^{1/5}$

(v) $\sqrt[3]{x^9y^6z^{12}}$

(vi) $\sqrt[4]{(p^4q^{-12})^3}$

9. Express the following in positive indices only :

(i) $(xy^{-1})^{-2}$

(ii) $\frac{a^2 \times b^{-2} \times c}{a^{-3} \times b^5 \times c}$

(iii) $(3x^{-3}y^{-1})^3 \div x^{-4}y^2z^{-1}$

(iv) $(x^{-2}y)^{1/2}(xy^{-3})^{1/3}$

10. Show that : $(x^{a+b})^{a-b} \times (x^{b+c})^{b-c} \times (x^{c+a})^{c-a} = 1$.

11. Simplify the following :

(i) $\frac{x^{2n+3} \times (x^2)^{n-1}}{x^{3n-5}}$

(ii) $\frac{5^{2(n+6)} \times (25)^{2n-7}}{(125)^{2n}}$

(iii) $\frac{x^{m+n} \times x^{n+l} \times x^{l+m}}{(x^m \times x^n \times x^l)^2}$

12. Simplify the following :

(i) $9^{5/2} - 3 \times 5^0 - \left(\frac{1}{81}\right)^{-1/2}$

(ii) $\left(\frac{1}{4}\right)^{-2} - 3 \times 8^{2/3} \times 4^0 + \left(\frac{9}{16}\right)^{-1/2}$

(iii) $(64)^{2/3} - \sqrt[3]{125} + \frac{1}{2^{-5}} + (27)^{-2/3} \times \left(\frac{25}{9}\right)^{-1/2}$

13. Show that : $\sqrt{x^{a-b}} \times \sqrt{x^{b-c}} \times \sqrt{x^{c-a}} = 1$.

14. Show that :

(i) $\left(\frac{x^m}{x^n}\right)^{\frac{1}{mn}} \left(\frac{x^n}{x^l}\right)^{\frac{1}{nl}} \left(\frac{x^l}{x^m}\right)^{\frac{1}{lm}} = 1$

(ii) $\frac{1}{1+x^{m-n}} + \frac{1}{1+x^{n-m}} = 1$

15. (i) If $2^n = \sqrt[3]{8} \div (2^3)^{2/3}$, find n .
 (ii) If $\frac{9^n \times 3^5 \times (27)^3}{3 \times (81)^4} = 27$, find n .
 (iii) If $5^{2x-1} = \frac{1}{(125)^{x-3}}$, find x .
 (iv) If $\left(\frac{a}{b}\right)^{3x+2} = \left(\frac{b}{a}\right)^{2-x}$, find x .

Summary

- If a is any real number and n is a natural number, then $a^n = a \times a \times a \dots n$ times where a is called the **base**, n is called the **exponent** or **index**.

In particular, $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$, $\frac{1}{a^{-n}} = a^n$ (where $a \neq 0$).

- For fractional indices : $\sqrt[n]{a} = a^{\frac{1}{n}}$ and $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$ ($a > 0$).

➤ Laws of exponents

If a and b are any two real numbers and m and n are any two integers, then

- | | |
|--|--|
| <input type="checkbox"/> $a^m \times a^n = a^{m+n}$ | <input type="checkbox"/> $a^m \div a^n = a^{m-n}$ ($a \neq 0$) |
| <input type="checkbox"/> $(a^m)^n = a^{mn}$ | <input type="checkbox"/> $(ab)^m = a^m b^m$ |
| <input type="checkbox"/> $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ ($b \neq 0$) | <input type="checkbox"/> $a^n = a^m \Rightarrow n = m$, provided $a > 0$ and $a \neq 1$. |

Check Your Progress

1. Find the value of :

(i) $3^0 + 3^{-1} + 3^{-2} + 3^{-3} + \sqrt[5]{(32)^2}$ (ii) $(81)^{3/4} - \left(\frac{1}{32}\right)^{-2/5} - (8)^{1/3} \times \left(\frac{1}{2}\right)^{-1} \times 5^0$.

2. Simplify : $7^{-20} - 7^{-21}$.

3. Simplify the following :

(i) $y^{2-a} \times y^{a-2}$ (ii) $\left(\frac{1}{3x}\right)^2 (2x)^3 (x^{-1})^0$ (iii) $\frac{a^{-1} + b^{-1}}{(ab)^{-1}}$.

4. Prove that $(a+b)^{-1} (a^{-1} + b^{-1}) = (ab)^{-1}$.

5. Show that $\left(\frac{x^m}{x^n}\right)^l \times \left(\frac{x^n}{x^l}\right)^m \times \left(\frac{x^l}{x^m}\right)^n = 1$.

6. If $\frac{2^{-n} \times 8^{2n+1} \times 16^{2n}}{4^{3n}} = \frac{1}{16}$, find n .