12

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

INTRODUCTION

So far, we were dealing in analytical geometry in a plane—two dimensional. But, as we live in a world of three dimensions, we must extend our knowledge of geometry to three dimensional space. In this chapter, we shall learn the basic extension to three dimensions of the concepts which we have already developed in two dimensional geometry.

12.1 CARTESIAN CO-ORDINATE SYSTEM

Recall that every point in a plane can be represented by an ordered pair of real numbers with reference to two perpendicular lines called x-axis and y-axis. Points in space (three dimensional) can be described in much the same way as in a plane.

Co-ordinate axes

Let X′OX, Y′OY and Z′OZ be three mutually perpendicular directed straight lines (whose positive directions are marked by arrow heads in fig. 12.1) intersecting at the fixed point O. *Co-ordinate* these lines by taking the common point O as origin and the same unit of length on each line, then

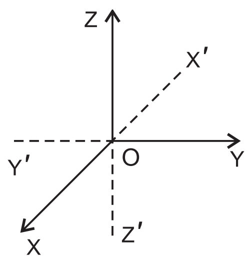

Fig. 12.1.

(*i*) X′OX is called the **axis of x or x-axis.**

(*ii*) Y′OY is called the **axis of y or y-axis.**

(*iii*) Z′OZ is called the **axis of z or z-axis.**

(*iv*) X′OX, Y′OY, Z′OZ taken together (in this particular order) are called **co-ordinate axes** or simply **axes.**

(*v*) O is called **origin**.

NOTE

As X′OX, Y′OY and Z′OZ are mutually at right angles to each other, these are called *rectangular axes.*

Co-ordinate planes

The plane containing the

(*i*) x-axis and y-axis is called XY-plane.

(*ii*) y-axis and z-axis is called YZ-plane.

(*iii*) z-axis and x-axis is called ZX-plane.

(*iv*) The planes YOZ, ZOX and XOY taken together (in this particular order) are called the **co-ordinate planes.**

Co-ordinates of a point

Let P be any point in space. Through P draw planes parallel to the co-ordinate planes to meet the axes in points A, B and C respectively. If the *directed distances* (positive or negative according as they are measured along positive or negative directions of the co-ordinate axes) OA, OB and OC are x, y and z respectively, then x, y, z taken in this particular order *i.e.* the ordered triplet (x, y, z) is called **cartesian co-ordinates of P** or simply **co-ordinates of P** and is written as P (x, y, z).

From the above definition, it follows that given any point P in space, the ordered triplet (x, y, z) of real numbers is uniquely determined. *Conversely*, let (x, y, z) be any given ordered triplet of real numbers, we can find points A, B and C on co-ordinate axes such that OA $= x$, OB $= y$, OC $= z$. Complete the parallelopiped OP having OA, OB, and OC as coterminus edges, then P is the unique point in space with co-ordinates (x, y, z).

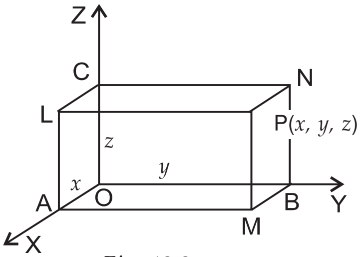

Fig. 12.2.

Alternatively

Given a point P in space, from P, draw a perpendicular PM on the XY-plane with M as the foot of this perpendicular (shown in fig. 12.3). Then, from the point M, draw a perpendicular MN to the x-axis with N as the foot of perpendicular. Let the directed distances ON, NM and MP be x, y and z respectively, then x, y, z taken in this particular order *i.e.* the triplet (x, y, z) is called **co-ordinates of P** and is written as P(x, y, z). From the above definition, it follows that given any point P in space, the ordered triplet of real numbers is uniquely determined. *Conversely*, let (x, y, z) be any given triplet of real numbers, we first fix the point N on the x-axis corresponding to x, then locate the point M in the XY-plane such that (x, y) are the co-ordinates of the point M in the XY-plane. Note that NM is perpendicular to x-axis (or is parallel to y-axis). Having located the point M, we erect a perpendicular MP to the XY-plane and locate on it the point P corresponding to z. The point P so obtained has the co-ordinates (x, y, z). Thus, given any triplet (x, y, z) of real numbers, there exists a unique point P in space with (x, y, z) as its co-ordinates.

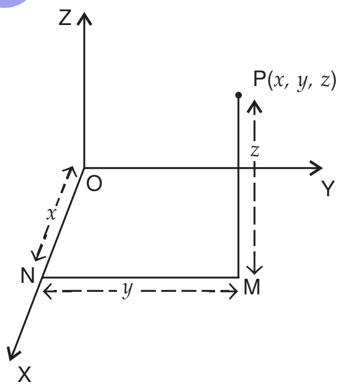

Fig. 12.3.

Hence, there is a one-one correspondence between the set of points in space and the set of ordered triplets of real numbers.

REMARKS

1. The co-ordinates of the origin are O (0, 0, 0).

2. It is clear from fig. 12.2 or 12.3 that

x = the directed distance of P from the YZ-plane,

y = the directed distance of P from the ZX-plane and

z = the directed distance of P from the XY-plane.

3. If P (x, y, z) is any point in space and A, B, C are the feet of perpendiculars drawn from P on the co-ordinate axes, then x = OA, y = OB and z = OC.

4. If a point P lies in the XY-plane, then by definition of co-ordinates of a point, z-coordinate of the point P is zero. Therefore, the co-ordinates of any point in the XY-plane may be taken as $(x, y, 0)$ and the equation of the XY-plane as $z = 0$.

Similarly, the co-ordinates of any point in the YZ-plane and ZX-plane may be taken as $(0, y, z)$ and $(x, 0, z)$ respectively, and their equations as $x = 0$ and $y = 0$ respectively.

5. If a point P lies on the x-axis, then its y and z co-ordinates are both zero. Therefore, the co-ordinates of any point on x-axis may be taken as $(x, 0, 0)$; and the equations of x-axis as $y = 0$, $z = 0$.

Similarly, the co-ordinates of any point on y-axis and z-axis may be taken as $(0, y, 0)$ and $(0, 0, z)$ respectively, and their equations as $x = 0$, $z = 0$ and $x = 0$, $y = 0$ respectively.

Octants

The three co-ordinate planes divide the space into eight parts called ***octants****.*

The signs of the co-ordinates of a point determine the octant in which the point lies. In other words, the signs of the co-ordinates of a point are determined by the octant in which it lies. The following table shows the signs of the co-ordinates of a point in eight octants :

Table 12.1

Octants → *Co-ordinates↓*	*I OXYZ*	*II OX′YZ*	*III OX′ Y′Z*	*IV OXY′Z*	*V OXYZ′*	*VI OX′YZ′*	*VII OX′Y′Z′*	*VIII OXY′Z′*
x	+ve	–ve	–ve	+ve	+ve	–ve	–ve	+ve
y	+ve	+ve	–ve	–ve	+ve	+ve	–ve	–ve
z	+ve	+ve	+ve	+ve	–ve	–ve	–ve	–ve

ILLUSTRATIVE EXAMPLES

Example 1. *In fig. 12.2, if the point P is (3, 2, 4), find the co-ordinates of the points M, N and L.*

Solution. For the point M, the directed distance measured along OZ is zero. Therefore, the co-ordinates of the point M are (3, 2, 0).

For the point N, the directed distance measured along OX is zero.

Therefore, the co-ordinates of the point N are (0, 2, 4).

Similarly, the co-ordinates of the point L are (3, 0, 4).

Example 2. *If L, M and N be the feet of perpendiculars drawn from a point P(3, 4, 5) on the x, y and z-axes respectively, then find the co-ordinates of L, M and N.* (*NCERT Examplar Problems*)

Solution. As L is the foot of perpendicular drawn from the point P(3, 4, 5) on x-axis, the directed distance measured along OX is 3 and its y and z co-ordinates are zero. Therefore, the co-ordinates of the point L are (3, 0, 0).

Similarly, the co-ordinates of M are (0, 4, 0) and the co-ordinates of the point N are (0, 0, 5).

Example 3. *In fig. 12.2, if the point P is (4, 2, 3), find the co-ordinates of the reflection (mirror image) of the point P in the XY-plane.*

Solution. Image of the point P(4, 2, 3) will be as much below the XY-plane as P is above it. Therefore, the co-ordinates of the reflection of P in the XY-plane will be (4, 2, –3).

Example 4. *Find the octants in which the points (–4, –2, 5) and (–3, 2, –1) lie.* *(NCERT)*

Solution. From table 12.1, we find that the point (–4, –2, 5) lies in the third octant and the point (–3, 2, –1) lies in the sixth octant.

Example 5. *What are the co-ordinates of the vertices of a cube whose edge is 2 units, one of whose vertices coincides with origin and the three edges passing through the origin coincides with the positive direction of axes through the origin.* (*NCERT Examplar Problems*)

Solution. A cube with edge 2 units, whose one vertex coinciding with origin and the three edges passing through origin coinciding with the positive direction of axes is shown in the adjoining fig. 12.4.

The co-ordinates of the vertices of the cube are O(0, 0, 0), A(2, 0, 0), B(0, 2, 0), C(0, 0, 2), D(2, 2, 0), E(0, 2, 2), F(2, 0, 2) and G(2, 2, 2).

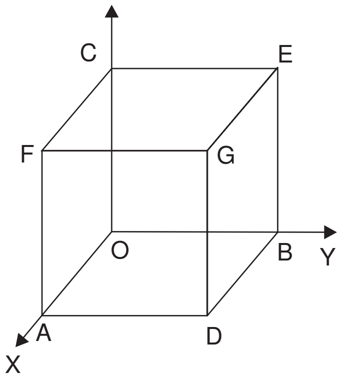

Fig. 12.4.

EXERCISE 12.1

Very short answer type questions (1 to 9) :

1. A point is on the x-axis. What are its y-coordinate and z-coordinate? (*NCERT*)
2. If a point lies on y-axis, then what are its x-coordinate and z-coordinate?
3. If a point lies in the YZ-plane, then what can you say about its x-coordinate?
4. A point is in the ZX-plane. What can you say about its y-coordinate? (*NCERT*)
5. Write the coordinates of the foot of perpendicular from the point (3, −2, 5) on
   (*i*) the x-axis (*ii*) the y-axis (*iii*) the z-axis.
6. Write the coordinates of the foot of perpendicular from the point (3, −4, 5) on the
   (*i*) XY-plane (*ii*) YZ-plane (*iii*) ZX-plane.
7. Write the co-ordinates of the image of the point (−3, 2, 7) in the
   (*i*) XY-plane (*ii*) YZ-plane (*iii*) ZX-plane.
8. Find the (mirror) image of the given point in the specified plane :
   (*i*) (−3, 4, 7) in the YZ-plane (*ii*) (−7, 2, −1) in the ZX-plane
   (*iii*) (5, 4, −3) in the XY-plane (*iv*) (−4, 0, 1) in the ZX-plane
   (*v*) (−2, 0, 0) in the XY-plane.
9. Name the octant in which the following points lie :
   (*i*) (1, 3, 5) (*ii*) (3, −2, 4) (*iii*) (−2, −3, 5) (*iv*) (−1, −2, −5)
   (*v*) (−3, 1, 2) (*vi*) (2, −1, −5) (*vii*) (5, 2, −3) (*viii*) (−1, 3, −2).
10. In fig. 12.2, if the point P is (a, b, c), write the co-ordinates of the points M, N, L, A, B and C.
11. Fill in the blanks :
    (*i*) The x-axis and y-axis taken together determine a plane known as ...... (*NCERT*)
    (*ii*) The co-ordinates of a point in the XY-plane are of the form .... (*NCERT*)
    (*iii*) Co-ordinate planes divide the space into .... octants. (*NCERT*)

12.2 DISTANCE FORMULA

To find the distance between two points whose co-ordinates are given.

Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be the two given points. Through P and Q draw planes parallel to the co-ordinate planes to form a rectangular *box* whose one diagonal is PQ.

Geometrically, to find the distance PQ is the computing of the length of the diagonal PQ of the box (shown in fig. 12.5) by means of Pythagoras theorem.

Since MQ is perpendicular to the plane PAMB and PM lies in this plane, so MQ $\perp$ PM $\Rightarrow \angle PMQ = 90^\circ$.

In Δ PMQ, $\angle$PMQ = 90°, therefore, by Pythagoras theorem, we get

$$PQ^2 = PM^2 + MQ^2 \quad \ldots(i)$$

Since AM $\perp$ AP, so $\angle$MAP = 90°

In Δ AMP, $\angle$MAP = 90°, therefore, by Pythagoras theorem, we get

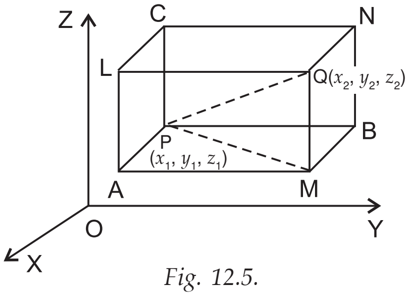

Fig. 12.5.

$$PM^2 = AP^2 + AM^2 \quad \ldots(ii)$$

From (*i*) and (*ii*), we get

$$PQ^2 = AP^2 + AM^2 + MQ^2 \quad \ldots(iii)$$

Clearly, from the figure

$$AP = x_2 - x_1, \; AM = PB = y_2 - y_1$$

and $\quad MQ = PC = z_2 - z_1$.

Substituting these values in (*iii*), we get

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$\Rightarrow \qquad \mathbf{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,.$$

Corollary. ***Distance from the origin***

The distance of the point P (x_1, y_1, z_1) from the origin O (0, 0, 0)

$$= \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2}$$

$$= \sqrt{x_1^2 + y_1^2 + z_1^2}\,.$$

Locus of a point.

*The **locus** of a point in space is the surface formed by the moving point under a given geometrical condition* (*or conditions*).

ILLUSTRATIVE EXAMPLES

Example 1. *Find the distance between the points P(–3, 7, 2) and Q(2, 4, –1).* (NCERT)

Solution. The distance between the points P(–3, 7, 2) and Q(2, 4, –1) is given by

$$PQ = \sqrt{(2-(-3))^2 + (4-7)^2 + (-1-2)^2} \text{ units}$$

$$= \sqrt{25+9+9} \text{ units} = \sqrt{43} \text{ units.}$$

Example 2. *If the distance between the points* (*a*, 2, 1) *and* (1, –1, 1) *is 5 units, find a.* (*NCERT Examplar Problems*)

Solution. Let the points A, B be (a, 2, 1), (1, –1, 1) respectively, then

$$AB = 5 \text{ (given)}$$

$$\Rightarrow \quad \sqrt{(1-a)^2 + (-1-2)^2 + (1-1)^2} = 5$$

$$\Rightarrow \quad 1 - 2a + a^2 + 9 + 0 = 25 \Rightarrow a^2 - 2a - 15 = 0$$

$$\Rightarrow \quad (a - 5)(a + 3) = 0 \Rightarrow a = 5, -3.$$

Example 3. *Find the length of perpendicular drawn from the point P*(3, 4, 5) *on y-axis.* (*NCERT Examplar Problems*)

Solution. Let M be the foot of perpendicular drawn from the point P(3, 4, 5) on y-axis, then the co-ordinates of point M are (0, 4, 0).

Length of perpendicular drawn from P(3, 4, 5) on y-axis

$= $ distance between points P(3, 4, 5) and M(0, 4, 0) = MP

$$= \sqrt{(3-0)^2 + (4-4)^2 + (5-0)^2} = \sqrt{9+0+25} = \sqrt{34}.$$

Example 4. *If a parallelopiped is formed by planes drawn through the points P(2, 3, 5) and Q(5, 9, 7) parallel to the co-ordinate planes, then find the lengths of edges of the parallelopiped and the length of a diagonal.* (*NCERT Examplar Problems*)

Solution. The planes passing through points P(2, 3, 5) and Q(5, 9, 7) drawn parallel to the co-ordinate planes form a cuboid (shown in the adjoining fig. 12.6).

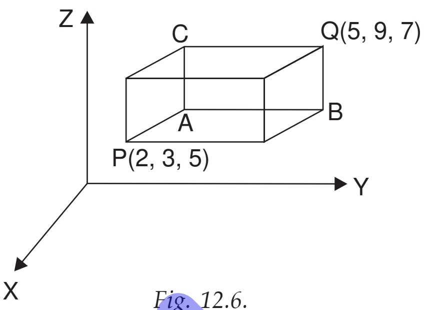

Fig. 12.6.

Length of edge parallel to x-axis

$= AP = x_2 - x_1 = 5 - 2 = 3,$

length of edge parallel to y-axis

$= AB = y_2 - y_1 = 9 - 3 = 6$ and

length of edge parallel to z-axis

$= AC = z_2 - z_1 = 7 - 2.$

Note that PQ is a diagonal of the cuboid.

$$\text{Length of a diagonal} = PQ = \sqrt{(5-2)^2 + (9-3)^2 + (7-5)^2}$$

$$= \sqrt{9+36+4} = \sqrt{49} = 7 \text{ units.}$$

Example 5. *By using distance formula, show that the points P(–2, 3, 5), Q(1, 2, 3) and R(7, 0, –1) are collinear.* (NCERT)

Solution. $PQ = \sqrt{(1-(-2))^2 + (2-3)^2 + (3-5)^2} = \sqrt{9+1+4} = \sqrt{14},$

$$QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2} = \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$$

and $PR = \sqrt{(7-(-2))^2 + (0-3)^2 + (-1-5)^2} = \sqrt{81+9+36} = \sqrt{126} = 3\sqrt{14}.$

Clearly, PR = PQ + QR, therefore, the given points are collinear.

Example 6. *Show that the points A (0, 7, 10), B (–1, 6, 6) and C (–4, 9, 6) are the vertices of an isosceles right angled triangle.* (NCERT)

Solution. $AB = \sqrt{(-1-0)^2 + (6-7)^2 + (6-10)^2} = \sqrt{1+1+16} = \sqrt{18}$

$$BC = \sqrt{(-4+1)^2 + (9-6)^2 + (6-6)^2} = \sqrt{9+9+0} = \sqrt{18}$$

and $CA = \sqrt{(-4-0)^2 + (9-7)^2 + (6-10)^2} = \sqrt{16+4+16} = 6.$

Clearly, AB = BC and $AB^2 + BC^2 = 18 + 18 = 36 = CA^2$, therefore, Δ ABC is an isosceles and right angled at B.

Example 7. *Are the points A (3, 6, 9), B (10, 20, 30) and C (25, –41, 5), the vertices of a right angled triangle?* (NCERT)

Solution.

$$AB = \sqrt{(10-3)^2 + (20-6)^2 + (30-9)^2} = \sqrt{49+196+441} = \sqrt{686},$$

$$BC = \sqrt{(25-10)^2 + (-41-20)^2 + (5-30)^2} = \sqrt{225+3721+625} = \sqrt{4571},$$

$$CA = \sqrt{(3-25)^2 + (6-(-41))^2 + (9-5)^2} = \sqrt{484+2209+16} = \sqrt{2709}.$$

Clearly BC > AB and BC > CA. Thus, BC is the longest side.

Here, $AB^2 + CA^2 = 686 + 2709 = 3395 \neq BC^2$.

Hence, the triangle ABC is not a right angled triangle.

CHAPTER TEST

1. By using distance formula, prove that the points A(3, –5, 1), B(–1, 0, 8) and C(7, –10, –6) are collinear.
2. Find the points on z-axis which are at a distance $\sqrt{21}$ from the point A(1, 2, 3).
3. Find the co-ordinates of the point which is equidistant from the four points with co-ordinates O (0, 0, 0), A (–3, 0, 0), B (0, 2, 0) and C (0, 0, –5). Find also the distance of this point from each of the given points.
4. A point C with z-coordinate 8 lies on the line segment joining the points A(2, –3, 4) and B(8, 0, 10). Find its co-ordinates.
5. In what ratio is the line joining the points A(–2, 3, 1) and B(3, –2, 5) divided by ZX-plane? Also find the co-ordinates of the point of division.
6. Show that the three points A (2, 3, 4), B (–1, 2, –3) and C (–4, 1, –10) are collinear. Also find the ratio in which C divides the segment AB. (*NCERT Examplar Problems*)
7. If the points P(0, –11, 4), Q(4, p, –2) and R(2, –3, 1) are collinear, find the value of p.
8. The mid-points of the sides of a triangle are (5, 7, 11), (0, 8, 5) and (2, 3, –1). Find the co-ordinates of the vertices of the triangle. (*NCERT Examplar Problems*)
9. If origin is the centroid of the triangle PQR with vertices P(2a, 2, 6), Q(–4, 3b, –10) and R(8, 14, 2c), then find the values of a, b and c. (*NCERT*)
10. If A and B are the points (–2, 2, 3) and (–1, 4, –3) respectively, then find the locus of P such that $3\,|PA| = 2\,|PB|$.
11. Find the coordinates of the point which is equidistant from the points (3, 2, 2), (–1, 2, 2), (0, 5, 6) and (2, 1, 2).
12. Prove that the points O (0, 0, 0), A (2, 0, 0), B $(1, \sqrt{3}, 0)$ and C $\left(1, \frac{1}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right)$ are the vertices of a regular tetrahedron.

   Hint. Show that $|OA| = |OB| = |OC| = |AB| = |BC| = |CA|$.

ANSWERS

EXERCISE 12.1

1. y-coordinate and z-coordinate are both zero
2. x-coordinate and z-coordinate are both zero **3.** x-coordinate is zero
4. y-coordinate is zero **5.** (*i*) (3, 0, 0) (*ii*) (0, –2, 0) (*iii*) (0, 0, 5)
6. (*i*) (3, –4, 0) (*ii*) (0, –4, 5) (*iii*) (3, 0, 5)
7. (*i*) (–3, 2, –7) (*ii*) (3, 2, 7) (*iii*) (–3, –2, 7)
8. (*i*) (3, 4, 7) (*ii*) (–7, –2, –1) (*iii*) (5, 4, 3) (*iv*) (–4, 0, 1) (*v*) (–2, 0, 0)
9. (*i*) I (*ii*) IV (*iii*) III (*iv*) VII (*v*) II (*vi*) VIII (*vii*) V (*viii*) VI
10. M(a, b, 0), N(0, b, c), L(a, 0, c), A(a, 0, 0), B(0, b, 0), C(0, 0, c)
11. (*i*) XY-plane (*ii*) (x, y, 0) (*iii*) eight

EXERCISE 12.2

1. 11 units (*ii*) 5 units
2. (*i*) $2\sqrt{5}$ units (*ii*) $3\sqrt{5}$ units (*iii*) $2\sqrt{5}$ units (*iv*) $2\sqrt{26}$ units
3. (*i*) 9, –15 (*ii*) 1, 9 **5.** 12 units
6. (*i*) $2\sqrt{5}$ units (*ii*) $\sqrt{13}$ units (*iii*) 5 units **7.** (6, 0, 0), (–2, 0, 0)
8. (0, 2, 0) **9.** (0, 0, 3), (0, 0, –5) **10.** $\left(0, \frac{5}{2}, 0\right)$