13

LIMITS AND DERIVATIVES

INTRODUCTION

The invention of calculus was one of the most far reaching events in the history of mathematics. It is that part of mathematics which mainly deals with the study of change in the value of a function as the value of the variable in the domain change. Calculus has a very wide range of uses in Sciences, Engineering, Economics and in many other walks of life. It is extensively used in graphical work, including the calculation of the slope of tangent to a curve at a point.

In this chapter, we shall introduce the concept of limit of a real function, study some algebra of limits and will evaluate limits of some algebraic and trigonometric functions. Then we shall define the derivative of a real function, give its geometrical and physical interpretation, study some algebra of derivatives and will obtain derivatives of some algebraic and trigonometric functions.

13.1 LIMITS

The concept of **limit** which is being introduced in this section is an essential notion for a genuine understanding of **calculus** and the reader is urged to attain a comprehensive knowledge of this idea.

Neighbourhood

The reader is familiar with various types of intervals on a real axis. Every interval of non-zero length contains infinitely many real numbers. If we consider any one of these real numbers, then the real numbers near about it are called its *neighbours*. This leads to :

Let c be any real number and δ be a small positive real number, then the open interval $(c - \delta, c + \delta)$ is called a (symmetric) ***neighbourhood*** *of c.*

Any neighbourhood of c not containing the number c is called a *deleted neighbourhood of c*. The interval $(c - \delta, c]$ is called a *left δ-neighbourhood* of c and the interval $[c, c + \delta)$ is called a *right δ-neighbourhood* of c. We can also define *deleted left* and *deleted right neighbourhood* of c.

13.1.1 Concept of Limit

Loosely speaking, a function f is said to have a *limit l* as x approaches c if $f(x)$ is *arbitrarily near* to l for all x *sufficiently near* to c. But what is meant by '*arbitrarily near*' and 'sufficiently near' ? To understand this, we study some examples giving us an intuitive feeling for what is at issue.

(*i*) Consider the function f defined by $f(x) = 3x$.

Clearly, domain of f = **R**, so that the function value $f(x)$ can be obtained for every real value of x.

Let us investigate the function values $f(x)$ when x is near to 1. Let x take on values nearer and nearer to 1 either from left side or from right side which we illustrate by means of tables 1 and 2 given below :

Table 1

x	0.9	0.99	0.999	0.9999	...
$f(x)$	2.7	2.97	2.997	2.9997	...

Table 2

x	1.1	1.01	1.001	1.0001	...
$f(x)$	3.3	3.03	3.003	3.0003	...

A portion of the graph of the function f is shown in fig. 13.1.

It is clear from the table 1 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value 3 for all x *sufficiently near* the number 1 (from left). We express this fact by saying that $\underset{x \to 1^-}{\text{Lt}} f(x) = 3$.

Also, it is clear from the table 2 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value 3 for all x *sufficiently near* the number 1 (from right). We express this fact by saying that $\underset{x \to 1^+}{\text{Lt}} f(x) = 3$.

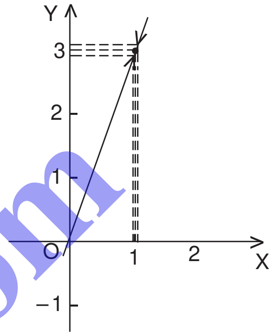

Fig. 13.1.

In fact, as $x \to 1$ either from left or from right, $f(x) \to 3$, as shown in fig. 13.1 by two arrow heads. We express this fact by saying that $\underset{x \to 1}{\text{Lt}} f(x) = 3$.

(*ii*) Consider the function f defined by $f(x) = x^2$.

Clearly, domain of $f = \mathbf{R}$, so that the function value $f(x)$ can be obtained for every value of x.

Let us investigate the function values $f(x)$ when x is near to 0. Let x take on values nearer and nearer to 0 either from left side or from right side which we illustrate by means of tables 3 and 4 given below :

Table 3

x	-0.5	-0.1	-0.01	-0.001	...
$f(x)$	0.25	0.01	0.0001	0.000001	...

Table 4

x	0.5	0.1	0.01	0.001	...
$f(x)$	0.25	0.01	0.0001	0.000001	...

A portion of the graph of the function f is shown in fig. 13.2.

It is clear from the table 3 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value 0 for all x *sufficiently near* the number 0 (from left). We express this fact by saying that $\underset{x \to 0^-}{\text{Lt}} f(x) = 0$.

Also, it is clear from the table 4 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value 0 for all x *sufficiently near* the number 0 (from right). We express this fact by saying that $\underset{x \to 0^+}{\text{Lt}} f(x) = 0$.

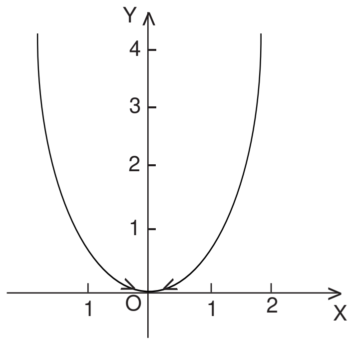

Fig. 13.2.

In fact, as $x \to 0$ either from left or from right, $f(x) \to 0$, as shown in fig. 13.2 by two arrow heads. We express this fact by saying that $\underset{x \to 0}{\text{Lt}} f(x) = 0$.

(*iii*) Consider the function f defined by $f(x) = \dfrac{2x^2 - 5x + 2}{x - 2}$.

Clearly, $D_f = \mathbf{R} - \{2\}$, so that the function value $f(x)$ can be obtained for every value of x except for $x = 2$. Moreover,

$$f(x) = \frac{2x^2 - 5x + 2}{x - 2} = \frac{(2x - 1)(x - 2)}{x - 2} = 2x - 1,\ x \neq 2.$$

Let us investigate the function values $f(x)$ when x is near to 2 but not equal to 2. Let x take on values nearer and nearer to 2 either from the left side or from the right side which we illustrate by means of tables 5 and 6 given below:

Table 5

x	1.9	1.99	1.999	1.9999	...
$f(x)$	2.8	2.98	2.998	2.9998	...

Table 6

x	2.1	2.01	2.001	2.0001	...
$f(x)$	3.2	3.02	3.002	3.0002	...

A portion of the graph of the function f is shown in fig. 13.3.

It is clear from the table 5 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value 3 for all x *sufficiently near* the number 2 (from left). We express this fact by saying that $\underset{x \to 2^-}{\text{Lt}} f(x) = 3$.

Also, it is clear from the table 6 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value 3 for all x *sufficiently near* the number 2 (from right). We express this fact by saying that $\underset{x \to 2^+}{\text{Lt}} f(x) = 3$.

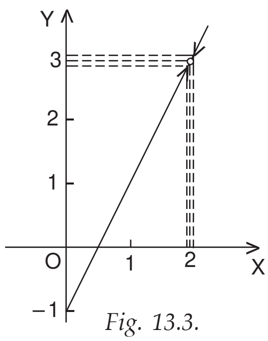

Fig. 13.3.

In fact, as $x \to 2$ either from the left or from the right, $f(x) \to 3$, as shown in fig. 13.3 by two arrow heads. We express this fact by saying that $\underset{x \to 2}{\text{Lt}} f(x) = 3$.

REMARK

For a function f to have a limit as $x \to c$, it is not necessary that the function f be defined at the point $x = c$. When finding the limit, we consider the values of the function f in the deleted neighbourhood of c only.

(*iv*) Consider the function f defined by $f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1}, & x \neq -1 \\ 2, & x = -1 \end{cases}$.

Clearly, $D_f = \mathbf{R}$, so that the function value $f(x)$ can be obtained for every value of x. Moreover,

$$f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x - 1)(x + 1)}{x + 1} = x - 1,\ x \neq -1.$$

Let us investigate the function values $f(x)$ when x is near to -1. Let x take on values nearer and nearer to -1 either from left side or from right side which we illustrate by means of tables 7 and 8 given on the next page :

Table 7

x	-1.1	-1.01	-1.001	-1.0001	...
$f(x)$	-2.1	-2.01	-2.001	-2.0001	...

Table 8

x	-0.9	-0.99	-0.999	-0.9999	...
$f(x)$	-1.9	-1.99	-1.999	-1.9999	...

A portion of the graph of the function f is shown in fig. 13.4.

It is clear from the table 7 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value -2 for all x *sufficiently near* the number -1 (from left). We express this fact by saying that $\underset{x \to -1^-}{\text{Lt}} f(x) = -2$.

Also, it is clear from the table 8 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value -2 for all x *sufficiently near* the number -1 (from right). We express this fact by saying that $\underset{x \to -1^+}{\text{Lt}} f(x) = -2$.

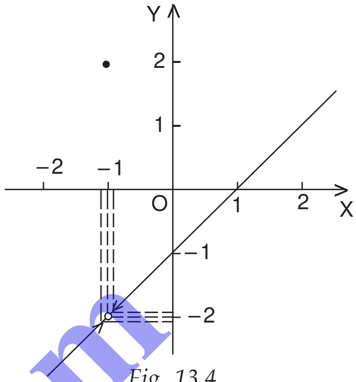

Fig. 13.4.

In fact, as $x \to -1$ either from left or from right, $f(x) \to -2$, as shown in fig. 13.4 by two arrow heads. We express this fact by saying that $\underset{x \to -1}{\text{Lt}} f(x) = -2$.

REMARK

In the above example, we notice that the function is defined at $x = -1$ and its value is 2 *i.e.* $f(-1) = 2$ (given) but $\underset{x \to -1}{\text{Lt}} f(x) = -2$. Thus, the limit of a function f as $x \to c$ may be different from the value of the function at $x = c$ *i.e.* $\underset{x \to c}{\text{Lt}} f(x)$ may not be equal to $f(c)$.

(*v*) Consider the function f defined by $f(x) = \begin{cases} x - 2 & , \ x < 0 \\ 0 & , \ x = 0 \\ x + 2 & , \ x > 0. \end{cases}$ (*NCERT*)

Clearly, $D_f = \mathbf{R}$, so that the function value $f(x)$ can be obtained for every real value of x.

Let us investigate the function values $f(x)$ when is near to 0 from left. We notice that when $x < 0$, the function values are dictated by $x - 2$, so we need to evaluate $x - 2$ when x is negative. Let x take on values nearer and nearer to 0 from left which we illustrate by means of table 9 given below :

Table 9

x	-0.5	-0.1	-0.01	-0.001	-0.0001	...
$f(x)$	-2.5	-2.1	-2.01	-2.001	-2.0001	...

A portion of the graph of the function f is shown in fig. 13.5.

It is clear from the table 9 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value -2 for all x *sufficiently near* the number 0 (from left). We express this fact by saying that $\underset{x \to 0^-}{\text{Lt}} f(x) = -2$.

Now, let us investigate the function values $f(x)$ when x is near to 0 from right. We notice that when $x > 0$, the function values are dictated by $x + 2$, so we need to evaluate $x + 2$ when x is positive. Let x take on

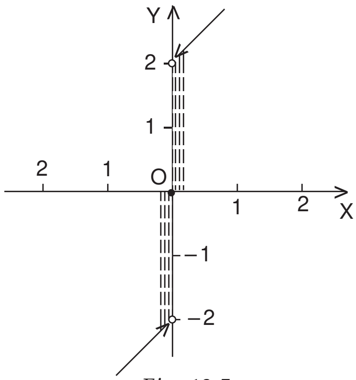

Fig. 13.5.

values nearer and nearer to 0 from right which we illustrate by means of table 10 given below :

Table 10

x	0.5	0.1	0.01	0.001	0.0001	...
$f(x)$	2.5	2.1	2.01	2.001	2.0001	...

It is clear from the table 10 or from the graph of the function f that $f(x)$ is *arbitrarily near* the value 2 for all x *sufficiently near* the number 0 (from right). We express this fact by saying that $\underset{x \to 0^+}{\text{Lt}} f(x) = 2$.

In this example, we observe that $\underset{x \to 0^-}{\text{Lt}} f(x) \neq \underset{x \to 0^+}{\text{Lt}} f(x)$, which we express by saying that $\underset{x \to 0}{\text{Lt}} f(x)$ does not exist.

Formally, a function f is said to tend to a ***limit*** *l as x approaches c iff the difference between f (x) and l can be made as small as we please by taking x sufficiently near c and we write it as* $\underset{x \to c}{\textbf{Lt}} \, f(x) = l$.

Theorem. $\underset{x \to c}{\textbf{Lt}} f(x) = l$ *iff* $\underset{x \to c^-}{\textbf{Lt}} f(x) = l = \underset{x \to c^+}{\textbf{Lt}} f(x)$.

We assert that for the existence of $\underset{x \to c}{\textbf{Lt}} f(x)$***, it is essential that*** $\underset{x \to c^-}{\textbf{Lt}} f(x)$ ***and*** $\underset{x \to c^+}{\textbf{Lt}} f(x)$ ***must both exist separately and be equal. This equal value is the limit of the function.***

13.1.2 Some standard results on limits

We state some standard results (without proof) which enable us to evaluate the limits in many problems :

1. ***(i)*** $\underset{x \to c}{\textbf{Lt}} \, \alpha = \alpha$, ***where*** α ***is a fixed real number.***

(ii) $\underset{x \to c}{\textbf{Lt}} \, x^n = c^n$, ***for all*** $n \in N$.

(iii) $\underset{x \to c}{\textbf{Lt}} \, f(x) = f(c)$, ***where*** $f(x)$ ***is a real polynomial in x.***

(iv) $\underset{x \to c}{\textbf{Lt}} \, |x| = |c|$.

2. Algebra of limits

Let f, g be two functions such that $\underset{x \to c}{\text{Lt}} f(x) = l$ and $\underset{x \to c}{\text{Lt}} g(x) = m$, then

(*i*) $\underset{x \to c}{\text{Lt}} (\alpha f(x)) = \alpha \,.\, \underset{x \to c}{\text{Lt}} f(x) = \alpha l$, for all $\alpha \in \mathbf{R}$.

(*ii*) $\underset{x \to c}{\text{Lt}} (f(x) + g(x)) = \underset{x \to c}{\text{Lt}} f(x) + \underset{x \to c}{\text{Lt}} g(x) = l + m$.

(*iii*) $\underset{x \to c}{\text{Lt}} (f(x) - g(x)) = \underset{x \to c}{\text{Lt}} f(x) - \underset{x \to c}{\text{Lt}} g(x) = l - m$.

(*iv*) $\underset{x \to c}{\text{Lt}} (f(x)\, g(x)) = \underset{x \to c}{\text{Lt}} f(x) \,.\, \underset{x \to c}{\text{Lt}} g(x) = lm$.

(*v*) $\underset{x \to c}{\text{Lt}} \dfrac{f(x)}{g(x)} = \dfrac{\underset{x \to c}{\text{Lt}} f(x)}{\underset{x \to c}{\text{Lt}} g(x)} = \dfrac{l}{m}$, provided $m \neq 0$.

(*vi*) $\underset{x \to c}{\text{Lt}} \dfrac{1}{f(x)} = \dfrac{1}{\underset{x \to c}{\text{Lt}} f(x)} = \dfrac{1}{l}$, provided $l \neq 0$.

(*vii*) $\underset{x \to c}{\text{Lt}} (f(x))^n = \left(\underset{x \to c}{\text{Lt}} f(x)\right)^n = l^n$, for all $n \in \mathbf{N}$.

REMARK

The converse of the above four basic results *i.e.* (*ii*) to (*v*) may not be true.

3. Sandwich Theorem (or squeeze principle)

If f, g, h are functions such that $f(x) \le g(x) \le h(x)$ for all x in some neighbourhood of c (except possibly at $x = c$) and if $\underset{x \to c}{Lt} f(x) = l = \underset{x \to c}{Lt} h(x)$, then $\underset{x \to c}{Lt} g(x) = l$.

4. If $\underset{x \to c}{\text{Lt}} f(x) = 0$ and $g(x)$ is bounded in a deleted neighbourhood of c, then

$$\underset{x \to c}{\text{Lt}} (f(x)\, g(x)) = 0.$$

A function f is said to be bounded in (a, b) iff there exist some real numbers k_1, k_2 such that $k_1 \le f(x) \le k_2$ for all $x \in (a, b)$.

5. (*i*) $\underset{x \to c^-}{\text{Lt}} f(x) = \underset{h \to 0^-}{\text{Lt}} f(c + h)$ (*ii*) $\underset{x \to c^+}{\text{Lt}} f(x) = \underset{h \to 0^+}{\text{Lt}} f(c + h)$

(*iii*) $\underset{x \to c}{\text{Lt}} f(x) = \underset{h \to 0}{\text{Lt}} f(c + h)$.

13.1.3 Some important theorems on limits

Theorem 1. $\underset{x \to 0^-}{\boldsymbol{Lt}} \boldsymbol{f(x)} = \underset{x \to 0^+}{\boldsymbol{Lt}} \boldsymbol{f(-x)}$.

Proof. Let $x = -y$, so that when $x \to 0^-$, $y \to 0^+$.

$$\therefore \quad \underset{x \to 0^-}{\text{Lt}} f(x) = \underset{y \to 0^+}{\text{Lt}} f(-y) = \underset{x \to 0^+}{\text{Lt}} f(-x) \qquad \text{(by merely changing the variable)}$$

Theorem 2. $\underset{x \to a}{\boldsymbol{Lt}} \dfrac{\boldsymbol{x^n - a^n}}{\boldsymbol{x - a}} = \boldsymbol{na^{n-1}}$, $\boldsymbol{n \in}$ **N.**

Proof. Let $x = a + h$, so that when $x \to a$, $h \to 0$.

$$\therefore \quad \underset{x \to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = \underset{h \to 0}{\text{Lt}} \frac{(a + h)^n - a^n}{h}.$$

By using binomial expansion for positive integral index, we get

$$\underset{x \to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = \underset{h \to 0}{\text{Lt}} \frac{\left(a^n + n\,a^{n-1} h + \dfrac{n(n-1)}{\lfloor 2} a^{n-2} h^2 + \ldots + h^n\right) - a^n}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{h\left(n\,a^{n-1} + \dfrac{n(n-1)}{\lfloor 2} a^{n-2} h + \ldots + h^{n-1}\right)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \left(n\,a^{n-1} + \frac{n(n-1)}{\lfloor 2} a^{n-2} h + \ldots + h^{n-1}\right) \qquad (\because h \ne 0)$$

$$= n\,a^{n-1} + 0 = n\,a^{n-1}.$$

REMARK

The above result is true even when n is a rational number.

However, it is assumed that the function is defined in the neighbourhood of a (except possibly at $x = a$), otherwise, the question of finding the limit does not arise.

13.1.4 Evaluation of limits of algebraic functions

In this section, we shall use the above standard results and the theorems to evaluate limits of some algebraic functions.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following limits :*

(i) $\underset{x\to\pi}{Lt}\left(x-\frac{22}{7}\right)$ (NCERT) (ii) $\underset{x\to 1}{Lt}(x^3-x^2+1)$ (NCERT)

(iii) $\underset{x\to 3}{Lt}\, x(x+1)$ (NCERT) (iv) $\underset{x\to -1}{Lt}(1+x+x^2+\ldots+x^{10})$ (NCERT)

(v) $\underset{x\to -2}{Lt}((2x^2+5)^2-7)$ (vi) $\underset{x\to -1}{Lt}(x^{24}+3x^9+1)^{100}$.

Solution. (*i*) $\underset{x\to\pi}{\text{Lt}}\left(x-\frac{22}{7}\right)=\pi-\frac{22}{7}$.

(*ii*) $\underset{x\to 1}{\text{Lt}}(x^3-x^2+1)=1^3-1^2+1=1-1+1=1.$

(*iii*) $\underset{x\to 3}{\text{Lt}}\, x(x+1)=\underset{x\to 3}{\text{Lt}}(x^2+x)=3^2+3=9+3=12.$

(*iv*) $\underset{x\to -1}{\text{Lt}}(1+x+x^2+\ldots+x^{10})=1+(-1)+(-1)^2+(-1)^3+\ldots+(-1)^{10}$

$=1-1+1-1+\ldots+1=1.$

(*v*) $\underset{x\to -2}{\text{Lt}}((2x^2+5)^2-7)=(2(-2)^2+5)^2-7=13^2-7$

$=169-7=162.$

(*vi*) $\underset{x\to -1}{\text{Lt}}(x^{24}+3x^9+1)^{100}=((-1)^{24}+3(-1)^9+1)^{100}$

$=(1-3+1)^{100}=(-1)^{100}=1.$

Example 2. *Evaluate the following limits :*

(i) $\underset{x\to 4}{Lt}\frac{4x+3}{x-2}$ (NCERT) (ii) $\underset{x\to 1}{Lt}\frac{x^2+1}{x+100}$ (NCERT)

(iii) $\underset{x\to -1}{Lt}\frac{x^{10}+x^5+1}{x-1}$ (NCERT) (iv) $\underset{x\to 1}{Lt}\frac{ax^2+bx+c}{cx^2+bx+a}$, $a+b+c\neq 0$. (NCERT)

Solution. (*i*) $\underset{x\to 4}{\text{Lt}}\frac{4x+3}{x-2}=\frac{\underset{x\to 4}{\text{Lt}}(4x+3)}{\underset{x\to 4}{\text{Lt}}(x-2)}=\frac{4\times 4+3}{4-2}=\frac{19}{2}.$

(*ii*) $\underset{x\to 1}{\text{Lt}}\frac{x^2+1}{x+100}=\frac{\underset{x\to 1}{\text{Lt}}(x^2+1)}{\underset{x\to 1}{\text{Lt}}(x+100)}=\frac{1^2+1}{1+100}=\frac{2}{101}.$

(*iii*) $\underset{x\to -1}{\text{Lt}}\frac{x^{10}+x^5+1}{x-1}=\frac{\underset{x\to -1}{\text{Lt}}(x^{10}+x^5+1)}{\underset{x\to -1}{\text{Lt}}(x-1)}=\frac{(-1)^{10}+(-1)^5+1}{-1-1}=\frac{1-1+1}{-2}=-\frac{1}{2}.$

(*iv*) $\underset{x\to 1}{\text{Lt}}\frac{ax^2+bx+c}{cx^2+bx+a}=\frac{a\times 1^2+b\times 1+c}{c\times 1^2+b\times 1+a}=\frac{a+b+c}{a+b+c}=1.$

Example 3. *Evaluate the following limits :*

(i) $\underset{x\to 2}{Lt}\frac{3x^2-x-10}{x^2-4}$ (NCERT) (ii) $\underset{x\to 1}{Lt}\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$

(iii) $\underset{x\to 3}{Lt}\frac{x^4-81}{2x^2-5x-3}$ (NCERT) (iv) $\underset{x\to\sqrt{2}}{Lt}\frac{x^4-4}{x^2+3\sqrt{2}x-8}$. (*NCERT Examplar Problems*)

Solution. (*i*) $\underset{x\to 2}{\text{Lt}} \frac{3x^2 - x - 10}{x^2 - 4} = \underset{x\to 2}{\text{Lt}} \frac{(x-2)(3x+5)}{(x-2)(x+2)}$

$= \underset{x\to 2}{\text{Lt}} \frac{3x+5}{x+2}$ ($\because$ $x \neq 2$, so $x - 2$ can be cancelled)

$= \frac{3 \times 2 + 5}{2 + 2} = \frac{11}{4}$.

(*ii*) $\underset{x\to 1}{\text{Lt}} \frac{(2x-3)(\sqrt{x}-1)}{2x^2 + x - 3} = \underset{x\to 1}{\text{Lt}} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$

$= \underset{x\to 1}{\text{Lt}} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}-1)(\sqrt{x}+1)} = \underset{x\to 1}{\text{Lt}} \frac{2x-3}{(2x+3)(\sqrt{x}+1)}$

($\because$ $x \to 1 \Rightarrow \sqrt{x} \neq 1 \Rightarrow \sqrt{x} - 1 \neq 0$)

$= \frac{2 \times 1 - 3}{(2 \times 1 + 3)(\sqrt{1} + 1)} = \frac{-1}{5 \times 2} = -\frac{1}{10}$.

(*iii*) $\underset{x\to 3}{\text{Lt}} \frac{x^4 - 81}{2x^2 - 5x - 3} = \underset{x\to 3}{\text{Lt}} \frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$ ($\because$ $x^4 - 81 = (x^2)^2 - 9^2 = (x^2 - 9)\ (x^2 + 9)$)

$= \underset{x\to 3}{\text{Lt}} \frac{(x+3)(x^2+9)}{2x+1}$ ($\because$ $x \neq 3$, so $x - 3$ can be cancelled)

$= \frac{(3+3)(3^2+9)}{2 \times 3 + 1} = \frac{6 \times 18}{7} = \frac{108}{7}$.

(*iv*) $\underset{x\to \sqrt{2}}{\text{Lt}} \frac{x^4 - 4}{x^2 + 3\sqrt{2}\,x - 8} = \underset{x\to \sqrt{2}}{\text{Lt}} \frac{(x^2+2)(x^2-2)}{(x+4\sqrt{2})(x-\sqrt{2})} = \underset{x\to \sqrt{2}}{\text{Lt}} \frac{(x^2+2)(x+\sqrt{2})(x-\sqrt{2})}{(x+4\sqrt{2})(x-\sqrt{2})}$

$= \underset{x\to \sqrt{2}}{\text{Lt}} \frac{(x^2+2)(x+\sqrt{2})(x-\sqrt{2})}{(x+4\sqrt{2})(x-\sqrt{2})}$ ($\because$ $x \to \sqrt{2} \Rightarrow x - \sqrt{2} \neq 0$)

$= \frac{(2+2)(\sqrt{2}+\sqrt{2})}{\sqrt{2}+4\sqrt{2}} = \frac{8\sqrt{2}}{5\sqrt{2}} = \frac{8}{5}$.

Example 4. *Evaluate the following limits :*

(*i*) $\underset{x\to -2}{Lt} \frac{\frac{1}{x} + \frac{1}{2}}{x+2}$ *(NCERT)* (*ii*) $\underset{x\to 1}{Lt} \left[\frac{x-2}{x^2 - x} - \frac{1}{x^3 - 3x^2 + 2x}\right]$. *(NCERT)*

Solution. (*i*) $\underset{x\to -2}{\text{Lt}} \frac{\frac{1}{x} + \frac{1}{2}}{x+2} = \underset{x\to -2}{\text{Lt}} \frac{\frac{2+x}{2x}}{x+2} = \underset{x\to -2}{\text{Lt}} \left(\frac{x+2}{2x} \times \frac{1}{x+2}\right)$

$= \underset{x\to -2}{\text{Lt}} \frac{1}{2x} = \frac{1}{2 \times (-2)} = -\frac{1}{4}$.

(*ii*) $\underset{x\to 1}{\text{Lt}} \left[\frac{x-2}{x^2 - x} - \frac{1}{x^3 - 3x^2 + 2x}\right] = \underset{x\to 1}{\text{Lt}} \left[\frac{x-2}{x(x-1)} - \frac{1}{x(x-1)(x-2)}\right]$

$= \underset{x\to 1}{\text{Lt}} \frac{(x-2)^2 - 1}{x(x-1)(x-2)} = \underset{x\to 1}{\text{Lt}} \frac{(x-2+1)(x-2-1)}{x(x-1)(x-2)}$

$= \underset{x\to 1}{\text{Lt}} \frac{(x-1)(x-3)}{x(x-1)(x-2)} = \underset{x\to 1}{\text{Lt}} \frac{x-3}{x(x-2)}$

$= \frac{1-3}{1(1-2)} = \frac{-2}{-1} = 2.$

Example 5. *Evaluate the following limits :*

(i) $\underset{x\to 2}{\text{Lt}} \dfrac{x^4 - 16}{x - 2}$ (ii) $\underset{x\to 1}{\text{Lt}} \dfrac{x^{15} - 1}{x^{10} - 1}$ (NCERT)

(iii) $\underset{z\to 1}{\text{Lt}} \dfrac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$ (NCERT) (iv) $\underset{x\to \frac{1}{2}}{\text{Lt}} \dfrac{8x^3 - 1}{16x^4 - 1}$

(v) $\underset{x\to -3}{\text{Lt}} \dfrac{x^3 + 27}{x^5 + 243}$ (NCERT Examplar Problems)

(vi) $\underset{x\to 0}{\text{Lt}} \dfrac{(1 + x)^5 - 1}{x}$ (NCERT) (vii) $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{1 + x} - 1}{x}$ (NCERT)

Solution. (*i*) $\underset{x\to 2}{\text{Lt}} \dfrac{x^4 - 16}{x - 2} = \underset{x\to 2}{\text{Lt}} \dfrac{x^4 - 2^4}{x - 2}$

$= 4 \times 2^{4-1}$ $\left(\text{Using } \underset{x\to a}{\text{Lt}} \dfrac{x^n - a^n}{x - a} = na^{n-1}\right)$

$= 4 \times 2^3 = 4 \times 8 = 32.$

(*ii*) $\underset{x\to 1}{\text{Lt}} \dfrac{x^{15} - 1}{x^{10} - 1} = \underset{x\to 1}{\text{Lt}} \left(\dfrac{x^{15} - 1}{x - 1} \times \dfrac{x - 1}{x^{10} - 1}\right) = \dfrac{\underset{x\to 1}{\text{Lt}} \dfrac{x^{15} - 1}{x - 1}}{\underset{x\to 1}{\text{Lt}} \dfrac{x^{10} - 1}{x - 1}}$

$= \dfrac{15 \times 1^{15-1}}{10 \times 1^{10-1}}$ $\left(\text{Using } \underset{x\to a}{\text{Lt}} \dfrac{x^n - a^n}{n - a} = na^{n-1}\right)$

$= \dfrac{15 \times 1}{10 \times 1} = \dfrac{3}{2}.$

(*iii*) $\underset{z\to 1}{\text{Lt}} \dfrac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = \underset{z\to 1}{\text{Lt}} \left(\dfrac{z^{\frac{1}{3}} - 1}{z - 1} \times \dfrac{z - 1}{z^{\frac{1}{6}} - 1}\right) = \dfrac{\underset{z\to 1}{\text{Lt}} \dfrac{z^{\frac{1}{3}} - 1}{z - 1}}{\underset{z\to 1}{\text{Lt}} \dfrac{z^{\frac{1}{6}} - 1}{z - 1}}$

$= \dfrac{\frac{1}{3} \times (1)^{\frac{1}{3}-1}}{\frac{1}{6} \times (1)^{\frac{1}{6}-1}} = \dfrac{\frac{1}{3} \times 1}{\frac{1}{6} \times 1} = \dfrac{6}{3} = 2.$

(*iv*) $\underset{x\to \frac{1}{2}}{\text{Lt}} \dfrac{8x^3 - 1}{16x^4 - 1} = \underset{x\to \frac{1}{2}}{\text{Lt}} \dfrac{8\left(x^3 - \frac{1}{8}\right)}{16\left(x^4 - \frac{1}{16}\right)} = \dfrac{1}{2} \underset{x\to \frac{1}{2}}{\text{Lt}} \dfrac{x^3 - \left(\frac{1}{2}\right)^3}{x^4 - \left(\frac{1}{2}\right)^4} = \dfrac{1}{2} \cdot \dfrac{\underset{x\to \frac{1}{2}}{\text{Lt}} \dfrac{x^3 - \left(\frac{1}{2}\right)^3}{x - \frac{1}{2}}}{\underset{x\to \frac{1}{2}}{\text{Lt}} \dfrac{x^4 - \left(\frac{1}{2}\right)^4}{x - \frac{1}{2}}}$

$= \dfrac{1}{2} \cdot \dfrac{3 \times \left(\frac{1}{2}\right)^{3-1}}{4 \times \left(\frac{1}{2}\right)^{4-1}} = \dfrac{1}{2} \cdot \dfrac{3 \times \frac{1}{4}}{4 \times \frac{1}{8}} = \dfrac{1}{2} \times \dfrac{3}{4} \times \dfrac{2}{1} = \dfrac{3}{4}.$

(*v*) $\underset{x \to -3}{\text{Lt}} \dfrac{x^3 + 27}{x^5 + 243} = \underset{x \to -3}{\text{Lt}} \dfrac{x^3 - (-27)}{x^5 - (-243)} = \underset{x \to -3}{\text{Lt}} \dfrac{x^3 - (-3)^3}{x^5 - (-3)^5} = \underset{x \to -3}{\text{Lt}} \dfrac{\dfrac{x^3 - (-3)^3}{x - (-3)}}{\dfrac{x^5 - (-3)^5}{x - (-3)}}$

$$= \frac{3.(-3)^{3-1}}{5.(-3)^{5-1}} = \frac{3}{5} \cdot \frac{(-3)^2}{(-3)^4} = \frac{3}{5} \cdot \frac{9}{81} = \frac{1}{15}.$$

(*vi*) Let $1 + x = h \Rightarrow x = h - 1$, so that when $x \to 0$, $h \to 1$.

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \frac{(1+x)^5 - 1}{x} = \underset{h \to 1}{\text{Lt}} \frac{h^5 - 1}{h - 1} = \underset{h \to 1}{\text{Lt}} \frac{h^5 - 1^5}{h - 1}$$

$$= 5 \times (1)^{5-1} = 5 \times 1 = 5.$$

(*vii*) Let $1 + x = h \Rightarrow x = h - 1$, so that when $x \to 0$, $h \to 1$.

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \frac{\sqrt{1+x} - 1}{x} = \underset{h \to 1}{\text{Lt}} \frac{\sqrt{h} - 1}{h - 1} = \underset{h \to 1}{\text{Lt}} \frac{h^{\frac{1}{2}} - 1^{\frac{1}{2}}}{h - 1} = \frac{1}{2} \times (1)^{\frac{1}{2} - 1} = \frac{1}{2} \times 1 = \frac{1}{2}.$$

Alternatively

$$\underset{x \to 0}{\text{Lt}} \frac{\sqrt{1+x} - 1}{x} = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{1+x} - 1}{x} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} = \underset{x \to 0}{\text{Lt}} \frac{(1+x) - 1}{x(\sqrt{1+x} + 1)} = \underset{x \to 0}{\text{Lt}} \frac{x}{x(\sqrt{1+x} + 1)}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{1+1} = \frac{1}{2}.$$

Example 6. *Evaluate the following limits:*

(*i*) $\underset{x \to 0}{Lt} \dfrac{\sqrt{1+3x} - \sqrt{1-3x}}{x}$ (*ii*) $\underset{h \to 0}{Lt} \dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}\right)$.

Solution. (*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{1+3x} - \sqrt{1-3x}}{x} = \underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{1+3x} - \sqrt{1-3x}}{x} \times \dfrac{\sqrt{1+3x} + \sqrt{1-3x}}{\sqrt{1+3x} + \sqrt{1-3x}}$

$$= \underset{x \to 0}{\text{Lt}} \frac{(1+3x) - (1-3x)}{x(\sqrt{1+3x} + \sqrt{1-3x})} = \underset{x \to 0}{\text{Lt}} \frac{6x}{x(\sqrt{1+3x} + \sqrt{1-3x})}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{6}{\sqrt{1+3x} + \sqrt{1-3x}} = \frac{6}{\sqrt{1 + 3 \times 0} + \sqrt{1 - 3 \times 0}} = \frac{6}{2} = 3.$$

(*ii*) $\underset{h \to 0}{\text{Lt}} \dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}\right) = \underset{h \to 0}{\text{Lt}} \dfrac{1}{h} \cdot \dfrac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\,\sqrt{x}}$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{h} \cdot \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\,\sqrt{x}} \times \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} = \underset{h \to 0}{\text{Lt}} \frac{1}{h} \cdot \frac{x - (x+h)}{\sqrt{x+h}\,\sqrt{x}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{-1}{\sqrt{x+h}\,\sqrt{x}\,(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x}\,\sqrt{x}\,(\sqrt{x} + \sqrt{x})} = -\frac{1}{2x^{3/2}}.$$

Example 7. *Evaluate the following limits:*

(*i*) $\underset{x \to -3}{Lt} \dfrac{x^2 - 9}{\sqrt{x^2 + 16} - 5}$ (*ii*) $\underset{x \to a}{Lt} \dfrac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}}$.

(*NCERT Examplar Problems*)

Solution. (*i*) $\underset{x \to -3}{\text{Lt}} \dfrac{x^2 - 9}{\sqrt{x^2 + 16} - 5} = \underset{x \to -3}{\text{Lt}} \dfrac{x^2 - 9}{\sqrt{x^2 + 16} - 5} \times \dfrac{\sqrt{x^2 + 16} + 5}{\sqrt{x^2 + 16} + 5}$

$$= \underset{x \to -3}{\text{Lt}} \frac{(x^2 - 9)(\sqrt{x^2 + 16} + 5)}{(x^2 + 16) - 25} = \underset{x \to -3}{\text{Lt}} (\sqrt{x^2 + 16} + 5)$$

$$= \sqrt{9 + 16} + 5 = 5 + 5 = 10.$$

(*ii*) $\underset{x\to a}{\text{Lt}} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} = \underset{x\to a}{\text{Lt}} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \times \dfrac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{3a+x}+2\sqrt{x}} \times \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}$

$$= \underset{x\to a}{\text{Lt}} \frac{(a+2x)-3x}{(3a+x)-4x} \times \frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}$$

$$= \underset{x\to a}{\text{Lt}} \frac{a-x}{3a-3x} \times \frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{2a+x}+\sqrt{3x}} = \underset{x\to a}{\text{Lt}} \frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{2a+x}+\sqrt{3x})}$$

$$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3(\sqrt{2a+a}+\sqrt{3a})} = \frac{4\sqrt{a}}{3.2\sqrt{3a}} = \frac{2}{3\sqrt{3}}.$$

(In this problem, it is understood that $a > 0$, otherwise the function will not be defined in the neighbourhood of a and the question of evaluating limit does not arise.)

Example 8. *If* $G(x) = -\sqrt{25-x^2}$, *evaluate* $\underset{x\to 1}{Lt} \dfrac{G(x)-G(1)}{x-1}$.

Solution. Given $G(x) = -\sqrt{25-x^2} \Rightarrow G(1) = -\sqrt{25-1} = -\sqrt{24}$.

$\therefore \quad \underset{x\to 1}{\text{Lt}} \dfrac{G(x)-G(1)}{x-1} = \underset{x\to 1}{\text{Lt}} \dfrac{-\sqrt{25-x^2}-(-\sqrt{24})}{x-1}$

$$= \underset{x\to 1}{\text{Lt}} \frac{\sqrt{24}-\sqrt{25-x^2}}{x-1} = \underset{x\to 1}{\text{Lt}} \frac{\sqrt{24}-\sqrt{25-x^2}}{x-1} \times \frac{\sqrt{24}+\sqrt{25-x^2}}{\sqrt{24}+\sqrt{25-x^2}}$$

$$= \underset{x\to 1}{\text{Lt}} \frac{24-(25-x^2)}{(x-1)(\sqrt{24}+\sqrt{25-x^2})} = \underset{x\to 1}{\text{Lt}} \frac{x^2-1}{(x-1)(\sqrt{24}+\sqrt{25-x^2})}$$

$$= \underset{x\to 1}{\text{Lt}} \frac{x+1}{(\sqrt{24}+\sqrt{25-x^2})} = \frac{1+1}{\sqrt{24}+\sqrt{25-1}} = \frac{1}{\sqrt{24}}.$$

Example 9. *Evaluate the following limits :*

(*i*) $\underset{x\to 0}{Lt} \dfrac{(1-x)^n-1}{x}$ (*ii*) $\underset{x\to 0}{Lt} \dfrac{(1+x)^6-1}{(1+x)^2-1}$. (*NCERT Examplar Problems*)

Solution. (*i*) Let $1-x = h \Rightarrow x = 1-h$, so that when $x \to 0$, $h \to 1$.

$$\therefore \quad \underset{x\to 0}{\text{Lt}} \frac{(1-x)^n-1}{x} = \underset{h\to 1}{\text{Lt}} \frac{h^n-1}{1-h} = -\underset{h\to 1}{\text{Lt}} \frac{h^n-1^n}{h-1} = -n\,.\,1^{n-1} = -n.$$

(*ii*) Let $1+x = h$, so that when $x \to 0$, $h \to 1$.

$$\therefore \quad \underset{x\to 0}{\text{Lt}} \frac{(1+x)^6-1}{(1+x)^2-1} = \underset{h\to 1}{\text{Lt}} \frac{h^6-1}{h^2-1} = \frac{\underset{h\to 1}{\text{Lt}} \dfrac{h^6-1}{h-1}}{\underset{h\to 1}{\text{Lt}} \dfrac{h^2-1}{h-1}} = \frac{6\times 1^{6-1}}{2.1^{2-1}} = \frac{6}{2} = 3.$$

Example 10. *Evaluate the following limits :*

(*i*) $\underset{x\to 0}{Lt} \dfrac{(x+2)^{1/3}-2^{1/3}}{x}$ (*NCERT Examplar Problems*)

(*ii*) $\underset{x\to 1}{Lt} \dfrac{x^4-\sqrt{x}}{\sqrt{x}-1}$ (*NCERT Examplar Problems*)

(*iii*) $\underset{x\to a}{Lt} \dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$ (*NCERT Examplar Problems*)

Solution. (*i*) Let $x + 2 = h \Rightarrow x = h - 2$, so that when $x \to 0$, $h \to 2$.

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \frac{(x+2)^{1/3} - 2^{1/3}}{x} = \underset{h \to 2}{\text{Lt}} \frac{h^{1/3} - 2^{1/3}}{h - 2} = \frac{1}{3} \cdot 2^{-2/3}.$$

(*ii*) Let $\sqrt{x} = h$, so that when $x \to 1$, $h \to 1$.

$$\therefore \quad \underset{x \to 1}{\text{Lt}} \frac{x^4 - \sqrt{x}}{\sqrt{x} - 1} = \underset{h \to 1}{\text{Lt}} \frac{h^8 - h}{h - 1} = \underset{h \to 1}{\text{Lt}}\, h \cdot \frac{h^7 - 1}{h - 1} = 1 \cdot 7 \cdot 1^{7-1} = 7.$$

(*iii*) Let $x + 2 = y$ and $a + 2 = h \Rightarrow x - a = y - h$, so that when $x \to a$, $y \to h$.

$$\therefore \quad \underset{x \to a}{\text{Lt}} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x - a} = \underset{y \to h}{\text{Lt}} \frac{y^{5/3} - h^{5/3}}{y - h}$$

$$= \frac{5}{3} \cdot (h)^{\frac{5}{3} - 1} = \frac{5}{3} h^{\frac{2}{3}} = \frac{5}{3}(a+2)^{2/3}.$$

Example 11. *If* $\underset{x \to 2}{Lt} \dfrac{x^n - 2^n}{x - 2} = 80$ *and* $n \in \mathbf{N}$, *then find* n. (*NCERT Examplar Problems*)

Solution. Given $\underset{x \to 2}{\text{Lt}} \dfrac{x^n - 2^n}{x - 2} = 80 \Rightarrow n \cdot 2^{n-1} = 80$

$\Rightarrow \quad n \cdot 2^{n-1} = 5 \cdot 2^{5-1} \Rightarrow n = 5.$

Example 12. *If* $f(x) = \begin{cases} x^2 - 1 & , \ x \le 1 \\ -x^2 - 1 & , \ x > 1 \end{cases}$, *does* $\underset{x \to 1}{Lt} f(x)$ *exist?* (*NCERT*)

Solution. $\underset{x \to 1^-}{\text{Lt}} f(x) = \underset{x \to 1^-}{\text{Lt}} (x^2 - 1)$ $\quad (\because f(x) = x^2 - 1 \text{ for } x \le 1)$

$= 1^2 - 1 = 1 - 1 = 0$

and $\underset{x \to 1^+}{\text{Lt}} f(x) = \underset{x \to 1^+}{\text{Lt}} (-x^2 - 1)$ $\quad (\because f(x) = -x^2 - 1 \text{ for } x > 1)$

$= -1^2 - 1 = -1 - 1 = -2$

$\Rightarrow \quad \underset{x \to 1^-}{\text{Lt}} f(x) \ne \underset{x \to 1^+}{\text{Lt}} f(x) \Rightarrow \underset{x \to 1}{\text{Lt}} f(x)$ does not exist.

Example 13. *Find* $\underset{x \to 0}{Lt} f(x)$ *and* $\underset{x \to 1}{Lt} f(x)$, *where* $f(x) = \begin{cases} 2x + 3 & , \ x \le 0 \\ 3(x+1) & , \ x > 0 . \end{cases}$ (*NCERT*)

Solution. $\underset{x \to 0^-}{\text{Lt}} f(x) = \underset{x \to 0^-}{\text{Lt}} (2x + 3)$ $\quad (\because f(x) = 2x + 3 \text{ for } x \le 0)$

$= 2 \times 0 + 3 = 3$

and $\underset{x \to 0^+}{\text{Lt}} f(x) = \underset{x \to 0^+}{\text{Lt}} 3(x + 1)$ $\quad (\because f(x) = 3(x + 1) \text{ for } x > 0)$

$= 3(0 + 1) = 3$

$\Rightarrow \quad \underset{x \to 0^-}{\text{Lt}} f(x) = 3 = \underset{x \to 0^+}{\text{Lt}} f(x) \Rightarrow \underset{x \to 0}{\text{Lt}} f(x) = 3.$

$\underset{x \to 1}{\text{Lt}} f(x) = \underset{x \to 1}{\text{Lt}} 3(x + 1)$ $\quad (\because f(x) = 3(x + 1) \text{ for } x > 0)$

$= 3(1 + 1) = 6.$

Example 14. *Let* $f(x) = \begin{cases} x+2 & , \ x \le -1 \\ cx^2 & , \ x > -1 \end{cases}$, *find c if* $\underset{x \to -1}{\text{Lt}} f(x)$ *exists.*

(*NCERT Examplar Problems*)

Solution. $\underset{x \to -1^-}{\text{Lt}} f(x) = \underset{x \to -1^-}{\text{Lt}} (x + 2)$ $(\because f(x) = x + 2 \text{ for } x \le -1)$

$= -1 + 2 = 1$

and $\underset{x \to -1^+}{\text{Lt}} f(x) = \underset{x \to -1^+}{\text{Lt}} cx^2$ $(\because f(x) = cx^2 \text{ for } x > -1)$

$= c \cdot (-1)^2 = c.$

Since $\underset{x \to -1}{\text{Lt}} f(x)$ exists (given),

$\therefore \quad \underset{x \to -1^-}{\text{Lt}} f(x) = \underset{x \to -1^+}{\text{Lt}} f(x) \Rightarrow 1 = c.$

Hence, $c = 1$.

Example 15. *Let* $f(x) = \begin{cases} a+bx & , \ x < 1 \\ 4 & , \ x = 1 \\ b-ax & , \ x > 1 \end{cases}$ *and if* $\underset{x \to 1}{\text{Lt}} f(x) = f(1)$, *what are the possible values of a and b?* (*NCERT*)

Solution. Given $f(x) = 4$ when $x = 1$ *i.e.* $f(1) = 4$.

$\underset{x \to 1^-}{\text{Lt}} f(x) = \underset{x \to 1^-}{\text{Lt}} (a + bx)$ $(\because f(x) = a + bx \text{ for } x < 1)$

$= a + b \times 1 = a + b$

and $\underset{x \to 1^+}{\text{Lt}} f(x) = \underset{x \to 1^+}{\text{Lt}} (b - ax)$ $(\because f(x) = b - ax \text{ for } x > 1)$

$= b - a \times 1 = b - a.$

Since $\underset{x \to 1}{\text{Lt}} f(x) = f(1)$ (given) $\Rightarrow \underset{x \to 1^-}{\text{Lt}} f(x) = \underset{x \to 1^+}{\text{Lt}} f(x) = f(1)$

$\Rightarrow \quad a + b = b - a = 4$

$\Rightarrow \quad a + b = 4$ and $b - a = 4$.

Solving these equations simultaneously, we get $a = 0$, $b = 4$.

Hence, $a = 0$ and $b = 4$.

Example 16. *Let f be a function defined by* $f(x) = \begin{cases} \dfrac{x}{|x|} & , \ x \ne 0 \\ 0 & , \ x = 0 \end{cases}$.

Does $\underset{x \to 0}{\text{Lt}} f(x)$ *exist?* (*NCERT*)

Solution. $\underset{x \to 0^-}{\text{Lt}} f(x) = \underset{x \to 0^-}{\text{Lt}} \dfrac{x}{|x|} = \underset{x \to 0^-}{\text{Lt}} \dfrac{x}{-x}$ $(\because x \to 0^- \Rightarrow x < 0 \Rightarrow |x| = -x)$

$= \underset{x \to 0^-}{\text{Lt}} (-1) = -1$

and $\underset{x \to 0^+}{\text{Lt}} f(x) = \underset{x \to 0^+}{\text{Lt}} \dfrac{x}{|x|} = \underset{x \to 0^+}{\text{Lt}} \dfrac{x}{x}$ $(\because x \to 0^+ \Rightarrow x > 0 \Rightarrow |x| = x)$

$= \underset{x \to 0^+}{\text{Lt}} (1) = 1$

$\Rightarrow \quad \underset{x \to 0^-}{\text{Lt}} f(x) \ne \underset{x \to 0^+}{\text{Lt}} f(x) \Rightarrow \underset{x \to 0}{\text{Lt}} f(x)$ does not exist.

Example 17. *Let f be a function defined by* $f(x) = \begin{cases} \dfrac{5x}{|x| - 2x^2} & , x \neq 0 \\ 0 & , x = 0. \end{cases}$

Does $\underset{x \to 0}{\text{Lt}}\ f(x)$ *exist?*

Solution. $\underset{x \to 0^-}{\text{Lt}}\ f(x) = \underset{x \to 0^-}{\text{Lt}}\ \dfrac{5x}{|x| - 2x^2} = \underset{x \to 0^-}{\text{Lt}}\ \dfrac{5x}{-x - 2x^2}$ $\quad (\because x \to 0^- \Rightarrow x < 0 \Rightarrow |x| = -x)$

$$= \underset{x \to 0^-}{\text{Lt}}\ \frac{5}{-1 - 2x} = \frac{5}{-1 - 2 \times 0} = -5$$

and $\underset{x \to 0^+}{\text{Lt}}\ f(x) = \underset{x \to 0^+}{\text{Lt}}\ \dfrac{5x}{|x| - 2x^2} = \underset{x \to 0^+}{\text{Lt}}\ \dfrac{5x}{x - 2x^2}$ $\quad (\because x \to 0^+ \Rightarrow x > 0 \Rightarrow |x| = x)$

$$= \underset{x \to 0^+}{\text{Lt}}\ \frac{5}{1 - 2x} = \frac{5}{1 - 2 \times 0} = 5$$

$\Rightarrow \quad \underset{x \to 0^-}{\text{Lt}}\ f(x) \neq \underset{x \to 0^+}{\text{Lt}}\ f(x) \Rightarrow \underset{x \to 0}{\text{Lt}}\ f(x)$ does not exist.

Example 18. *Show that* $\underset{x \to 4}{\text{Lt}}\ \dfrac{|x - 4|}{x - 4}$ *does not exist.* *(NCERT Examplar Problems)*

Solution. $\underset{x \to 4^-}{\text{Lt}}\ \dfrac{|x - 4|}{x - 4} = \underset{x \to 4^-}{\text{Lt}}\ \dfrac{-(x - 4)}{x - 4}$ $\quad (\because x \to 4^- \Rightarrow x - 4 < 0 \Rightarrow |x - 4| = -(x - 4))$

$$= \underset{x \to 4^-}{\text{Lt}}\ (-1) = -1$$

and $\underset{x \to 4^+}{\text{Lt}}\ \dfrac{|x - 4|}{x - 4} = \underset{x \to 4^+}{\text{Lt}}\ \dfrac{x - 4}{x - 4}$ $\quad (\because x \to 4^+ \Rightarrow x - 4 > 0 \Rightarrow |x - 4| = x - 4)$

$$= \underset{x \to 4^+}{\text{Lt}}\ (1) = 1$$

$\Rightarrow \quad \underset{x \to 4^-}{\text{Lt}}\ \dfrac{|x - 4|}{x - 4} \neq \underset{x \to 4^+}{\text{Lt}}\ \dfrac{|x - 4|}{x - 4} \Rightarrow \underset{x \to 4}{\text{Lt}}\ \dfrac{|x - 4|}{x - 4}$ does not exist.

Example 19. *Let f be a function defined by* $f(x) = \begin{cases} |x| + 1 & , x < 0 \\ 0 & , x = 0. \\ |x| - 1 & , x > 0 \end{cases}$ *(NCERT)*

For what value(s) of a does $\underset{x \to a}{\text{Lt}}\ f(x)$ *exist?*

Solution. As a is a real number, the following three cases arise :

Case I. When $a > 0$.

$\underset{x \to a}{\text{Lt}}\ f(x) = \underset{x \to a}{\text{Lt}}\ (|x| - 1)$ $\quad (\because f(x) = |x| - 1 \text{ for } x > 0)$

$= \underset{x \to a}{\text{Lt}}\ (x - 1)$ $\quad (\because x \to a \text{ and } a > 0 \Rightarrow x > 0 \Rightarrow |x| = x)$

$= a - 1.$

Case II. When $a < 0$.

$\underset{x \to a}{\text{Lt}}\ f(x) = \underset{x \to a}{\text{Lt}}\ (|x| + 1)$ $\quad (\because f(x) = |x| + 1 \text{ for } x < 0)$

$= \underset{x \to a}{\text{Lt}}\ (-x + 1)$ $\quad (\because x \to a \text{ and } a < 0 \Rightarrow x < 0 \Rightarrow |x| = -x)$

$= -a + 1.$

Case III. When $a = 0$.

$$\underset{x\to a^-}{\text{Lt}}\ f(x) = \underset{x\to 0^-}{\text{Lt}}\ f(x) = \underset{x\to 0^-}{\text{Lt}}\ (|x| + 1) \qquad (\because f(x) = |x| + 1 \text{ for } x < 0)$$

$$= \underset{x\to 0^-}{\text{Lt}}\ (-x + 1) = 0 + 1 = 1$$

and $$\underset{x\to a^+}{\text{Lt}}\ f(x) = \underset{x\to 0^+}{\text{Lt}}\ f(x) = \underset{x\to 0^+}{\text{Lt}}\ (|x| - 1) \qquad (\because f(x) = |x| - 1 \text{ for } x > 0)$$

$$= \underset{x\to 0^+}{\text{Lt}}\ (x - 1) = 0 - 1 = -1$$

$$\Rightarrow \quad \underset{x\to a^-}{\text{Lt}}\ f(x) \neq \underset{x\to a^+}{\text{Lt}}\ f(x) \Rightarrow \underset{x\to a}{\text{Lt}}\ f(x) \text{ does not exist.}$$

On combining all cases, we conclude that $\underset{x\to a}{\text{Lt}}\ f(x)$ exists for all real values of a except when $a = 0$.

Example 20. *Find the left limit of the function f defined by $f(x) = \sqrt{5 - x}$ as $x \to 5$. Does the right limit exist when $x \to 5$?*

Solution. $\underset{x\to 5^-}{\text{Lt}}\ f(x) = \underset{x\to 5^-}{\text{Lt}}\ \sqrt{5 - x} = 0.$

Further, as $x \to 5$ through values $x > 5$, $5 - x$ is negative, therefore, the function is not defined on the right of 5.

Hence, $\underset{x\to 5^+}{\text{Lt}}\ f(x)$ does not exist.

Example 21. *If the function $f(x)$ satisfies $\underset{x\to 1}{\text{Lt}}\ \dfrac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\underset{x\to 1}{\text{Lt}}\ f(x)$.* (NCERT)

Solution. Given $\underset{x\to 1}{\text{Lt}}\ \dfrac{f(x) - 2}{x^2 - 1} = \pi$ ...(*i*)

But $\underset{x\to 1}{\text{Lt}}\ (x^2 - 1) = 1^2 - 1 = 0.$

Since $\underset{x\to 1}{\text{Lt}}\ \dfrac{f(x) - 2}{x^2 - 1}$ exists, therefore, it is essential that

$$\underset{x\to 1}{\text{Lt}}\ (f(x) - 2) = 0$$

$$\therefore \quad \underset{x\to 1}{\text{Lt}}\ (f(x) - 2) = 0 \Rightarrow \underset{x\to 1}{\text{Lt}}\ f(x) - \underset{x\to 1}{\text{Lt}}\ 2 = 0$$

$$\Rightarrow \quad \underset{x\to 1}{\text{Lt}}\ f(x) - 2 = 0 \Rightarrow \underset{x\to 1}{\text{Lt}}\ f(x) = 2.$$

EXERCISE 13.1

Very short answer type questions (1 to 11) :

Evaluate the following (1 to 9) limits :

1. (*i*) $\underset{x\to 1}{\text{Lt}}\ (x^2 + x)$ (*ii*) $\underset{x\to 2}{\text{Lt}}\ (3x^3 - 5x + 2)$.
2. (*i*) $\underset{x\to 3}{\text{Lt}}\ (x + 3)$ (NCERT) (*ii*) $\underset{r\to 1}{\text{Lt}}\ \pi r^2$. (NCERT)
3. (*i*) $\underset{x\to 1}{\text{Lt}}\ ((2x - 1)^2 + 5)$ (*ii*) $\underset{x\to 1}{\text{Lt}}\ (x^{40} - 3x^{12} + 1)^{132}$.
4. (*i*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{x + 2}{x - 3}$ (*ii*) $\underset{x\to 2}{\text{Lt}}\ \dfrac{x^2 - 9}{x + 2}$.

5. (*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{ax+b}{cx+1}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{1+x}+\sqrt{1-x}}{3-x}$.

6. (*i*) $\underset{x \to 3}{\text{Lt}} \dfrac{x^2-9}{x+2}$ (*ii*) $\underset{x \to -1}{\text{Lt}} \dfrac{x^3-3x+1}{x-1}$.

7. (*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{ax+b}{cx+d}$ (*NCERT*) (*ii*) $\underset{x \to 2}{\text{Lt}} \dfrac{x^3+2x-5}{x+2}$.

8. (*i*) $\underset{x \to 3}{\text{Lt}} \dfrac{x^2-9}{x-3}$ (*ii*) $\underset{x \to -1}{\text{Lt}} \dfrac{2x^2+3x+1}{x^2-1}$.

(*NCERT Examplar Problems*)

9. (*i*) $\underset{x \to 2}{\text{Lt}} \dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$ (*ii*) $\underset{x \to 2}{\text{Lt}} \dfrac{x^5-32}{x-2}$.

10. If $f(x) = \begin{cases} x-2, & x<0 \\ x+2, & x \geq 0 \end{cases}$, find

(*i*) $\underset{x \to 1}{\text{Lt}} f(x)$ (*ii*) $\underset{x \to -1}{\text{Lt}} f(x)$ (*iii*) $\underset{x \to 0}{\text{Lt}} f(x)$.

11. Let f be a function defined by $f(x) = \begin{cases} 1, & x \leq 0 \\ 2, & x > 0 \end{cases}$. Does $\underset{x \to 0}{\text{Lt}} f(x)$ exist ?

Evaluate the following (12 to 25) limits :

12. (*i*) $\underset{x \to \frac{1}{2}}{\text{Lt}} \dfrac{4x^2-1}{2x-1}$ (*ii*) $\underset{x \to 1}{\text{Lt}} \dfrac{x-1}{2x^2-7x+5}$.

(*NCERT Examplar Problems*)

13. (*i*) $\underset{x \to 2}{\text{Lt}} \dfrac{x^3-8}{x-2}$ (*ii*) $\underset{x \to 3}{\text{Lt}} \dfrac{x^3-27}{x^2-9}$.

14. (*i*) $\underset{x \to 1}{\text{Lt}} \dfrac{x^2-3x+2}{x^2-1}$ (*ii*) $\underset{x \to 5}{\text{Lt}} \dfrac{x^2-9x+20}{x^2-6x+5}$.

15. (*i*) $\underset{x \to 2}{\text{Lt}} \dfrac{x^3-4x^2+4x}{x^2-4}$ (*NCERT*) (*ii*) $\underset{x \to 2}{\text{Lt}} \dfrac{x^3-2x^2}{x^2-5x+6}$. (*NCERT*)

16. (*i*) $\underset{x \to \frac{1}{2}}{\text{Lt}} \left(\dfrac{8x-3}{2x-1} - \dfrac{4x^2+1}{4x^2-1}\right)$ (*ii*) $\underset{x \to 2}{\text{Lt}} \left(\dfrac{1}{x-2} - \dfrac{2(2x-3)}{x^3-3x^2+2x}\right)$.

(*NCERT Examplar Problems*) (*NCERT Examplar Problems*)

17. (*i*) $\underset{x \to \sqrt{2}}{\text{Lt}} \dfrac{x^2-2}{x^2+\sqrt{2}\,x-4}$ (*ii*) $\underset{x \to \sqrt{3}}{\text{Lt}} \dfrac{x^4-9}{x^2+4\sqrt{3}\,x-15}$.

18. (*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{x}{\sqrt{1+x}-1}$.

(*NCERT Examplar Problems*)

19. (*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}$ (*ii*) $\underset{h \to 0}{\text{Lt}} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

(*NCERT Examplar Problems*) (*NCERT Examplar Problems*)

20. (*i*) $\underset{x \to 2}{\text{Lt}} \dfrac{\sqrt{3-x}-1}{2-x}$ (*ii*) $\underset{x \to 3}{\text{Lt}} \dfrac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$.

21. (*i*) $\underset{x \to 3}{\text{Lt}} \dfrac{(x+1)-\sqrt{x+13}}{x-3}$ (*ii*) $\underset{x \to -3}{\text{Lt}} \dfrac{x^2-9}{\sqrt{x^2+16}-5}$.

22. (*i*) $\underset{x \to 2}{\text{Lt}} \dfrac{x^8 - 256}{x - 2}$ (*ii*) $\underset{x \to 5}{\text{Lt}} \dfrac{x^5 - 3125}{x - 5}$.

23. (*i*) $\underset{x \to 4}{\text{Lt}} \dfrac{x^{3/2} - 8}{x - 4}$ (*ii*) $\underset{x \to 3}{\text{Lt}} \dfrac{x^5 - 243}{x^2 - 9}$.

Hint. (*i*) $\underset{x \to 4}{\text{Lt}} \dfrac{x^{3/2} - 8}{x - 4} = \underset{x \to 4}{\text{Lt}} \dfrac{x^{3/2} - 4^{3/2}}{x - 4}$.

24. (*i*) $\underset{x \to 2}{\text{Lt}} \dfrac{x^{10} - 1024}{x^5 - 32}$ (*ii*) $\underset{x \to a}{\text{Lt}} \dfrac{x^m - a^m}{x^n - a^n}$.

25. (*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)^n - 1}{x}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{1+x} - 1}{\sqrt[3]{1+x} - 1}$.

26. If $\underset{x \to 3}{\text{Lt}} \dfrac{x^n - 3^n}{x - 3} = 108$ and $n \in \mathbf{N}$, find n. (*NCERT Examplar Problems*)

27. If $\underset{x \to 1}{\text{Lt}} \dfrac{x^4 - 1}{x - 1} = \underset{x \to k}{\text{Lt}} \dfrac{x^3 - k^3}{x^2 - k^2}$, find all values of k. (*NCERT Examplar Problems*)

28. If $\underset{x \to a}{\text{Lt}} \dfrac{x^9 - a^9}{x - a} = \underset{x \to 5}{\text{Lt}} (x + 4)$, find all possible real values of a.

29. If $a_1, a_2, \ldots, a_n$ are fixed real numbers and a function f is defined by $f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$, find $\underset{x \to a_1}{\text{Lt}} f(x)$.

For some $a \neq a_1, a_2, \ldots, a_n$, compute $\underset{x \to a}{\text{Lt}} f(x)$.

30. (*i*) If $f(x) = \begin{cases} 1 + x^2 & , \ 0 \le x \le 1 \\ 2 - x & , \ x > 1 \end{cases}$, does $\underset{x \to 1}{\text{Lt}} f(x)$ exist?

(*ii*) If $f(x) = \begin{cases} 5x - 4 & , \ x \le 1 \\ 4x^2 - 3x & , \ x > 1 \end{cases}$, find $\underset{x \to 1}{\text{Lt}} f(x)$.

31. (*i*) If f is defined by $f(x) = \begin{cases} x & , \ x \le \frac{1}{2} \\ 1 - x & , \ x > \frac{1}{2} \end{cases}$, does $\underset{x \to \frac{1}{2}}{\text{Lt}} f(x)$ exist? If so, find it.

(*ii*) If $f(x) = \begin{cases} x + 1, & \text{if } x > 0 \\ x - 1, & \text{if } x \le 0 \end{cases}$, show that $\underset{x \to 0}{\text{Lt}} f(x)$ does not exist.

32. Let $f(x) = \begin{cases} x & , \ 0 \le x < \frac{1}{2} \\ 0 & , \ x = \frac{1}{2} \\ x - 1 & , \ \frac{1}{2} < x \le 1 \end{cases}$, show that $\underset{x \to \frac{1}{2}}{\text{Lt}} f(x)$ does not exist.

33. Let f be defined by $f(x) = \begin{cases} 3x - 1 & , \ x < 0 \\ 0 & , \ x = 0 \\ 2x + 5 & , \ x > 0 \end{cases}$. Evaluate

(*i*) $\underset{x \to 2}{\text{Lt}} f(x)$ (*ii*) $\underset{x \to -3}{\text{Lt}} f(x)$. Does $\underset{x \to 0}{\text{Lt}} f(x)$ exist? If no, explain.

34. Find k so that $\underset{x \to 2}{\text{Lt}} f(x)$ may exist where $f(x) = \begin{cases} 4x - 5, & x \le 2 \\ x - k, & x > 2 \end{cases}$.

35. Let f be a function defined by $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$. Does $\underset{x \to 0}{\text{Lt}} f(x)$ exist? (*NCERT*)

36. If $f(x) = |x| - 5$, evaluate the following limits :

(*i*) $\underset{x \to 5^+}{\text{Lt}} f(x)$ (*ii*) $\underset{x \to 5^-}{\text{Lt}} f(x)$ (*iii*) $\underset{x \to 5}{\text{Lt}} f(x)$ (*NCERT*) (*iv*) $\underset{x \to -5}{\text{Lt}} f(x)$.

37. Let $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x \ne 0 \\ -2, & x = 0 \end{cases}$, show that $\underset{x \to 0}{\text{Lt}} f(x)$ does not exist.

13.1.5 Some standard theorems on limits of trigonometric functions

Theorem 3. (i) $\underset{x \to 0}{\text{Lt}} \sin x = 0$ ***(ii)*** $\underset{x \to 0}{\text{Lt}} \cos x = 1$ ***(iii)*** $\underset{x \to 0}{\text{Lt}} \tan x = 0$.

Proof. (*i*) First, let $x \to 0^+$, so that $0 < x < \frac{\pi}{2}$. Consider a circle of radius 1 unit having centre at O. Let A, P be two points on the circle (as shown in fig. 13.6) such that the length of the arc AP (measured in anticlockwise direction) equals x, then radian measure of $\angle AOP = \dfrac{\text{arc AP}}{\text{radius}} = \dfrac{x}{1} = x$.

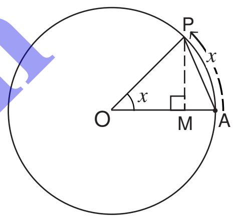

Fig. 13.6.

From P, draw MP ⊥ OA and join A, P. Then

$$\sin x = \frac{MP}{OP} = \frac{MP}{1} = MP \quad \text{...(1)}$$

From the figure, clearly

area of Δ OAP < area of sector OAP

$\Rightarrow \quad \frac{1}{2} . OA . MP < \frac{1}{2} (\text{radius})^2 . x$

$\Rightarrow \quad \frac{1}{2} . 1 . \sin x < \frac{1}{2} . 1^2 . x$ (using (1))

$\Rightarrow \quad \sin x < x$

But $\sin x > 0$ $\left(\because 0 < x < \frac{\pi}{2} \Rightarrow \sin x > 0\right)$

$\therefore \quad 0 < \sin x < x$

Also $\underset{x \to 0^+}{\text{Lt}} 0 = 0$ and $\underset{x \to 0^+}{\text{Lt}} x = 0$, therefore, by squeeze principle,

$$\underset{x \to 0^+}{\text{Lt}} \sin x = 0 \quad \text{...(2)}$$

Further, $\underset{x \to 0^-}{\text{Lt}} \sin x = \underset{x \to 0^+}{\text{Lt}} \sin(-x)$ (By theorem 1)

$= \underset{x \to 0^+}{\text{Lt}} (-\sin x) = -\underset{x \to 0^+}{\text{Lt}} \sin x$

$= -0$ (using (2))

$= 0.$

Thus, $\underset{x \to 0^+}{\text{Lt}} \sin x = 0 = \underset{x \to 0^-}{\text{Lt}} \sin x$.

$\therefore \quad \underset{x \to 0}{\text{Lt}} \sin x = 0.$

(*ii*) $\lim\limits_{x \to 0} \cos x = \lim\limits_{x \to 0} \left(1 - 2\sin^2 \frac{x}{2}\right)$

$= \lim\limits_{x \to 0} 1 - 2\left(\lim\limits_{\frac{x}{2} \to 0} \sin \frac{x}{2}\right)^2$ $\quad (\because \text{ as } x \to 0, \frac{x}{2} \to 0)$

$= 1 - 2 \times 0$ $\quad (\text{using } \lim\limits_{x \to 0} \sin x = 0)$

$= 1.$

(*iii*) $\lim\limits_{x \to 0} \tan x = \lim\limits_{x \to 0} \frac{\sin x}{\cos x} = \frac{\lim\limits_{x \to 0} \sin x}{\lim\limits_{x \to 0} \cos x}$

$= \frac{0}{1} = 0.$

Theorem 4. (i) $\lim\limits_{x \to c} \sin x = \sin c$ ***(ii)*** $\lim\limits_{x \to c} \cos x = \cos c.$

Proof. Put $x = c + h$, when $x \to c$, $h \to 0$.

(*i*) $\lim\limits_{x \to c} \sin x = \lim\limits_{h \to 0} \sin (c + h) = \lim\limits_{h \to 0} (\sin c \cos h + \cos c \sin h)$

$= \sin c \lim\limits_{h \to 0} \cos h + \cos c \lim\limits_{h \to 0} \sin h$

$= \sin c \times 1 + \cos c \times 0 = \sin c.$

(*ii*) $\lim\limits_{x \to c} \cos x = \lim\limits_{h \to 0} \cos (c + h) = \lim\limits_{h \to 0} (\cos c \cos h - \sin c \sin h)$

$= \cos c \lim\limits_{h \to 0} \cos h - \sin c \lim\limits_{h \to 0} \sin h$

$= \cos c \times 1 - \sin c \times 0 = \cos c.$

Theorem 5. ***(i)*** $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ ***(ii)*** $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$ ***(iii)*** $\lim\limits_{x \to 0} \frac{1 - \cos x}{x} = 0.$

Proof. (*i*) First, let $x \to 0^+$, so that $0 < x < \frac{\pi}{2}$. Consider a circle of radius 1 unit having centre at O. Let A, P be two points on the circle (as shown in fig. 13.7) such that the length of arc AP (measured in anti-clockwise direction) equals x, then

radian measure of $\angle AOP = \frac{\text{arc AP}}{\text{radius}} = \frac{x}{1} = x.$

Draw AQ ⊥ OA at A to meet OP produced at Q. Let M be the foot of perpendicular from P on OA. Then

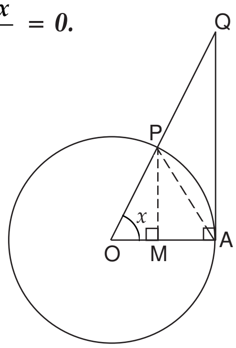

Fig. 13.7.

$\sin x = \frac{MP}{OP} = \frac{MP}{1} = MP$ ...(1)

and $\tan x = \frac{AQ}{OA} = \frac{AQ}{1} = AQ.$ ...(2)

From the figure, clearly

area of ΔOAP < area of sector OAP < area of ΔOAQ

$\Rightarrow \frac{1}{2} . OA . MP < \frac{1}{2} (\text{radius})^2 . x < \frac{1}{2} . OA . AQ$

$\Rightarrow \frac{1}{2} . 1 . \sin x < \frac{1}{2} . 1^2 . x < \frac{1}{2} . 1 . \tan x$ (using (1) and (2))

$\Rightarrow \sin x < x < \tan x$

$\Rightarrow 1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$ $\quad (\because 0 < x < \frac{\pi}{2} \Rightarrow \sin x > 0)$

$\Rightarrow \quad 1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x} \Rightarrow 1 > \dfrac{\sin x}{x} > \cos x$

$\Rightarrow \quad \cos x < \dfrac{\sin x}{x} < 1.$

Also $\underset{x \to 0^+}{\text{Lt}}\ 1 = 1$ and $\underset{x \to 0^+}{\text{Lt}} \cos x = 1$, therefore, by squeeze principle,

$$\underset{x \to 0^+}{\text{Lt}} \frac{\sin x}{x} = 1 \qquad \ldots(3)$$

Further, $\underset{x \to 0^-}{\text{Lt}} \dfrac{\sin x}{x} = \underset{x \to 0^+}{\text{Lt}} \dfrac{\sin(-x)}{-x}$ (By theorem 1)

$$= \underset{x \to 0^+}{\text{Lt}} \left(\frac{-\sin x}{-x}\right) = \underset{x \to 0^+}{\text{Lt}} \frac{\sin x}{x}$$

$= 1$ (using (3))

$\therefore \quad \underset{x \to 0^+}{\text{Lt}} \dfrac{\sin x}{x} = 1 = \underset{x \to 0^-}{\text{Lt}} \dfrac{\sin x}{x} \Rightarrow \underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x} = 1.$

(*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{\tan x}{x} = \underset{x \to 0}{\text{Lt}} \left(\dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x}\right) = \underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x} \cdot \underset{x \to 0}{\text{Lt}} \dfrac{1}{\cos x}$

$$= 1 \cdot \frac{1}{1} = 1.$$

(*iii*) $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \cos x}{x} = \underset{x \to 0}{\text{Lt}} \dfrac{2\sin^2 \frac{x}{2}}{x} = \underset{x \to 0}{\text{Lt}} \left(\dfrac{\sin \frac{x}{2}}{\frac{x}{2}} \cdot \sin \dfrac{x}{2}\right)$

$$= \underset{\frac{x}{2} \to 0}{\text{Lt}} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \cdot \underset{\frac{x}{2} \to 0}{\text{Lt}} \sin \frac{x}{2} = 1 \times 0 = 0.$$

13.1.6 Evaluation of limits involving trigonometric functions

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following limits :*

(*i*) $\underset{x \to 0}{Lt} \dfrac{\sin ax}{bx}$ *(NCERT)* (*ii*) $\underset{x \to 0}{Lt} \dfrac{\sin^2 x}{5x}$ (*iii*) $\underset{x \to 0}{Lt} \dfrac{\sin ax}{\sin bx}$. *(NCERT)*

Solution. (*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin ax}{bx} = \underset{x \to 0}{\text{Lt}} \left(\dfrac{\sin ax}{ax} \cdot \dfrac{a}{b}\right) = \dfrac{a}{b} \cdot \underset{ax \to 0}{\text{Lt}} \dfrac{\sin ax}{ax}$ (as $x \to 0$, $ax \to 0$)

$$= \frac{a}{b} \times 1 = \frac{a}{b}.$$

(*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin^2 x}{5x} = \underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{5} \cdot \dfrac{\sin x}{x} \cdot \sin x\right) = \dfrac{1}{5} \underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x} \cdot \underset{x \to 0}{\text{Lt}} \sin x$

$$= \frac{1}{5} \times 1 \times 0 = 0.$$

(*iii*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin ax}{\sin bx} = \underset{x \to 0}{\text{Lt}} \left(\dfrac{\sin ax}{ax} \cdot \dfrac{bx}{\sin bx} \cdot \dfrac{a}{b}\right)$

$= \dfrac{a}{b} \underset{ax \to 0}{\text{Lt}} \dfrac{\sin ax}{ax} \cdot \dfrac{1}{\underset{bx \to 0}{\text{Lt}} \dfrac{\sin bx}{bx}}$ $(x \to 0 \Rightarrow ax \to 0, bx \to 0)$

$$= \frac{a}{b} \times 1 \times \frac{1}{1} = \frac{a}{b}.$$

9. If for the function f defined by $f(x) = kx^2 + 7x - 4$, $f'(5) = 97$, find the value of k (use definition).

10. Find the derivatives of the following functions from first principles :

(*i*) $2x^3 + 5$ (*ii*) $\frac{3x-1}{4x+3}$ (*iii*) $\sin(ax+b)$.

11. Show that the derivative of the function f, given by

$f(x) = 2x^3 - 9x^2 + 12x + 9$, at $x = 1$ and at $x = 2$ are equal.

12. If $y = \sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}$, prove that $2xy\frac{dy}{dx} = \frac{x}{a} - \frac{a}{x}$.

13. If $y = \frac{x}{x+a}$, prove that $x\frac{dy}{dx} = y(1-y)$.

14. Find the derivative of sin $2x$ at $x = \frac{\pi}{3}$ from definition.

Find the derivatives of the following (15 to 19) functions :

15. (*i*) $\left(x - \frac{1}{\sqrt{x}}\right)^3$ (*ii*) $\left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)$.

16. (*i*) $5\tan x - 3\sin x + 4x^{3/2}$ (*ii*) $(1 + x^2)\cos x$.

17. (*i*) $(x^2 - 5x + 6)\sec x$ (*ii*) $\frac{x + \sin x}{x + \cos x}$.

18. (*i*) $\frac{1 - \tan x}{1 + \tan x}$ (*ii*) $\frac{\sin(x+a)}{\cos x}$. (*NCERT*)

19. (*i*) $\frac{a + \sin x}{1 + a\sin x}$ (*ii*) $\frac{\sin x - x\cos x}{x\sin x + \cos x}$.

ANSWERS

EXERCISE 13.1

1. (*i*) 2 (*ii*) 16 **2.** (*i*) 6 (*ii*) π **3.** (*i*) 6 (*ii*) 1

4. (*i*) $-\frac{2}{3}$ (*ii*) $-\frac{5}{4}$ **5.** (*i*) b (*ii*) $\frac{2}{3}$ **6.** (*i*) 0 (*ii*) $-\frac{3}{2}$

7. (*i*) $\frac{b}{d}$ (*ii*) $\frac{7}{4}$ **8.** (*i*) 6 (*ii*) $\frac{1}{2}$ **9.** (*i*) $-\frac{1}{4}$ (*ii*) 80

10. (*i*) 3 (*ii*) -3 (*iii*) does not exist **11.** No

12. (*i*) 2 (*ii*) $-\frac{1}{3}$ **13.** (*i*) 12 (*ii*) $\frac{9}{2}$ **14.** (*i*) $-\frac{1}{2}$ (*ii*) $\frac{1}{4}$

15. (*i*) 0 (*ii*) -4 **16.** (*i*) $\frac{7}{2}$ (*ii*) $-\frac{1}{2}$ **17.** (*i*) $\frac{2}{3}$ (*ii*) 2

18. (*i*) $\frac{1}{2\sqrt{2}}$ (*ii*) 2 **19.** (*i*) 0 (*ii*) $\frac{1}{2\sqrt{x}}$ **20.** (*i*) $\frac{1}{2}$ (*ii*) 1

21. (*i*) $\frac{7}{8}$ (*ii*) 10 **22.** (*i*) 1024 (*ii*) 3125 **23.** (*i*) 3 (*ii*) $\frac{135}{2}$

24. (*i*) 64 (*ii*) $\frac{m}{n}a^{m-n}$ **25.** (*i*) n (*ii*) $\frac{3}{2}$ **26.** 4

27. $\frac{8}{3}$ **28.** 1, -1 **29.** 0; $(a - a_1)(a - a_2) \ldots (a - a_n)$