7 Factorization

7.1 POLYNOMIAL AND RELATED TERMS

An expression of the form $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$, *where* $a_0, a_1, a_2, \ldots, a_n$ *are real numbers and* n *is a non-negative integer, is called a* **polynomial** *in the variable* x.

The polynomials in the variable x are usually denoted by the symbols $f(x)$, $g(x)$, $h(x)$ etc. Thus,

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n.$$

(*i*) If $a_0 \neq 0$, then n is called the **degree** of the polynomial $f(x)$, is written as $\deg f(x) = n$. Degree of a polynomial can never be negative.

(*ii*) $a_0 x^n, a_1 x^{n-1}, a_2 x^{n-2}, \ldots, a_{n-1} x, a_n$ are called the **terms** of the polynomial $f(x)$; a_n is called the **constant term.**

(*iii*) $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are called the **coefficients** of the polynomial $f(x)$.

(*iv*) If $a_0 \neq 0$, then $a_0 x^n$ is called the **leading term** and a_0 is called the **leading coefficient** of the polynomial.

(*v*) If all the coefficients $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are zero, then $f(x)$ is called a **zero polynomial**; it is denoted by the symbol 0. The degree of the zero polynomial is never defined.

(*vi*) The degree of a polynomial is zero if and only if it is a **non-zero constant polynomial**. Thus, if $\deg f(x) = 0$, then $f(x)$ is a (non-zero) constant polynomial; it is usually denoted by c *i.e.* $f(x) = c$, $c \neq 0$.

For example :

(*i*) $2x + 7$ is a polynomial of degree 1, called a *linear polynomial.*

(*ii*) $3x^2 - 5x + \sqrt{2}$ is a polynomial of degree 2, called a *quadratic polynomial.*

(*iii*) $5x^3 + 7x^2 - \frac{4}{5}x + 11$ is a polynomial of degree 3, called a *cubic polynomial.*

(*iv*) $7x^4 - 3x^2 + \sqrt{2}\,x - \frac{2}{3}$ is a polynomial of degree 4, called a *biquadratic polynomial.*

(*v*) 7 is a polynomial of degree 0, it is a (non-zero) *constant polynomial.*

Polynomial equation

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$, $a_0 \neq 0$, be a polynomial in x of degree n, then $f(x) = 0$ i.e. $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$ is called a ***polynomial equation of degree n.*** Thus,

(*i*) $3x^2 - \sqrt{5}\,x + 7 = 0$ is a polynomial equation of degree 2.

(*ii*) $7x^3 - 3x^2 + 5x + 11 = 0$ is a polynomial equation of degree 3.

Equality of two polynomials

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ and

$g(x) = b_0 x^m + b_1 x^{m-1} + b_2 x^{m-2} + \ldots + b_{m-1} x + b_m$

be two polynomials in the variable x, then $f(x)$, $g(x)$ are said to be **equal**, written as $f(x) = g(x)$, if and only if deg $f(x)$ = deg $g(x)$ *i.e.* $n = m$ and $a_i = b_i$ for all i *i.e.* all their corresponding coefficients are equal.

Division algorithm for polynomials

If a polynomial $f(x)$ is divided by a non-zero polynomial $g(x)$ then there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)\,q(x) + r(x)$ where either $r(x) = 0$ or deg $r(x)$ < deg $g(x)$.

Here, dividend = $f(x)$, divisor = $g(x)$, quotient = $q(x)$ and remainder = $r(x)$.

Remarks

Let $f(x)$ be any polynomial and it be divided by a non-zero polynomial $g(x)$ and $r(x)$ be the remainder.

- If $g(x)$ is a quadratic polynomial then $r(x)$ is of the form $ax + b$, where a, b may be zero.
- If $g(x)$ is a linear polynomial then $r(x)$ is a constant polynomial *i.e.* $r(x) = c$, where c may be zero.

For example :

Let us divide $2x^3 - 7x^2 + 5x - 9$ by $2x - 3$.

Here, quotient = $q(x)$

$= x^2 - 2x - \frac{1}{2}$

and remainder = $r(x) = -\frac{21}{2}$.

$$\begin{array}{r|l} & x^2 - 2x - \frac{1}{2} \\ \hline 2x - 3 & 2x^3 - 7x^2 + 5x - 9 \\ & 2x^3 - 3x^2 \\ & - \quad + \\ \hline & -4x^2 + 5x \\ & -4x^2 + 6x \\ & + \quad - \\ \hline & -x - 9 \\ & -x + \frac{3}{2} \\ & + \quad - \\ \hline & -\frac{21}{2} \end{array}$$

Factor of a polynomial

A non-zero polynomial $g(x)$ is called a ***factor*** *of any polynomial $f(x)$ iff there exists some polynomial $q(x)$ such that $f(x) = g(x)\,q(x)$.*

Thus, a non-zero polynomial $g(x)$ is a factor of a polynomial $f(x)$ iff on dividing $f(x)$ by $g(x)$, the remainder = 0.

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For example :

1. As $2x^2 + 7x + 6 = (x + 2)(2x + 3)$, therefore, $x + 2$ is a factor of $2x^2 + 7x + 6$.
2. As $x^3 - 5x^2 + 7x - 3 = (x - 3)(x^2 - 2x + 1)$, therefore, $x - 3$ is a factor of $x^3 - 5x^2 + 7x - 3$.

Value of a polynomial $f(x)$ at $x = \alpha$

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$ be a polynomial in x and α be a real number, then the real number

$$a_0 \alpha^n + a_1 \alpha^{n-1} + a_2 \alpha^{n-2} + \dots + a_{n-1}\alpha + a_n$$

is called the ***value*** *of the polynomial $f(x)$ at $x = \alpha$; it is denoted by $f(\alpha)$.*

For example :

1. Let $f(x) = 3x^2 - 5x + 2$ be a quadratic polynomial in x, then
   (i) $f(2) = 3.2^2 - 5.2 + 2 = 12 - 10 + 2 = 4$
   (ii) $f(-3) = 3.(-3)^2 - 5.(-3) + 2 = 27 + 15 + 2 = 44$
   (iii) $f(1) = 3.1^2 - 5.1 + 2 = 3 - 5 + 2 = 0$ etc.
2. Let $f(x) = 2x^3 - 7x^2 - 5x + 4$ be a cubic polynomial in x, then
   (i) $f(0) = 2.0^3 - 7.0^2 - 5.0 + 4 = 0 - 0 - 0 + 4 = 4$
   (ii) $f(1) = 2.1^3 - 7.1^2 - 5.1 + 4 = 2 - 7 - 5 + 4 = -6$
   (iii) $f(-1) = 2.(-1)^3 - 7.(-1)^2 - 5.(-1) + 4 = -2 - 7 + 5 + 4 = 0$ etc.

Root of a polynomial equation

Let $f(x) = 0$, where $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$, be a polynomial equation in x and α be a real number then α is a ***root*** *of the polynomial equation $f(x) = 0$ iff $f(\alpha) = 0$ i.e. iff $a_0 \alpha^n + a_1 \alpha^{n-1} + a_2 \alpha^{n-2} + \dots + a_{n-1}\alpha + a_n = 0$.*

For example :

1. Let $f(x) = 0$, where $f(x) = 3x^2 - 5x + 2$, be a polynomial equation, then $f(1) = 3.1^2 - 5.1 + 2 = 3 - 5 + 2 = 0$, therefore, 1 is a root of the polynomial equation $f(x) = 0$ *i.e.* 1 is a root of the quadratic equation $3x^2 - 5x + 2 = 0$.
2. Let $f(x) = 0$, where $f(x) = 2x^3 - 7x^2 - 5x + 4$, be a polynomial equation, then $f(-1) = 2.(-1)^3 - 7.(-1)^2 - 5.(-1) + 4 = -2 - 7 + 5 + 4 = 0$, therefore, -1 is a root of the polynomial equation $f(x) = 0$ *i.e.* -1 is a root of the cubic equation $2x^3 - 7x^2 - 5x + 4 = 0$.

7.2 REMAINDER THEOREM

If a polynomial $f(x)$ is divided by $(x - \alpha)$, then remainder $= f(\alpha)$.

Proof. By division algorithm, for $f(x)$ and $(x - \alpha)$, there exist unique quotient $q(x)$ and constant remainder, say c, such that

$$f(x) = (x - \alpha) . q(x) + c.$$

On putting $x = \alpha$, we get

$$f(\alpha) = (\alpha - \alpha) . q(\alpha) + c$$
$$= 0 . q(\alpha) + c = 0 + c = c.$$

Hence, remainder $= f(\alpha)$.

Thus, when a polynomial $f(x)$ is divided by $(x - \alpha)$, then

remainder $= f(\alpha)$
$=$ the value of the polynomial $f(x)$ at $x = \alpha$.

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Corollary 1. *If a polynomial $f(x)$ is divided by $(x + \alpha)$, then remainder $= f(-\alpha)$.*

Proof. Since $x + \alpha = x - (-\alpha)$, therefore, when a polynomial $f(x)$ is divided by $(x + \alpha)$ *i.e.* by $(x - (-\alpha))$, then

$$\text{remainder} = f(-\alpha)$$
$$= \text{the value of the polynomial at } x = -\alpha.$$

Corollary 2. *If a polynomial $f(x)$ is divided by $(ax + b)$, $a \neq 0$, then remainder $= f\left(-\frac{b}{a}\right)$.*

Proof. To find the remainder on dividing $f(x)$ by $(ax + b)$ *i.e.* by $a\left(x+\frac{b}{a}\right)$ *i.e.* by $a\left(x-\left(-\frac{b}{a}\right)\right)$, let $Q(x)$ and r be the quotient and remainder respectively on dividing $f(x)$ by $\left(x-\left(-\frac{b}{a}\right)\right)$, then

$$f(x) = \left(x-\left(-\frac{b}{a}\right)\right) Q(x) + r$$
$$= \left(x+\frac{b}{a}\right) Q(x) + r$$
$$= (ax + b) \cdot \frac{Q(x)}{a} + r.$$

This shows that when $f(x)$ is divided by $(ax + b)$, the remainder $= r$, which is the same as obtained on dividing $f(x)$ by $\left(x-\left(-\frac{b}{a}\right)\right)$.

Hence, the required remainder $= f\left(-\frac{b}{a}\right)$

$$= \text{the value of } f(x) \text{ at } x = -\frac{b}{a}.$$

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ILLUSTRATIVE EXAMPLES

Example 1. *Find the remainder (without division) on dividing $f(x)$ by $(x + 3)$ where*

(i) $f(x) = 2x^2 - 7x - 1$ *(ii)* $f(x) = 3x^3 - 7x^2 + 11x + 1$.

Solution. Since $x + 3 = x - (-3)$, by cor. 1 to remainder theorem :

(*i*) remainder $= f(-3)$
$= 2.(-3)^2 - 7.(-3) - 1$
$= 2.9 + 21 - 1 = 18 + 21 - 1 = 38.$

(*ii*) remainder $= f(-3)$
$= 3.(-3)^3 - 7.(-3)^2 + 11.(-3) + 1$
$= 3.(-27) - 7.9 - 33 + 1$
$= -81 - 63 - 33 + 1 = -176.$

Example 2. *Using remainder theorem, find the remainder on dividing $3x^2 + 5x - 11$ by $2x + 5$.*

Solution. Let $f(x) = 3x^2 + 5x - 11$.

Since $2x + 5 = 2\left(x+\frac{5}{2}\right) = 2\left(x-\left(-\frac{5}{2}\right)\right)$, by cor. 2 to remainder theorem, we get

the required remainder $= f\left(-\frac{5}{2}\right)$

$$= 3.\left(-\frac{5}{2}\right)^2 + 5.\left(-\frac{5}{2}\right) - 11$$

$$= 3 \cdot \frac{25}{4} - \frac{25}{2} - 11 = \frac{75}{4} - \frac{25}{2} - 11$$

$$= \frac{75 - 50 - 44}{4} = -\frac{19}{4} = -4\frac{3}{4}.$$

Example 3. *Find the remainder (without division) on dividing $3x^3 + 5x^2 - 11x - 4$ by $3x + 1$.*

Solution. Let $f(x) = 3x^3 + 5x^2 - 11x - 4$.

Since $3x + 1 = 3\left(x + \frac{1}{3}\right) = 3\left(x - \left(-\frac{1}{3}\right)\right)$, by cor. 2 to remainder theorem, we get

$$\text{the required remainder} = f\left(-\frac{1}{3}\right)$$

$$= 3 \cdot \left(-\frac{1}{3}\right)^3 + 5 \cdot \left(-\frac{1}{3}\right)^2 - 11 \cdot \left(-\frac{1}{3}\right) - 4$$

$$= 3 \cdot \left(-\frac{1}{27}\right) + 5 \cdot \frac{1}{9} + \frac{11}{3} - 4$$

$$= -\frac{1}{9} + \frac{5}{9} + \frac{11}{3} - 4 = \frac{-1 + 5 + 33 - 36}{9} = \frac{1}{9}.$$

Example 4. *When $x^3 + 3x^2 - kx + 4$ is divided by $x - 2$, the remainder is k. Find the value of the constant k.*

Solution. Let $f(x) = x^3 + 3x^2 - kx + 4$.

By remainder theorem, when $f(x)$ is divided by $(x - 2)$,

$$\text{the remainder} = f(2)$$

$$= 2^3 + 3.2^2 - k.2 + 4$$

$$= 8 + 12 - 2k + 4 = 24 - 2k.$$

According to given, $24 - 2k = k$

$\Rightarrow 3k = 24 \Rightarrow k = 8.$

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Example 5. *What number should be added to $2x^3 - 3x^2 - 8x$ so that the resulting polynomial leaves the remainder 10 when divided by $2x + 1$?*

Solution. Let the number to be added be k and the resulting polynomial be $f(x)$, then $f(x) = 2x^3 - 3x^2 - 8x + k$.

By cor. 2 to remainder theorem, when $f(x)$ is divided by $2x + 1$,

$$\text{the remainder} = f\left(-\frac{1}{2}\right) \qquad \left|\ 2x + 1 = 2\left(x - \left(-\frac{1}{2}\right)\right)\right.$$

$$= 2 \cdot \left(-\frac{1}{2}\right)^3 - 3 \cdot \left(-\frac{1}{2}\right)^2 - 8\left(-\frac{1}{2}\right) + k$$

$$= 2 \cdot \left(-\frac{1}{8}\right) - 3 \cdot \frac{1}{4} + 4 + k.$$

$$= -\frac{1}{4} - \frac{3}{4} + 4 + k = 3 + k.$$

According to given, $3 + k = 10 \Rightarrow k = 7.$

Hence, the number to be added is 7.

Example 6. *The polynomials $ax^3 - 7x^2 + 7x - 2$ and $x^3 - 2ax^2 + 8x - 8$ when divided by $x - 2$ leave the same remainder. Find the value of a.*

Solution. Let $f(x) = ax^3 - 7x^2 + 7x - 2$ and $g(x) = x^3 - 2ax^2 + 8x - 8$.

By remainder theorem, when $f(x)$ is divided by $(x - 2)$, remainder $= f(2)$, and when $g(x)$ is divided by $(x - 2)$, the remainder $= g(2)$.

Since the polynomials $f(x)$ and $g(x)$ when divided by $(x-2)$ leave the same remainder,

$f(2) = g(2)$

$\Rightarrow$ $a.2^3 - 7.2^2 + 7.2 - 2 = 2^3 - 2a.2^2 + 8.2 - 8$

$\Rightarrow$ $8a - 28 + 14 - 2 = 8 - 8a + 16 - 8$

$\Rightarrow$ $16a = 32$ $\Rightarrow a = 2.$

7.3 FACTOR THEOREM

If $f(x)$ is a polynomial and α is a real number, then $(x-\alpha)$ is a factor of $f(x)$ iff $f(\alpha) = 0$.

Proof. By remainder theorem, when $f(x)$ is divided by $(x-\alpha)$, then remainder $= f(\alpha)$.

Now, $(x-\alpha)$ is a factor of $f(x)$ iff remainder $= 0$ *i.e.* iff $f(\alpha) = 0$.

Hence, $(x-\alpha)$ is a factor of $f(x)$ iff $f(\alpha) = 0$.

Corollary 1. *If $f(x)$ is a polynomial and α is a real number, then $(x+\alpha)$ is a factor of $f(x)$ iff $f(-\alpha) = 0$.*

Corollary 2. *If $f(x)$ is a polynomial and $a \neq 0$, b are real numbers, then $(ax+b)$ is a factor of $f(x)$ iff $f\left(-\frac{b}{a}\right) = 0$.*

Corollary 3. *If $f(x)$ is a polynomial and α is a real number, then $(x-\alpha)$ is a factor of $f(x)$ iff α is a root of the equation $f(x) = 0$.*

Proof. By factor theorem, $(x-\alpha)$ is a factor of $f(x)$ iff $f(\alpha) = 0$

i.e. iff α is a root of the equation $f(x) = 0$.

Hence $(x-\alpha)$ is a factor of $f(x)$ iff α is a root of the equation $f(x) = 0$.

7.4 USE OF FACTOR THEOREM

The following examples illustrate the use of the factor theorem.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that $(x-5)$ and $(2x-1)$ are factors of $2x^2 - 11x + 5$.*

Solution. Let $f(x) = 2x^2 - 11x + 5$ ...(*i*)

Putting $x = 5$ in (*i*), we get

$$f(5) = 2.5^2 - 11.5 + 5 = 2.25 - 55 + 5 = 50 - 55 + 5 = 0.$$

$\therefore$ By cor. 2 to factor theorem, $(x-5)$ is a factor of $f(x)$.

As $2x - 1 = 2\left(x - \frac{1}{2}\right)$, putting $x = \frac{1}{2}$ in (*i*), we get

$$f\left(\frac{1}{2}\right) = 2.\left(\frac{1}{2}\right)^2 - 11.\frac{1}{2} + 5 = 2.\frac{1}{4} - \frac{11}{2} + 5 = \frac{1}{2} - \frac{11}{2} + 5 = 0.$$

$\therefore$ By cor. 2 to factor theorem, $2x - 1$ is a factor of $f(x)$.

Hence, $(x-5)$ and $(2x-1)$ are factors of $2x^2 - 11x + 5$.

Example 2. *Show that $(x+2)$ and $(3x+1)$ are both factors of $6x^3 + 11x^2 - 3x - 2$.*

Solution. Let $f(x) = 6x^3 + 11x^2 - 3x - 2$ ...(*i*)

As $x + 2 = x - (-2)$, putting $x = -2$ in (*i*), we get

$$f(-2) = 6.(-2)^3 + 11.(-2)^2 - 3.(-2) - 2 = -48 + 44 + 6 - 2 = 0.$$

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$\therefore$ By cor. 1 to factor theorem, $(x + 2)$ is a factor of $f(x)$.

Since $3x + 1 = 3\left(x - \left(-\frac{1}{3}\right)\right)$, putting $x = -\frac{1}{3}$ in (i), we get

$$f\left(-\frac{1}{3}\right) = 6 \cdot \left(-\frac{1}{3}\right)^3 + 11 \cdot \left(-\frac{1}{3}\right)^2 - 3 \cdot \left(-\frac{1}{3}\right) - 2$$

$$= -\frac{2}{9} + \frac{11}{9} + 1 - 2 = 0.$$

$\therefore$ By cor. 2 to factor theorem, $(3x + 1)$ is factor of $f(x)$.

Hence, $(x + 2)$ and $(3x + 1)$ are both factors of $f(x)$.

Example 3. *Show that $(x + 3)$ is a factor of $2x^2 - x - 21$. Hence factorise $2x^2 - x - 21$.*

Solution. Let $f(x) = 2x^2 - x - 21$ ...(i)

As $x + 3 = x - (-3)$, putting $x = -3$ in (i), we get

$$f(-3) = 2 \cdot (-3)^2 - (-3) - 21 = 2.9 + 3 - 21$$

$$= 18 + 3 - 21 = 0.$$

$\therefore$ By cor. 1 to factor theorem, $x + 3$ is a factor of $2x^2 - x - 21$.

Dividing $2x^2 - x - 21$ by $x + 3$, we get $2x - 7$ as quotient and remainder = 0,

$\therefore$ $2x^2 - x - 21 = (x + 3)(2x - 7)$.

$$\begin{array}{r|l} & 2x - 7 \\ x + 3 & 2x^2 - x - 21 \\ & 2x^2 + 6x \\ & - \quad - \\ \hline & -7x - 21 \\ & -7x - 21 \\ & + \quad + \\ \hline & \times \end{array}$$

Example 4. *Show that $(x - 1)$ is a factor of $x^3 - 7x^2 + 14x - 8$. Hence, completely factorise the above expression.* (2007)

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Solution. Let $f(x) = x^3 - 7x^2 + 14x - 8$ ...(i)

Putting $x = 1$ in (i), we get

$$f(1) = 1^3 - 7.1^2 + 14.1 - 8 = 1 - 7 + 14 - 8$$

$$= 0.$$

$\therefore$ By factor theorem, $x - 1$ is a factor of $x^3 - 7x^2 + 14x - 8$.

On dividing $x^3 - 7x^2 + 14x - 8$ by $x - 1$, we get $x^2 - 6x + 8$ as the quotient and remainder = 0.

$\therefore$ The other factors of $f(x)$ are the factors of $x^2 - 6x + 8$.

Now $x^2 - 6x + 8 = x^2 - 2x - 4x + 8$

$= x(x - 2) - 4(x - 2)$

$= (x - 2)(x - 4)$.

Hence $x^3 - 7x^2 + 14x - 8 = (x - 1)(x - 2)(x - 4)$.

$$\begin{array}{r|l} & x^2 - 6x + 8 \\ x - 1 & x^3 - 7x^2 + 14x - 8 \\ & x^3 - x^2 \\ & - \quad + \\ \hline & -6x^2 + 14x \\ & -6x^2 + 6x \\ & + \quad - \\ \hline & 8x - 8 \\ & 8x - 8 \\ & - \quad + \\ \hline & \times \end{array}$$

Example 5. *Show that $2x + 7$ is a factor of $2x^3 + 7x^2 - 4x - 14$. Hence factorise $2x^3 + 7x^2 - 4x - 14$.*

Solution. Let $f(x) = 2x^3 + 7x^2 - 4x - 14$ ...(i)

As $2x + 7 = 2\left(x + \frac{7}{2}\right) = 2\left(x - \left(-\frac{7}{2}\right)\right)$, putting $x = -\frac{7}{2}$ in (i), we get

$$f\left(-\frac{7}{2}\right) = 2 \cdot \left(-\frac{7}{2}\right)^3 + 7\left(-\frac{7}{2}\right)^2 - 4 \cdot \left(-\frac{7}{2}\right) - 14$$

$= 2 \cdot \left(-\frac{343}{8}\right) + 7 \cdot \frac{49}{4} + 14 - 14$

$= -\frac{343}{4} + \frac{343}{4} + 14 - 14 = 0.$

$\therefore$ By cor. 2 to factor theorem, $2x + 7$ is a factor of $2x^3 + 7x^2 - 4x - 14$.

On dividing $2x^3 + 7x^2 - 4x - 14$ by $2x + 7$, we get $x^2 - 2$ as the quotient and remainder = 0.

$$\begin{array}{r|l} & x^2 - 2 \\ 2x + 7 & 2x^3 + 7x^2 - 4x - 14 \\ & 2x^3 + 7x^2 \\ & - \quad - \\ \hline & -4x - 14 \\ & -4x - 14 \\ & + \quad + \\ \hline & \times \end{array}$$

$\therefore$ The other factors of $f(x)$ are the factors of $x^2 - 2$.

Now $\quad x^2 - 2 = x^2 - (\sqrt{2})^2$

$= (x + \sqrt{2})(x - \sqrt{2}).$

Hence, $2x^3 + 7x^2 - 4x - 14 = (2x + 7)(x + \sqrt{2})(x - \sqrt{2})$.

Example 6. *Use factor theorem to factorise completely* $x^3 + x^2 - 4x - 4$ (2004)

Solution. Let $f(x) = x^3 + x^2 - 4x - 4$. ...(*i*)

Putting $\quad x = 2$ in (*i*), we get

$f(2) = 2^3 + 2^2 - 4.2 - 4$

$= 8 + 4 - 8 - 4 = 0.$

$\therefore$ By factor theorem, $x - 2$ is a factor of $f(x)$.

On dividing $x^3 + x^2 - 4x - 4$ by $x - 2$, we get $x^2 + 3x + 2$ as the quotient and remainder = 0.

$$\begin{array}{r|l} & x^2 + 3x + 2 \\ x - 2 & x^3 + x^2 - 4x - 4 \\ & x^3 - 2x^2 \\ & - \quad + \\ \hline & 3x^2 - 4x \\ & 3x^2 - 6x \\ & - \quad + \\ \hline & 2x - 4 \\ & 2x - 4 \\ & - \quad + \\ \hline & \times \end{array}$$

$\therefore$ The other factors of $f(x)$ are the factors of $x^2 + 3x + 2$.

Now $\quad x^2 + 3x + 2 = x^2 + x + 2x + 2$

$= x(x + 1) + 2(x + 1)$

$= (x + 1)(x + 2).$

Hence, $\quad x^3 + x^2 - 4x - 4 = (x - 2)(x + 1)(x + 2)$.

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Note

A factor of $f(x)$ can sometimes be found by 'trials'. For that, calculate $f(1)$, $f(-1)$, $f(2)$, $f(-2)$ etc. and see which one out of these is zero. If $f(1) = 0$ then $x - 1$ is a factor of $f(x)$ etc.

In the above example, $f(2) = 0$, so $x - 2$ is a factor of $f(x)$.

Example 7. *Find the value of a, if* $(x - a)$ *is a factor of* $x^3 - ax^2 + x + 2$. (2003)

Solution. Let $f(x) = x^3 - ax^2 + x + 2$.

Given $(x - a)$ is factor of $f(x)$, by cor. 2 to factor theorem, $f(a) = 0$

$\Rightarrow \quad a^3 - a.a^2 + a + 2 = 0$

$\Rightarrow \quad a^3 - a^3 + a + 2 = 0$

$\Rightarrow \quad a + 2 = 0 \Rightarrow a = -2.$

Example 8. *Find the value of 'k' if* $(x - 2)$ *is a factor of* $x^3 + 2x^2 - kx + 10$*. Hence determine whether* $(x + 5)$ *is also a factor.* (2011)

Solution. Let $f(x) = x^3 + 2x^2 - kx + 10$ ...(*i*)

Given $(x - 2)$ is a factor of $f(x)$, by factor theorem, $f(2) = 0$

$\Rightarrow \quad 2^3 + 2 \times 2^2 - k \times 2 + 10 = 0$

$\Rightarrow \quad 8 + 8 - 2k + 10 = 0$

$\Rightarrow \quad 26 - 2k = 0 \Rightarrow 2k = 26 \Rightarrow k = 13.$

Hence, the value of k is 13.

On putting $k = 13$ in (*i*), we get

$$f(x) = x^3 + 2x^2 - 13x + 10 \quad \text{...(ii)}$$

Putting $x = -5$ in (*ii*), we get

$$f(-5) = (-5)^3 + 2 \times (-5)^2 - 13 \times (-5) + 10$$
$$= -125 + 50 + 65 + 10 = 0.$$

$\therefore$ By factor theorem $(x + 5)$ is a factor of $f(x)$.

Example 9. *What number should be subtracted from $2x^3 - 5x^2 + 5x$ so that the resulting polynomial has a factor $2x - 3$?*

Solution. Let the number to be subtracted be k and the resulting polynomial be $f(x)$, then

$f(x) = 2x^3 - 5x^2 + 5x - k.$

Since $2x - 3$ is a factor of $f(x)$, by cor. 2 to factor theorem,

$$f\left(\frac{3}{2}\right) = 0 \qquad \left| 2x - 3 = 2\left(x - \frac{3}{2}\right)\right.$$

$$\Rightarrow \quad 2 \cdot \left(\frac{3}{2}\right)^3 - 5 \cdot \left(\frac{3}{2}\right)^2 + 5 \cdot \frac{3}{2} - k = 0$$

$$\Rightarrow \quad \frac{27}{4} - \frac{45}{4} + \frac{15}{2} - k = 0$$

$$\Rightarrow \quad 27 - 45 + 30 - 4k = 0$$

$$\Rightarrow \quad 4k = 12 \Rightarrow k = 3.$$

Hence, the number to be subtracted is 3.

Example 10. *Given that $x + 2$ and $x + 3$ are factors of $2x^3 + ax^2 + 7x - b$. Determine the values of a and b.* *(2009)*

Solution. Let $f(x) = 2x^3 + ax^2 + 7x - b$ ...(*i*)

Given $x + 2$ is a factor of $f(x)$, by cor. 1 to factor theorem, $f(-2) = 0$

$$\Rightarrow \quad 2 \cdot (-2)^3 + a \cdot (-2)^2 + 7 \cdot (-2) - b = 0$$

$$\Rightarrow \quad -16 + 4a - 14 - b = 0$$

$$\Rightarrow \quad 4a - b - 30 = 0 \quad \text{...(ii)}$$

Also $x + 3$ is a factor of $f(x)$, by cor. 1 to factor theorem, $f(-3) = 0$

$$\Rightarrow \quad 2 \cdot (-3)^3 + a \cdot (-3)^2 + 7 \cdot (-3) - b = 0$$

$$\Rightarrow \quad -54 + 9a - 21 - b = 0$$

$$\Rightarrow \quad 9a - b - 75 = 0 \quad \text{...(iii)}$$

Subtracting (*ii*) from (*iii*), we get

$$5a - 45 = 0 \Rightarrow a = 9.$$

Putting $a = 9$ in (*ii*), we get $36 - b - 30 = 0 \Rightarrow b = 6$.

Hence, $a = 9$ and $b = 6$.

Example 11. *Given that $(x + 1)$ and $(x - 2)$ are factors of $x^3 + ax^2 - bx - 6$, find the values of a and b. With these values of a and b, factorise the given expression completely.*

Solution. Let $f(x) = x^3 + ax^2 - bx - 6$ ...(*i*)

Given $x + 1$ is a factor of $f(x)$, by cor. 1 to factor theorem, $f(-1) = 0$

$$\Rightarrow \quad (-1)^3 + a \cdot (-1)^2 - b(-1) - 6 = 0$$

$$\Rightarrow \quad a + b - 7 = 0 \quad \text{...(ii)}$$

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Also, as $x - 2$ is a factor of $f(x)$, by factor theorem, $f(2) = 0$

$\Rightarrow \quad 2^3 + a.2^2 - b.2 - 6 = 0$

$\Rightarrow \quad 4a - 2b + 2 = 0$

$\Rightarrow \quad 2a - b + 1 = 0$ ...(*iii*)

Adding (*ii*) and (*iii*), we get $3a - 6 = 0 \Rightarrow a = 2$.

Putting $a = 2$ in (*ii*), we get $2 + b - 7 = 0 \Rightarrow b = 5$.

Hence, $a = 2, b = 5$.

Substituting these values of a and b in (*i*), we get

given expression $= f(x) = x^3 + 2x^2 - 5x - 6$ ...(*iv*)

Since $(x + 1)$ and $(x - 2)$ are factors of $f(x)$, $f(x)$ is exactly divisible by $(x + 1)(x - 2)$ *i.e.* by $x^2 - x - 2$.

On dividing $f(x)$ by $x^2 - x - 2$, we get $x + 3$ as the quotient and remainder = 0.

$$
\begin{array}{r|l}
 & x + 3 \\
x^2 - x - 2 & x^3 + 2x^2 - 5x - 6 \\
 & x^3 - 2x^2 - 2x \\
 & - \quad + \quad + \\
\hline
 & 3x^2 - 3x - 6 \\
 & 3x^2 - 3x - 6 \\
 & - \quad + \quad + \\
\hline
 & \times
\end{array}
$$

$\therefore$ The remaining factor is $(x + 3)$.

Hence, $x^3 + 2x^2 - 5x - 6 = (x + 1)(x - 2)(x + 3)$.

Example 12. *If $2x^3 + ax^2 + bx - 2$ has a factor $(x + 2)$ and leaves a remainder 7 when divided by $2x - 3$, find the values of a and b. With these values of a and b, factorise the given polynomial completely.*

Solution. Let $f(x) = 2x^3 + ax^2 + bx - 2$ ...(*i*)

Given $(x + 2)$ is a factor of $f(x)$, by cor. 1 to factor theorem, $f(-2) = 0$

$\Rightarrow \quad 2.(-2)^3 + a.(-2)^2 + b.(-2) - 2 = 0$

$\Rightarrow \quad -16 + 4a - 2b - 2 = 0 \Rightarrow 4a - 2b - 18 = 0$

$\Rightarrow \quad 2a - b - 9 = 0$ ...(*ii*)

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Also, when $f(x)$ is divided by $(2x - 3)$ *i.e.* by $2\left(x - \frac{3}{2}\right)$,

the remainder $= f\left(\frac{3}{2}\right)$ (By cor. 2 to remainder theorem)

$$= 2.\left(\frac{3}{2}\right)^3 + a.\left(\frac{3}{2}\right)^2 + b.\frac{3}{2} - 2$$

$$= \frac{27}{4} + \frac{9}{4}a + \frac{3b}{2} - 2 = \frac{9}{4}a + \frac{3}{2}b + \frac{19}{4}.$$

According to given, $\frac{9}{4}a + \frac{3}{2}b + \frac{19}{4} = 7$

$\Rightarrow \quad 9a + 6b - 9 = 0$

$\Rightarrow \quad 3a + 2b - 3 = 0$ ... (*iii*)

Multiplying (*ii*) by 2 and adding it to (*iii*), we get

$7a - 21 = 0 \Rightarrow a = 3$.

Putting $a = 3$ in (*ii*), we get

$2.3 - b - 9 = 0 \Rightarrow -b - 3 = 0 \Rightarrow b = -3$.

Hence, $a = 3, b = -3$.

Substituting these values of a and b in (*i*), the given polynomial $= f(x) = 2x^3 + 3x^2 - 3x - 2$.

On dividing $f(x)$ by $x + 2$, we get $2x^2 - x - 1$ as the quotient and remainder = 0.

$$
\begin{array}{r|l}
 & 2x^2 - x - 1 \\
x + 2 & 2x^3 + 3x^2 - 3x - 2 \\
 & 2x^3 + 4x^2 \\
 & - \quad - \\
\hline
 & -x^2 - 3x \\
 & -x^2 - 2x \\
 & + \quad + \\
\hline
 & -x - 2 \\
 & -x - 2 \\
 & + \quad + \\
\hline
 & \times
\end{array}
$$

$\therefore$ The other factors of $f(x)$ are the factors of $2x^2 - x - 1$.

Now $2x^2 - x - 1 = 2x^2 - 2x + x - 1$

$= 2x(x - 1) + 1(x - 1) = (x - 1)(2x + 1).$

Hence, $2x^3 + 3x^2 - 3x - 2 = (x + 2)(x - 1)(2x + 1).$

Exercise 7

1. Find the remainder (without division) on dividing $f(x)$ by $(x - 2)$ where
   (*i*) $f(x) = 5x^2 - 7x + 4$ (*ii*) $f(x) = 2x^3 - 7x^2 + 3.$
2. Using remainder theorem, find the remainder on dividing $f(x)$ by $(x + 3)$ where
   (*i*) $f(x) = 2x^2 - 5x + 1$ (*ii*) $f(x) = 3x^3 + 7x^2 - 5x + 1.$
3. Find the remainder (without division) on dividing $f(x)$ by $(2x + 1)$ where
   (*i*) $f(x) = 4x^2 + 5x + 3$ (*ii*) $f(x) = 3x^3 - 7x^2 + 4x + 11.$
4. (*i*) Find the remainder (without division) when $2x^3 - 3x^2 + 7x - 8$ is divided by $x - 1$. (2000)
   (*ii*) Find the remainder (without division) on dividing $3x^2 + 5x - 9$ by $(3x + 2)$.
5. When $kx^3 + 9x^2 + 4x - 10$ is divided by $(x + 1)$, the remainder is 2. Find the value of k.
6. Using remainder theorem, find the value of a if the division of
   $x^3 + 5x^2 - ax + 6$ by $(x - 1)$ leaves the remainder $2a$.
7. (*i*) What number must be subtracted from $2x^2 - 5x$ so that the resulting polynomial leaves the remainder 2 when divided by $2x + 1$?
   (*ii*) What number must be added to $2x^3 - 7x^2 + 2x$ so that the resulting polynomial leaves the remainder -2 when divided by $2x - 3$?
8. (*i*) When divided by $x - 3$ the polynomials $x^3 - px^2 + x + 6$ and $2x^3 - x^2 - (p + 3)x - 6$ leave the same remainder. Find the value of 'p'. (2010)
   (*ii*) The polynomials $kx^3 + 3x^2 - 4$ and $2x^3 - 5x + 4k$ when divided by $x + 3$ leave the same remainder. Find the value of k.
9. By factor theorem, show that $(x + 3)$ and $(2x - 1)$ are factors of $2x^2 + 5x - 3$.
10. Show that $(x - 2)$ is a factor of $3x^2 - x - 10$. Hence factorise $3x^2 - x - 10$.
11. Show that $(x - 1)$ is a factor of $x^3 - 5x^2 - x + 5$. Hence factorise $x^3 - 5x^2 - x + 5$.
12. Show that $(x - 3)$ is a factor of $x^3 - 7x^2 + 15x - 9$. Hence factorise $x^3 - 7x^2 + 15x - 9$. (2002)
13. Show that $(2x + 1)$ is a factor of $4x^3 + 12x^2 + 11x + 3$. Hence factorise $4x^3 + 12x^2 + 11x + 3$.
14. Show that $2x + 7$ is a factor of $2x^3 + 5x^2 - 11x - 14$. Hence factorise the given expression completely, using the factor theorem. (2006)
15. Use factor theorem to factorise the following polynomials completely :
    (*i*) $x^3 + 2x^2 - 5x - 6$ (*ii*) $x^3 - 13x - 12.$
16. (*i*) Use the Remainder Theorem to factorise the following expression :
    $2x^3 + x^2 - 13x + 6.$ (2010)
    (*ii*) Using the Remainder Theorem, factorise completely the following polynomial:
    $3x^3 + 2x^2 - 19x + 6$ (2012)
17. If $(2x + 1)$ is a factor of $6x^3 + 5x^2 + ax - 2$, find the value of a.
18. If $(3x - 2)$ is a factor of $3x^3 - kx^2 + 21x - 10$, find the value of k.
19. What number must be added to $4x^3 - 8x^2 + 3x$ so that the resulting polynomial has a factor $2x + 1$?
20. If $(x - 2)$ is a factor of $2x^3 - x^2 - px - 2$, then
    (*i*) find the value of p.
    (*ii*) with this value of p, factorise the above expression completely. (2008)

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21. Find the value of the constants a and b, if $(x - 2)$ and $(x + 3)$ are both factors of the expression $x^3 + ax^2 + bx - 12$. *(2001)*

22. If $(x + 2)$ and $(x - 3)$ are factors of $x^3 + ax + b$, find the values of a and b. With these values of a and b, factorise the given expression.

23. $(x - 2)$ is a factor of the expression $x^3 + ax^2 + bx + 6$. When this expression is divided by $(x - 3)$, it leaves the remainder 3. Find the values of a and b. *(2005)*

24. If $(x - 2)$ is a factor of the expression $2x^3 + ax^2 + bx - 14$ and when the expression is divided by $(x - 3)$, it leaves a remainder 52, find the values of a and b. *(2013)*

25. If $ax^3 + 3x^2 + bx - 3$ has a factor $(2x + 3)$ and leaves remainder -3 when divided by $(x + 2)$, find the values of a and b. With these values of a and b, factorise the given expression.

26. Given $f(x) = ax^2 + bx + 2$ and $g(x) = bx^2 + ax + 1$. If $x - 2$ is a factor of $f(x)$ but leaves the remainder -15 when it divides $g(x)$, find the values of a and b. With these values of a and b, factorise the expression

$$f(x) + g(x) + 4x^2 + 7x.$$

Hint

$f(2) = 0$ and $g(2) = -15$

$\Rightarrow$ $4a + 2b + 2 = 0$ and $4b + 2a + 1 = -15$

$\Rightarrow$ $2a + b + 1 = 0$ and $a + 2b + 8 = 0 \Rightarrow a = 2, b = -5$.

$\therefore$ $f(x) + g(x) + 4x^2 + 7x = (2x^2 - 5x + 2) + (-5x^2 + 2x + 1) + 4x^2 + 7x$

$= x^2 + 4x + 3.$

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CHAPTER TEST

1. Find the remainder when $2x^3 - 3x^2 + 4x + 7$ is divided by
   (*i*) $x - 2$ (*ii*) $x + 3$ (*iii*) $2x + 1$.
2. When $2x^3 - 9x^2 + 10x - p$ is divided by $(x + 1)$, the remainder is -24. Find the value of p.
3. If $(2x - 3)$ is a factor of $6x^2 + x + a$, find the value of a. With this value of a, factorise the given expression.
4. When $3x^2 - 5x + p$ is divided by $(x - 2)$, the remainder is 3. Find the value of p. Also factorise the polynomial $3x^2 - 5x + p - 3$.
5. Prove that $(5x + 4)$ is a factor of $5x^3 + 4x^2 - 5x - 4$. Hence, factorise the given polynomial completely.
6. Use factor theorem to factorise the following polynomials completely :
   (*i*) $4x^3 + 4x^2 - 9x - 9$ (*ii*) $x^3 - 19x - 30$.
7. If $x^3 - 2x^2 + px + q$ has a factor $(x + 2)$ and leaves a remainder 9 when divided by $(x + 1)$, find the values of p and q. With these values of p and q, factorise the given polynomial completely.
8. If $(x + 3)$ and $(x - 4)$ are factors of $x^3 + ax^2 - bx + 24$, find the values of a and b. With these values of a and b, factorise the given expression.
9. If $2x^3 + ax^2 - 11x + b$ leaves remainders 0 and 42 when divided by $(x - 2)$ and $(x - 3)$ respectively, find the values of a and b. With these values of a and b, factorise the given expression.
10. If $(2x + 1)$ is a factor of both the expressions $2x^2 - 5x + p$ and $2x^2 + 5x + q$, find the values of p and q. Hence find the other factors of both the polynomials.
11. When a polynomial $f(x)$ is divided by $(x - 1)$, the remainder is 5 and when it is divided by $(x - 2)$, the remainder is 7. Find the remainder when it is divided by $(x - 1)(x - 2)$.

Hint

According to given, $f(1) = 5$ and $f(2) = 7$.

Let $f(x) = (x - 1)(x - 2)\, q(x) + ax + b$ ...(*i*)

where $q(x)$ is quotient.

Putting $x = 1$, $x = 2$ we get

$f(1) = a + b$ and $f(2) = 2a + b \Rightarrow a + b = 5$ and $2a + b = 7$.

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