11

CONIC SECTIONS

INTRODUCTION

The curves known as *conics* were named after their historical discovery as the intersection of a plane with a *right circular cone.* Appllonius (before 200 B.C.) realised that a conic (or conic section) is a curve of intersection of a plane with a right circular cone of two nappes, and the three curves so obtained are *parabola, hyperbola* and *ellipse.*

In this chapter, we will see how the intersection of a plane with a double napped right circular cone results in different types of curves. We will also derive the standard equations of parabola, ellipse, hyperbola and circle and will study their simple properties.

11.1 SECTIONS OF A CONE

Let l be a fixed line and m be another line intersecting it at a fixed point V and making an angle α with it as shown in fig. 11.1.

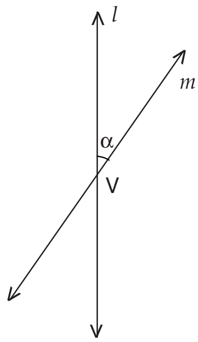

Fig. 11.1.

Now, we rotate the line m around l in such a way that the angle α remains constant. Then the surface generated is a double napped right circular hollow cone extending indefinitely far in both directions. A portion of a right circular cone of two nappes is shown in fig. 11.2. This surface is usually referred as right circular cone. The fixed point V is called the **vertex** of the cone; the fixed line l is called the **axis** of the cone. The rotating line m is called a generator of the cone. The vertex separates the cone into two parts called **nappes**.

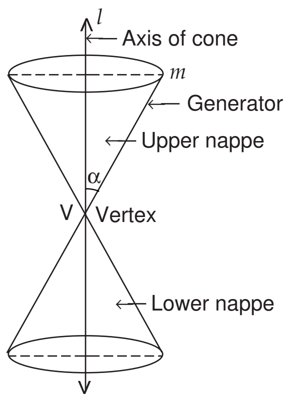

Fig. 11.2.

If we take the intersection of a plane with a cone, the section so obtained is called a conic section (or conic). Thus conic sections are the curves obtained by the intersection of a right circular cone and a plane.

We obtain different kinds of conics depending upon the position of the intersecting plane with respect to cone and by the angle made by it with the axis of the cone. Let β $(0 < \beta < \frac{\pi}{2})$ be the angle made by the intersecting plane with the axis of the cone as shown in fig 11.3.

The plane may intersect the cone either at its vertex or at any other part of the nappe either above or below the vertex.

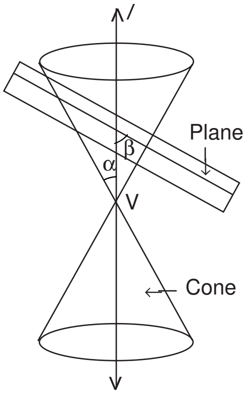

Fig. 11.3.

11.1.1 Degenerated conic sections

When the plane cuts the cone at its vertex, we have the following different cases :

Case I. When $\alpha < \beta < \frac{\pi}{2}$.

In this case, the section is a ***point*** as shown in fig. 11.4.

Case II. When $\alpha = \beta$.

In this case, the section is a ***straight line*** as shown in fig. 11.5.

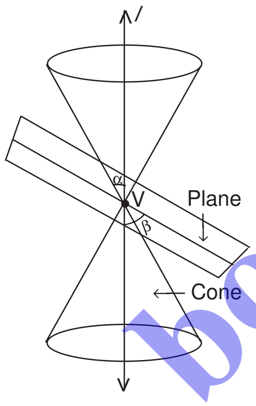

Fig. 11.4.

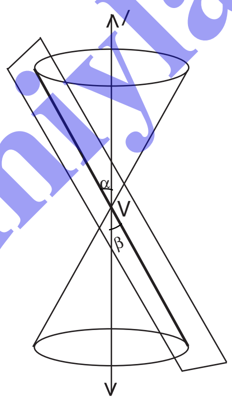

Fig. 11.5.

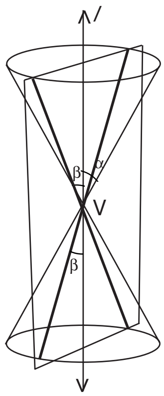

Fig. 11.6.

Case III. When $0 < \beta < \alpha$.

In this case, the section is a ***pair of intersecting straight lines*** as shown in fig. 11.6. In fact, it is a degenerated case of a hyperbola.

11.1.2 Different conic sections

When the intersection of a plane with a cone does not contain the vertex of the cone, then we have the following cases :

Case I. When $\beta = \frac{\pi}{2}$.

In this case, the section is a ***circle*** as shown in fig. 11.7.

Case II. When $\alpha < \beta < \frac{\pi}{2}$.

In this case, the section is an ***ellipse*** as shown in fig. 11.8.

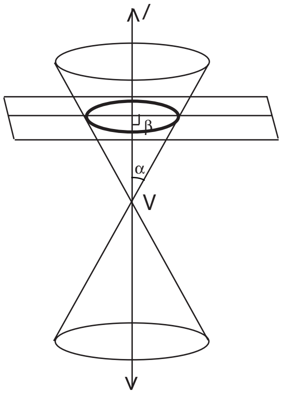

Fig. 11.7.

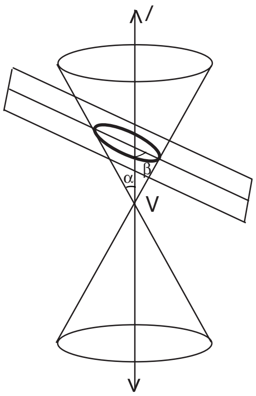

Fig. 11.8.

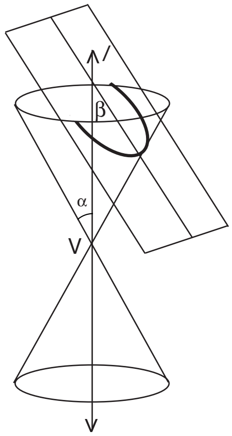

Fig. 11.9.

Case III. When $\beta = \alpha$.

In this case, the section is a ***parabola*** as shown in fig. 11.9.

Case IV. When $0 < \beta < \alpha$.

In this case, the plane cuts through both the nappes and the curves of intersection is a ***hyperbola*** as shown in fig. 11.10.

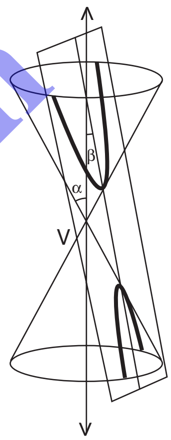

Fig. 11.10.

11.2 CIRCLE

*A **circle** is the set of all points in a plane, each of which is at a constant distance from a fixed point in the plane.*

*In other words, a **circle** is the locus of a point which moves in a plane so that it remains at a constant distance from a fixed point in the plane. The fixed point is called the **centre** and the constant distance is called **radius**. Radius is always positive.*

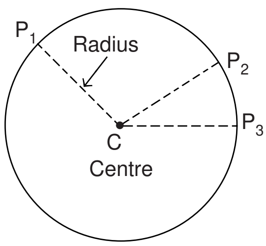

Fig. 11.11.

If P_1, P_2, P_3, … are points on the circle with centre C and radius r, then

$$CP_1 = CP_2 = CP_3 = \ldots = r.$$

Equation of a circle is simplest if its centre is at the origin.

11.2.1 Standard (or simplest) form

Let O (0, 0) be the centre of the circle and r (> 0) be its radius. Let P (x, y) be a point in the plane, then P lies on the circle iff

$$OP = r$$

i.e. iff $\sqrt{(x-0)^2 + (y-0)^2} = r$

i.e. iff $\boldsymbol{x^2 + y^2 = r^2}$, which is the equation of the circle.

This is known as ***standard*** (or ***simplest***) *form.*

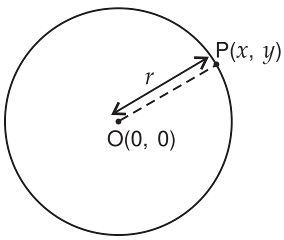

Fig. 11.12.

11.2.2 Central form

Let C (h, k) be the centre of the circle and r (> 0) be its radius. Let P (x, y) be a point in the plane, then P lies on the circle iff

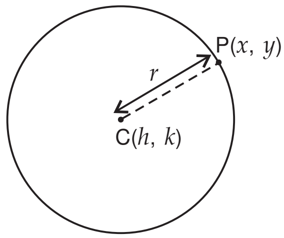

Fig. 11.13.

$$\text{CP} = r$$

i.e. iff $\sqrt{(x-h)^2+(y-k)^2} = r$

i.e. iff $\boldsymbol{(x - h)^2 + (y - k)^2 = r^2}$,

which is the equation of the circle.

This is known as ***central form***.

11.2.3 Diameter form

Let A (x_1, y_1) and B (x_2, y_2) be the extremities of a diameter of the circle.

Let P (x, y), different from A and B, be a point on the circle, then

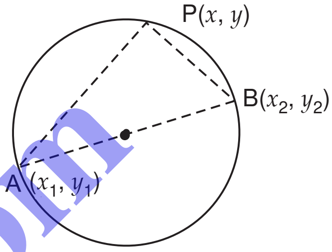

Fig. 11.14.

slope of line AP $= \dfrac{y - y_1}{x - x_1}$ and

slope of line BP $= \dfrac{y - y_2}{x - x_2}$.

Now P will lie on the circle iff $\angle$APB = 90°

i.e. iff the lines AP and BP are perpendicular to each other

i.e. iff $\dfrac{y - y_1}{x - x_1} \cdot \dfrac{y - y_2}{x - x_2} = -1$

i.e. $(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

i.e. iff $\boldsymbol{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0}$, which is the equation of the circle.

This is known as ***diameter form***.

11.2.4 General form

We know that the equation of the circle with centre (h, k) and radius r (> 0) is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0 \quad \ldots(i)$$

It can be written as

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(ii)$$

where $g = -h$, $f = -k$ and $c = h^2 + k^2 - r^2$ such that

$$g^2 + f^2 - c = (-h)^2 + (-k)^2 - (h^2 + k^2 - r^2) = r^2 > 0 \quad (\because r > 0)$$

Conversely, if we consider any equation $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(*iii*)

with $g^2 + f^2 - c > 0$, then on adding $g^2 + f^2$ to both sides of (*iii*), we get

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) + c = g^2 + f^2$$

$$\Rightarrow \quad (x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

$$\Rightarrow \quad (x - (-g))^2 + (y - (-f))^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

$$\Rightarrow \quad \sqrt{(x-(-g))^2 + (y-(-f))^2} = \sqrt{g^2 + f^2 - c}$$

$\Rightarrow$ the distance of the point (x, y) from the point $(-g, -f)$ is a fixed positive real number $r\ (= \sqrt{g^2 + f^2 - c})$ $(\because\ g^2 + f^2 - c > 0 \Rightarrow \sqrt{g^2 + f^2 - c}$ is a real number)

$\Rightarrow$ the locus of (*iii*) is the set of the all points (x, y) which are at a constant distance $r\ (= \sqrt{g^2 + f^2 - c})$ from the fixed point $(-g, -f)$

$\Rightarrow$ the equation (*iii*) represents a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

Thus, we have proved that the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle iff $g^2 + f^2 - c > 0$.

Its centre is $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

This is known as **general form.**

REMARKS

1. If $g^2 + f^2 - c = 0$, then the above equation (*iii*) is satisfied by one and only one point $(-g, -f)$, therefore, it represents a single point set, known as a **degenerate** (or **point**) **circle**.
2. If $g^2 + f^2 - c < 0$, then the above equation (*iii*) is not satisfied by any real values of x, y *i.e.* it is not satisfied by the co-ordinates of any point in the plane, therefore, in this case it represents the **empty set**.
3. We observe that the general equation of a circle has the following characteristics :
   (*i*) It is an equation of second degree in x, y containing no product term xy.
   (*ii*) coeff. of x^2 = coeff. of $y^2 = 1$.
   (*iii*) $\left(\left(\frac{1}{2}\text{ coeff. of } x\right)^2 + \left(\frac{1}{2}\text{ coeff. of } y\right)^2 - \text{constant term}\right)$ is a positive real number.
4. ***The equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$ represents a circle iff***
   (i) $a \neq 0$ and (ii) $g^2 + f^2 - ac > 0$.

First, note that $a \neq 0$, for, if $a = 0$ then the given equation reduces to

$$2gx + 2fy + c = 0$$

which being a first degree equation in x, y represents a straight line,

Now, dividing the given equation by a, we get $x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0$.

To complete square on L.H.S., on adding $\frac{g^2}{a^2} + \frac{f^2}{a^2}$ to both sides, we get

$$\left(x^2 + \frac{2g}{a}x + \frac{g^2}{a^2}\right) + \left(y^2 + \frac{2f}{a}y + \frac{f^2}{a^2}\right) = \frac{g^2}{a^2} + \frac{f^2}{a^2} - \frac{c}{a}$$

$$\Rightarrow \quad \left(x + \frac{g}{a}\right)^2 + \left(y + \frac{f}{a}\right)^2 = \frac{g^2 + f^2 - ac}{a^2}$$

$$\Rightarrow \quad \left(x - \left(-\frac{g}{a}\right)\right)^2 + \left(y - \left(-\frac{f}{a}\right)\right)^2 = \left(\frac{\sqrt{g^2 + f^2 - ac}}{|a|}\right)^2,$$

which represents a circle if $g^2 + f^2 - ac > 0$.

Therefore, the given equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$ represents a circle iff

(*i*) $a \neq 0$ and (*ii*) $g^2 + f^2 - ac > 0$.

If these conditions are satisfied then the given equation represents a circle with centre $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius $\frac{\sqrt{g^2 + f^2 - ac}}{|a|}$.

This form of the equation is known as **most general form**.

5. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle iff

(*i*) $a = b \neq 0$ (*ii*) $h = 0$ and (*iii*) $g^2 + f^2 - ac > 0$.

Rule to write the centre and radius of a circle :

(*i*) Write the given equation so that

coeff. of x^2 = coeff. of y^2 = 1 on L.H.S. (R.H.S. being zero) ;

dividing throughout (if necessary) by the coeff. of x^2 (or y^2).

(*ii*) Compare this equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, then

$$g = \frac{1}{2} \text{ coeff. of } x, \quad f = \frac{1}{2} \text{ coeff. of } y, \quad c = \text{constant term.}$$

(*iii*) Check that $g^2 + f^2 - c > 0$, then centre is $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

Concentric circles. *Circles having same centre and different radii are called* ***concentric circles****.*

Equal circles. *Circles having equal radius are called* ***equal circles****.*

11.2.5 Geometrical condition for the intersection of a line and a circle

Let S be a circle with centre C and radius r (> 0). Let l be a line in the plane of the circle S and d be the perpendicular distance from C to the line l, then (see fig. 11.15)

(*i*) l intersects S in two distinct points iff $d < r$.

(*ii*) l intersects S in one and only one point iff $d = r$.

(*iii*) l does not intersect S iff $d > r$.

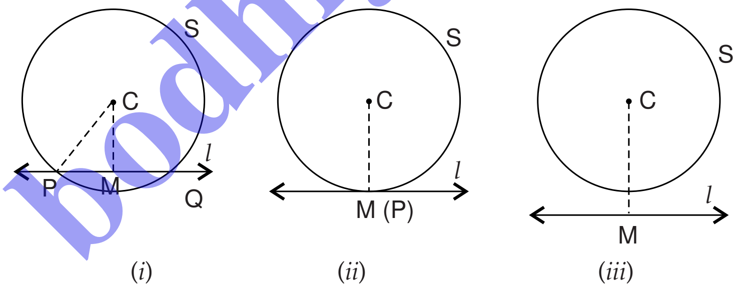

Fig. 11.15.

From the above, we note that if $d = r$ then the line l and the circle S have one and only one point in common, namely the foot of perpendicular M from C on l, and we say that l touches S or l is a ***tangent*** to S at M and the unique point M is called the ***point of contact.***

Thus, a line l touches a circle S iff the length of perpendicular from the centre of S to the line l is equal to the radius of S.

If a line l meets a circle in two distinct points A and B, then the length of the segment [*AB*] *i.e.* $|AB|$ *is called the* ***length of intercept*** *made by the circle S on the line l.*

11.2.6 Relative position of two circles

Let S_1, S_2 be two (non-concentric) circles with centres A, B and radii r_1, r_2 and d be the distance between their centres, then

(*i*) *one circle lies completely inside the other circle iff* $d < |r_1 - r_2|$

(*ii*) *the two circles touch internally iff* $d = |r_1 - r_2|$

(*iii*) *the two circles intersect in two points iff* $d > |r_1 - r_2|$ *and* $d < r_1 + r_2$

(*iv*) *the two circles touch externally iff* $d = r_1 + r_2$

(*v*) *one circle lies completely outside the other circle iff* $d > r_1 + r_2$.

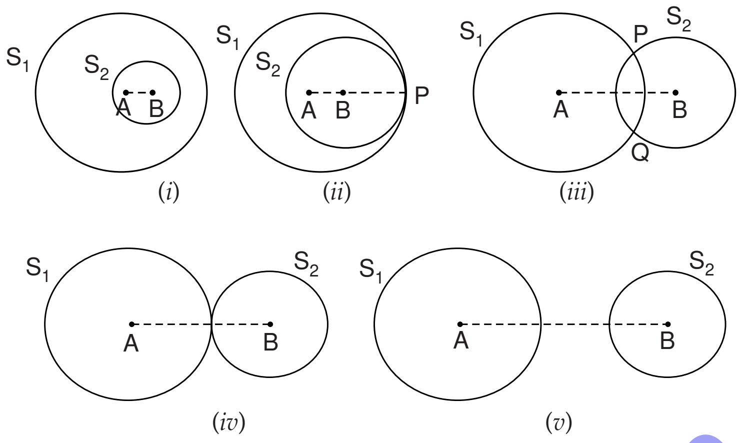

Fig. 11.16.

In fig. 11.16, we have taken $r_1 > r_2$. But we have similar situations when $r_1 < r_2$. Moreover, in case (*iii*), (*iv*) and (*v*), r_1 and r_2 may be equal.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the equation of a circle whose centre is (–2, 3) and radius is 4.* (*NCERT*)

Solution. Since the centre of the circle is (–2, 3) and its radius is 4, therefore, the equation of the circle is

$$(x - (-2))^2 + (y - 3)^2 = 4^2 \qquad \text{(central form)}$$

$$\Rightarrow \quad (x + 2)^2 + (y - 3)^2 = 16$$

$$\Rightarrow \quad x^2 + 4x + 4 + y^2 - 6y + 9 = 16$$

$$\Rightarrow \quad x^2 + y^2 + 4x - 6y - 3 = 0.$$

Example 2. *Find the equation of the circle with centre (2, 2) and which passes through the point (4, 5).* (*NCERT*)

Solution. The centre of the circle is C(2, 2) and it passes through the point P(4, 5)

$\therefore$ Radius of circle = CP

$$= \sqrt{(4-2)^2 + (5-2)^2}$$

$$= \sqrt{4+9} = \sqrt{13}.$$

$\therefore$ The equation of the circle is

$$(x - 2)^2 + (y - 2)^2 = (\sqrt{13})^2 \qquad \text{(central form)}$$

i.e. $x^2 + y^2 - 4x - 4y = 5$.

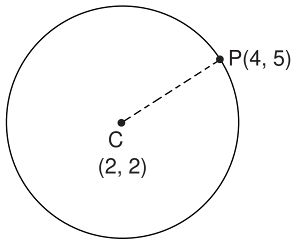

Fig. 11.17.

Example 3. *Find the equation of a circle having (1, –2) as its centre and passing through the intersection of the lines 3x + y = 14 and 2x + 5y = 18.* (*NCERT Examplar Problems*)

Solution. Given lines are

$$3x + y = 14 \quad \ldots(i) \qquad \text{and} \qquad 2x + 5y = 18 \quad \ldots(ii)$$

Solving (*i*) and (*ii*) simultaneously, we get $x = 4$, $y = 2$.

$\therefore$ The point of intersection, say P, of the given lines is (4, 2).

Since the centre of the circle is C (1, –2) and it passes through the point P (4, 2),

its radius = CP = $\sqrt{(4-1)^2+(2-(-2))^2} = \sqrt{9+16} = 5$.

$\therefore$ The equation of the circle is

$(x-1)^2 + (y+2)^2 = 5^2$ (Central Form)

or $x^2 + y^2 - 2x + 4y - 20 = 0$.

Example 4. *If two diameters of a circle lie along the lines $x - y - 9 = 0$ and $x - 2y - 7 = 0$ and the area of the circle is 38.5 sq units, find its equation.*

Solution. Since two diameters of the circle lie along the lines

$x - y - 9 = 0$ …(*i*) and $x - 2y - 7 = 0$ …(*ii*)

so their point of intersection is the centre of the circle.

Solving (*i*) and (*ii*) simultaneously, we get $x = 11$, $y = 2$.

$\therefore$ The centre of the circle is (11, 2).

Let r be the radius of the circle, then area $= \pi r^2 = 38.5$ square units (given)

$\Rightarrow \frac{22}{7} r^2 = \frac{77}{2} \Rightarrow r^2 = \frac{49}{4} \Rightarrow r = \frac{7}{2}$ $(\because r > 0)$

$\therefore$ The equation of the cirlce is

$(x-11)^2 + (y-2)^2 = \left(\frac{7}{2}\right)^2$ (central form)

or $x^2 + y^2 - 22x - 4y + 125 - \frac{49}{4} = 0$

or $4(x^2 + y^2) - 88x - 16y + 451 = 0$.

Example 5. *Find the equation of the circle which passes through the point (–2, –3) and has its centre on the negative direction of x-axis and is of radius 5 units.*

Solution. As the centre of the circle lies on the negative direction of x-axis, let its centre be C $(h, 0)$, $h < 0$.

Since the circle passes through A (–2, –3) and has radius 5,

CA $= 5 \Rightarrow (h+2)^2 + (0+3)^2 = 5^2$

$\Rightarrow (h+2)^2 = 25 - 9 = 16 \Rightarrow h + 2 = 4, -4$

$\Rightarrow h = 2, -6$ but $h < 0 \Rightarrow h = -6$.

$\therefore$ The centre of the circle is (–6, 0) and hence its equation is

$(x+6)^2 + (y-0)^2 = 5^2$ (central form)

i.e. $x^2 + y^2 + 12x + 11 = 0$.

Example 6. *Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2, 3).* *(NCERT)*

Solution. As the centre of the circle lies on x-axis, let its centre be C$(h, 0)$.

Since the circle passes through A (2, 3) and has radius 5,

CA $= 5 \Rightarrow (2-h)^2 + (3-0)^2 = 5^2$

$\Rightarrow (2-h)^2 = 16 \Rightarrow 2 - h = 4, -4 \Rightarrow h = -2, 6$.

$\therefore$ The centre of the circle is (–2, 0) or (6, 0).

The equation of the circle is

$(x+2)^2 + (y-0)^2 = 5^2$ or $(x-6)^2 + (y-0)^2 = 5^2$

i.e. $x^2 + y^2 + 4x - 21 = 0$ or $x^2 + y^2 - 12x + 11 = 0$.

There are two circles satisfying the given conditions.

Example 7. *Find the equation of the circle which touches both the axes in first quadrant and whose radius is a.* (*NCERT Examplar Problems*)

Solution. As the radius of the circle is a units and it touches both the axes in the first quadrant, its centre is C(a, a).

The equation of the circle is

$$(x - a)^2 + (y - a)^2 = a^2$$

or $\quad x^2 + y^2 - 2ax - 2ay + a^2 = 0.$

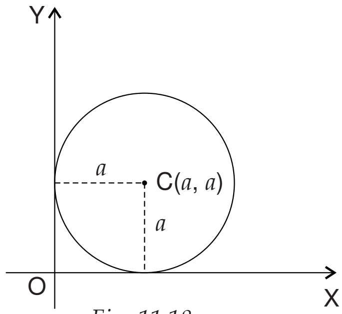

Fig. 11.18.

Example 8. *Find the equation of the circle which touches x-axis and whose centre is* (1, 2). (*NCERT Examplar Problems*)

Solution. Centre of the circle is C(1, 2).

As the circle touches the x-axis,

its radius = perpendicular distance from centre (1, 2) to the x-axis

= ordinate of point C = 2.

$\therefore$ The equation of the circle is $(x - 1)^2 + (y - 2)^2 = 2^2$

or $\quad x^2 + y^2 - 2x - 4y + 1 = 0.$

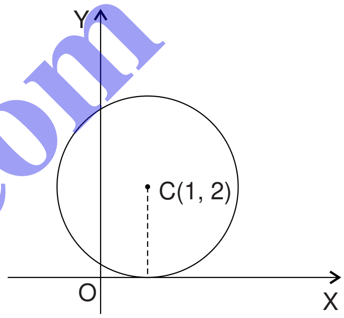

Fig. 11.19.

Example 9. *Find the equation of the circle whose centre is C(–2, 3) and which touches the line x – y + 7 = 0.*

Solution. The given line is $x - y + 7 = 0$ …(i)

Let r be the radius of the required circle, then

r = perpendicular distance from C(–2, 3) to the line (i)

$$= \frac{|-2-3+7|}{\sqrt{1^2+(-1)^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

$\therefore$ The equation of the circle is $(x + 2)^2 + (y - 3)^2 = (\sqrt{2})^2$

or $\quad x^2 + y^2 + 4x - 6y + 11 = 0.$

Example 10. *Find the equation of a circle which touches*

(i) the y-axis at origin and whose radius is 3 units

(ii) both the co-ordinate axes and the line x = 3.

Solution. (i) There are two circles satisfying given conditions. As the circles touch y-axis at the origin, their centres lie on x-axis. Since radius is 3 units, centres of the circles are (3, 0) and (–3, 0) and hence the equations of the circles are

$$(x \pm 3)^2 + (y - 0)^2 = 3^2$$

or $\quad x^2 + y^2 \pm 6x = 0.$

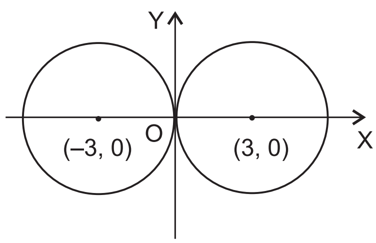

Fig. 11.20.

(*ii*) There are two circles satisfying the given conditions. From fig. 11.21, clearly, the centres of these circles are $\left(\frac{3}{2}, \frac{3}{2}\right)$ and $\left(\frac{3}{2}, -\frac{3}{2}\right)$, and radius of each circle is $\frac{3}{2}$.

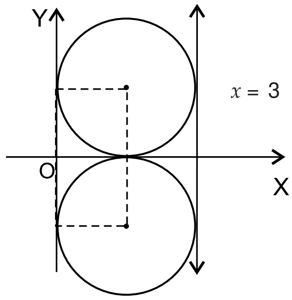

Fig. 11.21.

$\therefore$ The equations of these circles are

$$\left(x - \frac{3}{2}\right)^2 + \left(y \pm \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$

or $\quad x^2 + y^2 - 3x \pm 3y + \frac{9}{4} = 0$

or $\quad 4x^2 + 4y^2 - 12x \pm 12y + 9 = 0.$

Example 11. *Find the equation of a circle which touches both the axes and the line $3x - 4y + 8 = 0$ and lies in the third quadrant.* (*NCERT Examplar Problems*)

Solution. Given line is $3x - 4y + 8 = 0$ $\quad \ldots(i)$

Let the radius of the circle be a. As the circle lies in the third quadrant and touches both the axes, its centre is C$(-a, -a)$.

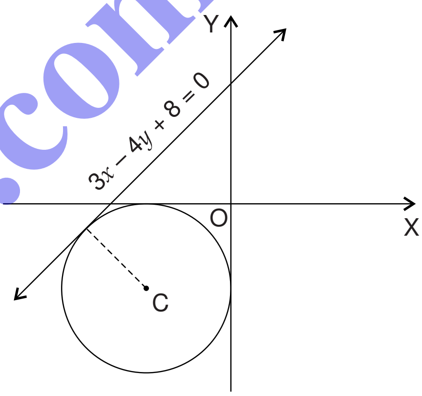

Fig. 11.22.

Since the line (*i*) touches the circle, the perpendicular distance from C$(-a, -a)$ to the line (*i*) = radius of circle.

$$\therefore \quad \frac{|-3a + 4a + 8|}{\sqrt{3^2 + (-4)^2}} = a \Rightarrow \frac{|a + 8|}{5} = a$$

$\Rightarrow \quad a + 8 = 5a$ or $a + 8 = -5a$

$\Rightarrow \quad a = 2$ or $a = -\frac{4}{3}$ but $a > 0$ (a being radius)

$\Rightarrow \quad a = 2.$

$\therefore$ The centre of circle is $(-2, -2)$ and radius = 2.

The equation of the circle is $(x + 2)^2 + (y + 2)^2 = 2^2$

or $\quad x^2 + y^2 + 4x + 4y + 4 = 0.$

Example 12. *Find the equation of a circle whose centre is $(3, -1)$ and which cuts off a chord of length 6 units on the line $2x - 5y + 18 = 0$.* (*NCERT Examplar Problems*)

Solution. The given line is $2x - 5y + 18 = 0$ $\ldots(i)$

and the centre of the circle is C$(3, -1)$.

Let the line (*i*) meet the required circle at points A and B. From C, draw CM $\perp$ AB then M is mid-point of segment AB.

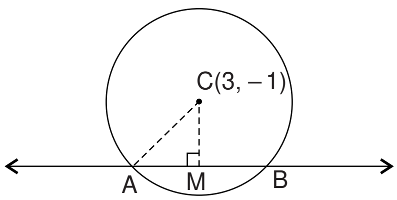

Fig. 11.23.

Given AB = 6 units $\Rightarrow$ 2 . AM = 6 units

$\Rightarrow \quad$ AM = 3 units

CM = perpendicular distance from C$(3, -1)$ to the line (*i*)

$$= \frac{|2.3 - 5.(-1) + 18|}{\sqrt{2^2 + (-5)^2}} \text{ units} = \frac{29}{\sqrt{29}} \text{ units}$$

$$= \sqrt{29} \text{ units.}$$

Let r units be the radius of the circle.

From rt. $\angle d$ ΔAMC, $CA^2 = AM^2 + CM^2$

$\Rightarrow \quad r^2 = 3^2 + (\sqrt{29})^2 = 38 \Rightarrow$ radius $= \sqrt{38}$ units.

$\therefore$ The equation of the circle is $(x-3)^2 + (y+1)^2 = (\sqrt{38})^2$

or $\quad x^2 + y^2 - 6x + 2y - 28 = 0.$

Example 13. *Find the equation of the circle which passes through two points on y-axis which are at a distance of 3 units from origin and has radius 5 units.*

Solution. There are two circles satisfying the given conditions.

Since the circles pass through the points on y-axis which are at distance of 3 units from origin, so OB = OB′ = 3. The centres of the circles lie on the right bisector of BB′ *i.e.* on x-axis. If C is centre of a circle, then CB is its radius.

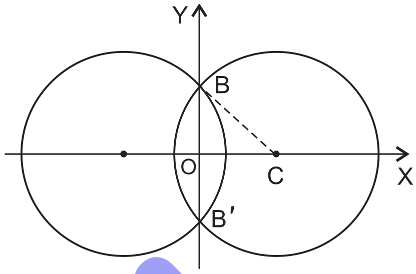

Fig. 11.24.

From rt. $\angle$d ΔOBC,

$OC^2 = BC^2 - OB^2 = 5^2 - 3^2 = 16$

$\Rightarrow \quad OC = 4.$

$\therefore \quad$ The centres of the circles are (4, 0) and (– 4, 0).

$\therefore \quad$ The equations of the circles are

$(x \pm 4)^2 + (y - 0)^2 = 5^2$

or $\quad x^2 + y^2 \pm 8x - 9 = 0.$

Example 14. *Find the equation of the circle when the end points of a diameter are A (– 2, 3) and B (3, – 5).*

Solution. Using diameter form, the equation of the circle having A (– 2, 3) and B (3, – 5) as the end points of a diameter is

$(x - (-2))(x - 3) + (y - 3)(y - (-5)) = 0$

or $\quad (x + 2)(x - 3) + (y - 3)(y + 5) = 0$

or $\quad x^2 - x - 6 + y^2 + 2y - 15 = 0$

or $\quad x^2 + y^2 - x + 2y - 21 = 0.$

Example 15. *Find the equation of a circle which has the portion of the line $3x + 4y = 14$ intercepted by the lines $x - y = 0$ and $11x - 4y = 0$ as a diameter.*

Solution. The given lines are

$3x + 4y - 14 = 0 \quad \ldots(i)$

$x - y = 0 \quad \ldots(ii)$

and $\quad 11x - 4y = 0 \quad \ldots(iii)$

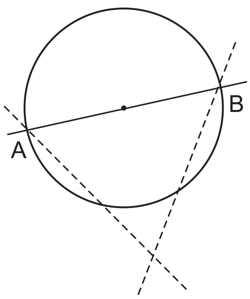

Fig. 11.25.

The intersection of (i) and (ii) is the point A (2, 2) and the intersection of (i) and (iii) is the point B $\left(1, \frac{11}{4}\right)$.

Using diameter form, the equation of the circle having AB as its diameter is

$(x - 2)(x - 1) + (y - 2)\left(y - \frac{11}{4}\right) = 0$

or $\quad x^2 - 3x + 2 + y^2 - 2y - \frac{11}{4}y + \frac{11}{2} = 0$

or $\quad x^2 + y^2 - 3x - \frac{19}{4}y + \frac{15}{2} = 0$

or $\quad 4x^2 + 4y^2 - 12x - 19y + 30 = 0.$

Example 29. *Find the equation of the circle passing through the point (7, 3) having radius 3 units and whose centre lies on the line $y = x - 1 = 0$.* (*NCERT Examplar Problems*)

Solution. The given line is $y = x - 1$ ...(*i*)

Let (h, k) be the centre of the circle. As centre lies on (*i*), we get

$$k = h - 1 \quad \text{...}(ii)$$

The equation of the circle is $(x - h)^2 + (y - k)^2 = 3^2$ ($\because$ radius = 3)

or $(x - h)^2 + (y - (h - 1))^2 = 9.$

Since the circle passes through the point (7, 3),

$\therefore$ $(7 - h)^2 + (3 - (h - 1))^2 = 9$

$\Rightarrow$ $(7 - h)^2 + (4 - h)^2 = 9$

$\Rightarrow$ $49 - 14h + h^2 + 16 - 8h + h^2 - 9 = 0$

$\Rightarrow$ $2h^2 - 22h + 56 = 0 \Rightarrow h^2 - 11h + 28 = 0$

$\Rightarrow$ $(h - 7)(h - 4) = 0 \Rightarrow h = 7, 4.$

When $h = 7$, $k = 7 - 1 = 6$; when $h = 4$, $k = 4 - 1 = 3$.

$\therefore$ The centre of circle is (7, 6) or (4, 3).

Thus, we have two circles satisfying given conditions. Their equations are

$$(x - 7)^2 + (y - 6)^2 = 3^2 \text{ or } (x - 4)^2 + (y - 3)^2 = 3^2$$

i.e. $x^2 + y^2 - 14x - 12y + 76 = 0$ or $x^2 + y^2 - 8x - 6y + 16 = 0$

Example 30. *Find the equation of the circle passing through the points (1, –2), (5, 4) and (10, 5).*

Solution. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...}(i)$$

As the circle passes through the points (1, –2), (5, 4) and (10, 5), we get

$$1 + 4 + 2g - 4f + c = 0 \quad \Rightarrow 2g - 4f + c + 5 = 0 \quad \text{...}(ii)$$

$$25 + 16 + 10g + 8f + c = 0 \quad \Rightarrow 10g + 8f + c + 41 = 0 \quad \text{...}(iii)$$

$$100 + 25 + 20g + 10f + c = 0 \Rightarrow 20g + 10f + c + 125 = 0 \quad \text{...}(iv)$$

Subtracting (*ii*) from (*iii*), we get

$$8g + 12f + 36 = 0 \quad \Rightarrow 2g + 3f + 9 = 0 \quad \text{...}(v)$$

Subtracting (*ii*) from (*iv*), we get

$$18g + 14f + 120 = 0 \quad \Rightarrow 9g + 7f + 60 = 0 \quad \text{...}(vi)$$

Solving (*v*) and (*vi*) simultaneously, we get $g = -9$, $f = 3$.

From (*ii*), we get $c = -5 - 2(-9) + 4(3) = 25$.

Substituting these values of g, f and c in (*i*), we get

$x^2 + y^2 - 18x + 6y + 25 = 0$, which is the equation of the required circle.

Example 31. *Show that the points (7, 5), (6, –2), (–1, –1) and (0, 6) are concyclic. Also find the radius and the centre of the circle on which they lie.*

Solution. Let us find the equation of the circle passing through the points (7, 5), (6, –2) and (–1, –1).

Let the equation of this circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...}(i)$$

As the points (7, 5), (6, –2) and (–1, –1) lie on it, we get

$$49 + 25 + 14g + 10f + c \quad \Rightarrow 14g + 10f + c + 74 = 0 \quad \text{...}(ii)$$

$$36 + 4 + 12g - 4f + c = 0 \Rightarrow 12g - 4f + c + 40 = 0 \quad \text{...}(iii)$$

$$1 + 1 - 2g - 2f + c = 0 \quad \Rightarrow 2g + 2f - c - 2 = 0 \quad \text{...}(iv)$$

Adding (*ii*) and (*iv*), we get

$$16g + 12f + 72 = 0 \Rightarrow 4g + 3f + 18 = 0 \qquad \ldots(v)$$

Adding (*iii*) and (*iv*), we get

$$14g - 2f + 38 = 0 \Rightarrow 7g - f + 19 = 0 \qquad \ldots(vi)$$

Solving (*v*) and (*vi*) simultaneously, we get $g = -3$, $f = -2$.

From (*ii*), we get $c = -14(-3) - 10(-2) - 74 = -12$.

Substituting these values of g, f and c in (*i*), we get

$$x^2 + y^2 - 6x - 4y - 12 = 0 \qquad \ldots(vii)$$

The fourth point (0, 6) will lie on (*vii*) if $0 + 36 - 0 - 24 - 12 = 0$ *i.e.* if $0 = 0$, which is true.

Hence, the given points are concyclic.

Also, (*vii*) is the equation of the circle on which these points lie.

Its centre is (3, 2) and radius $= \sqrt{9 + 4 - (-12)} = 5$.

Example 32. *Find the equation of the circle circumscribing the triangle formed by the straight lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$.*

Solution. Let the equations of the sides AB, BC and CA of ΔABC be

$$x + y = 6 \qquad \ldots(i)$$

$$2x + y = 4 \qquad \ldots(ii)$$

and $$x + 2y = 5 \qquad \ldots(iii)$$

respectively. Solving (*i*) and (*iii*); (*i*) and (*ii*); (*ii*) and (*iii*) simultaneously, we get the coordinates of the points A, B and C as (7, –1), (–2, 8) and (1, 2) respectivley.

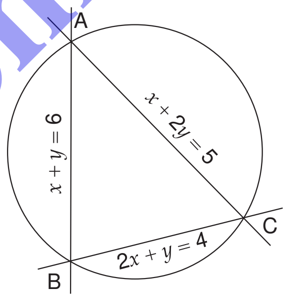

Fig. 11.32.

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots(iv)$$

As the circle passes through the points A (7, –1), B (–2, 8) and C (1, 2), we get

$$49 + 1 + 14g - 2f + c = 0 \quad i.e. \; 14g - 2f + c + 50 = 0 \qquad \ldots(v)$$

$$4 + 64 - 4g + 16f + c = 0 \quad i.e. \; -4g + 16f + c + 68 = 0 \qquad \ldots(vi)$$

$$1 + 4 + 2g + 4f + c = 0 \quad i.e. \; 2g + 4f + c + 5 = 0 \qquad \ldots(vii)$$

Subtracting (*vi*) from (*v*), we get

$$18g - 18f - 18 = 0 \quad i.e. \; g - f - 1 = 0 \qquad \ldots(viii)$$

Subtracting (*vii*) from (*v*), we get

$$12g - 6f + 45 = 0 \quad i.e. \; 4g - 2f + 15 = 0 \qquad \ldots(ix)$$

Solving (*viii*) and (*ix*), we get $g = -\frac{17}{2}$, $f = -\frac{19}{2}$.

Putting these values of g and f in (*vii*), we get $c = 50$.

Substituting these values of g, f and c in (*iv*), we get

$x^2 + y^2 - 17x - 19y + 50 = 0$, which is the equation of the required circle.

EXERCISE 11.1

Very short answer type questions (1 to 23) :

1. Find the equation of the circle whose :

(*i*) centre is at the origin and the radius is 5 units

(*ii*) centre is (0, 2) and radius 2 (*NCERT*) (*iii*) centre is (– 3, 2) and radius 4 (*NCERT*) (*iv*) centre is (1, 1) and radius $\sqrt{2}$ (*NCERT*) (*v*) centre is $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$ (*NCERT*)

(*vi*) centre is $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$. (*NCERT*)

2. Write the equation of a circle whose centre is origin and which passes through the point (3, –4).

3. Determine the equation of a circle whose centre is (8, –6) and which passes through the point (5, –2).

4. Show that the points (–3, 2) and (4, 3) lie on a circle with centre at the point (1, –1).

5. Find the equation of a circle of radius 6 units and whose centre lies on the negative direction of x-axis at a distance of 4 units from the origin.

6. Find the equation of the circle whose centre lies on the negative direction of y-axis at a distance 3 units from origin and whose radius is 4 units.

7. Find the equation of the circle which has A (1, 3) and B (4, 5) as opposite ends of a diameter.

8. Find the equation of the circle which has the points (–2, 3) and (0, –1) as opposite ends of a diameter.

9. Find the equation of the circle which passes through the origin and cuts off intercepts –2 and 3 from the co-ordinate axes.

10. Find the equation of the circle which passes through origin and cuts off intercepts 3 and –2 from the coordinate axes.

11. Does the point (–2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$? (*NCERT*)

Hint. Centre of the circle is O (0, 0) and its radius = 5. Distance from centre to the given point = $\sqrt{18.5}$, which is less than radius of the circle.

12. Does the (–3, 7) lie inside, outside or on the circle $x^2 + y^2 = 49$?

13. Which of the following equations represent a circle?

(*i*) $x^2 + y^2 + 6x - 14y + 5 = 0$ (*ii*) $x^2 + y^2 + 3x - 2y + 7 = 0$

(*iii*) $2x^2 + 3y^2 - 4x - 6y - 5 = 0$ (*iv*) $x^2 + y^2 - 10x + 2y + 20 = 0$?

14. What does the equation $x^2 + y^2 + 4x + 6y + 13 = 0$ represent?

15. Find the centre and the radius of the circle :

(*i*) $(x + 5)^2 + (y - 3)^2 = 36$ (*NCERT*) (*ii*) $x^2 + y^2 - 2x + 4y = 8$ (*NCERT Examplar Problems*)

(*iii*) $x^2 + y^2 + 8x + 10y - 8 = 0$ (*NCERT*) (*iv*) $x^2 + y^2 - 4x - 8y - 45 = 0$ (*NCERT*)

(*v*) $2x^2 + 2y^2 - 3x + 5y - 7 = 0$ (*vi*) $x^2 + y^2 - ax - by = 0$.

16. Which of the following equations represent a circle? If so, determine its centre and radius :

(*i*) $x^2 + y^2 + x - y = 0$ (*ii*) $x^2 + y^2 - 3x + 3y + 10 = 0$

(*iii*) $2x^2 + 2y^2 = 5x + 7y + 3$ (*iv*) $x^2 + y^2 + 2x + 10y + 26 = 0$.

17. Find the value of p so that $x^2 + y^2 + 8x + 10y + p = 0$ is the equation of a circle of radius 7 units.

18. Find the value(s) of k so that the equation $x^2 + y^2 - 2kx + 4y - 12 = 0$ may represent a circle of radius 5 units.

19. Find the equation of a circle concentric with the circle $x^2 + y^2 - 8x + 2y + 3 = 0$ and of radius 3 units.

20. Find the value of k for which the circles $x^2 + y^2 - 3x + ky - 5 = 0$ and

$4x^2 + 4y^2 - 12x - y - 9 = 0$ are concentric.

21. Find the equation of the circle which passes through the point (1, –2) and is concentric with the circle $x^2 + y^2 - 4x + 5y - 7 = 0$.

22. Find the shortest distance of the point (8, 1) from the circle $(x + 2)^2 + (y - 1)^2 = 25$.

41. Find the equation of the circle which passes through the centre of the circle $x^2 + y^2 + 8x + 10y + 7 = 0$ and is concentric with the circle

$2x^2 + 2y^2 - 8x - 12y - 9 = 0.$

42. Find the equation of the circle concentric with the circle

$x^2 + y^2 + 4x - 8y - 6 = 0$ and having radius double of its radius.

43. Find the equation of the circle concentric with the circle

$2x^2 + 2y^2 + 8x + 10y - 35 = 0$ and with area 16π square units.

44. Find the equation of the circle which is concentric with the circle

$x^2 + y^2 - 4x + 6y - 3 = 0$ and of double its

(*i*) circumference (*ii*) area

45. Prove that the centres of three circles

$x^2 + y^2 - 4x - 6y - 14 = 0$, $x^2 + y^2 + 2x + 4y - 5 = 0$ and

$x^2 + y^2 - 10x - 16y + 7 = 0$ are collinear.

46. Prove that the radii of the circles $x^2 + y^2 = 1$, $x^2 + y^2 - 2x - 6y - 6 = 0$ and $x^2 + y^2 - 4x - 12y - 9 = 0$ are in A.P.

47. Find the equation of the circle passing through the points (4, 1), (6, 5) and whose centre is on the line $4x + y = 16$. (*NCERT*)

48. Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the line $y - 4x + 3 = 0$. (*NCERT Examplar Problems*)

49. Find the equation of the circle which passes through the points (2, – 2), (3, 4) and whose centre is on the line $x + y = 2$. (*NCERT*)

50. Find the equation of the circle passing through the three points

(*i*) (0, 0), (0, 1) and (2, 3) (*ii*) (0, 0), (5, 0) and (3, 3)

(*iii*) (1, 0), (–1, 0) and (0, 1) (*iv*) (1, 2), (3, – 4) and (5, –6).

(*v*) (20, 3), (19, 8) and (2, –9) (*NCERT Examplar Problems*)

Also find its centre and radius.

51. Find the equation of the circle which is circumscribed about the triangle whose vertices are (– 2, 3), (5, 2) and (6, – 1).

52. Show that the points (7, 1), (–2, 4), (5, 5) and (6, 4) are concyclic. Also find the equation, centre and radius of the circle on which they lie.

11.3 SYMMETRY

Reflection of a point in a line

If a point P lies on the line *l*, then the **reflection** of P in *l* is defined as the point P itself. If P does not lie on *l*, let M be the foot of perpendicular from P on *l* and produce it to a point P′ such that MP = MP′, then P′ is called the **reflection** of P in *l* (shown in fig. 11.33).

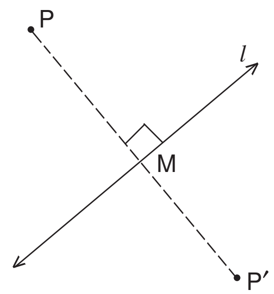

Fig. 11.33.

Note that if P′ is the reflection of P in *l*, then P is the reflection of P′ in *l*.

It follows that if P does not lie on *l*, then P′ is the reflection of P in *l* iff *l* is the perpendicular bisector of the segment PP′.

Reflection of a point in a point

The **reflection** of a point P in a fixed point M is the point P′ such that M is mid-point of the segment PP′ (shown in fig. 11.34).

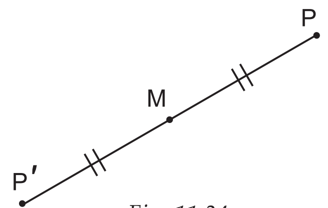

Fig. 11.34.

In particular, if P = M then P′ = M = P.

Note that if P′ is the reflection of P in a point M, then P is the reflection of P′ in M.

We leave it for the reader to see that the *reflection of a point* (α, β) *in the*

(*i*) *origin is the point* $(-\alpha, -\beta)$

(*ii*) *x-axis is the point* $(\alpha, -\beta)$

(*iii*) *y-axis is the point* $(-\alpha, \beta)$.

Symmetry of a curve about a line

A curve C is said to be ***symmetrical*** *about a line l iff for every point P on C, the reflection of P in l also lies on C. If a curve is symmetrical about a line l, then l is called a* ***line of symmetry*** *of C or an* ***axis*** *of C.*

Let $F(x, y) = 0$ be an equation of a curve C, then C is symmetrical about

(i) x-axis iff $F(x, y) = F(x, -y)$ ***(ii) y-axis iff*** $F(x, y) = F(-x, y)$.

Proof. (*i*) Let $P(\alpha, \beta)$ be a point on the curve C, then

$$F(\alpha, \beta) = 0 \quad \ldots(1)$$

Now the curve C, is symmetrical about x-axis iff the reflection of P in the x-axis *i.e.* the point $P'(\alpha, -\beta)$ lies on C *i.e.* iff $F(\alpha, -\beta) = 0$ ...(2)

From (1) and (2), it follows that C is symmetrical about x-axis iff

$$F(x, y) = F(x, -y).$$

We leave the proof of (*ii*) for the reader.

Symmetry of a curve about a point

A curve C is said to be ***symmetrical*** *about a point M iff for every point P on C, the reflection of P in M also lies on C. If a curve is symmetrical about a point M, then M is called* ***a centre of symmetry*** *of C or a* ***centre*** *of C.*

Let $F(x, y) = 0$ ***be an equation of a curve C, then C is symmetrical about the origin iff*** $F(x, y) = F(-x, -y)$.

Proof. Let $P(\alpha, \beta)$ be a point on the curve C, then

$$F(\alpha, \beta) = 0 \quad \ldots(1)$$

Now the curve C is symmetrical about the origin iff the reflection of P in the origin *i.e.* the point $P'(-\alpha, -\beta)$ lies on C

i.e. iff $F(-\alpha, -\beta) = 0$ ...(2)

From (1) and (2), it follows that C is symmetrical about origin iff

$$F(x, y) = F(-x, -y).$$

ILLUSTRATIVE EXAMPLE

Example. *Which of the following curves are symmetrical about the x-axis? or y-axis ? or origin ?*

(*i*) $7y^2 = 5x$ (*ii*) $3x^2 - 4y^2 + 7 = 0$.

Solution. (*i*) The equation of the given curve is

$$F(x, y) = 7y^2 - 5x = 0.$$

Here,
$$F(x, -y) = 7(-y)^2 - 5x = 7y^2 - 5x = F(x, y);$$
$$F(-x, y) = 7y^2 - 5(-x) = 7y^2 + 5x \neq F(x, y);$$
$$F(-x, -y) = 7(-y)^2 - 5(-x) = 7y^2 + 5x \neq F(x, y).$$

Therefore, the given curve is symmetrical about x-axis but neither about y-axis nor about the origin.

(*ii*) The equation of the given curve is

$$F(x, y) = 3x^2 - 4y^2 + 7 = 0.$$

Here, $F(x, -y) = 3x^2 - 4(-y)^2 + 7 = 3x^2 - 4y^2 + 7 = F(x, y);$

$F(-x, y) = 3(-x)^2 - 4y^2 + 7 = 3x^2 - 4y^2 + 7 = F(x, y);$

$F(-x, -y) = 3(-x)^2 - 4(-y)^2 + 7 = 3x^2 - 4y^2 + 7 = F(x, y).$

Therefore, the given curve is symmetrical about x-axis, y-axis and about the origin.

EXERCISE 11.2

Which of the following curves are symmetrical about the x-axis ? or y-axis ? or origin?

1. $5x^2 + 7y = 0.$

2. $3y^2 = 7x.$

3. $x^2 = 3y^2 + 7.$

4. $x^2 + y^2 = 25.$

11.4 PARABOLA

*A **parabola** is the set of all points in a plane which are equidistant from a fixed line and a fixed point (not on the line) in the plane.*

The fixed line (say l) is called the **directrix** of the parabola and the fixed point (say F) is called the **focus** of the parabola.

If P_1, P_2, P_3, ... are points on the parabola and M_1P_1, M_2P_2, M_3P_3 are perpendiculars to the directrix l (see fig. 11.35), then

$FP_1 = M_1P_1$, $FP_2 = M_2P_2$, $FP_3 = M_3P_3$ etc.

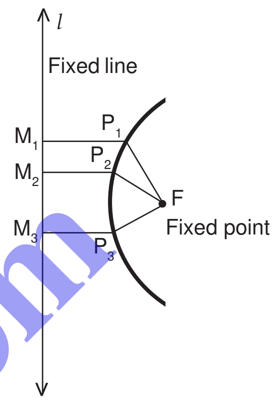

Fig. 11.35.

The line passing through the focus and perpendicular to the directrix is called the **axis of parabola**. The point of intersection of parabola with its axis is called **vertex** of parabola (see fig. 11.36).

The equation of a parabola is in simplest form if its vertex is at the origin and its axis lies along either x-axis or y-axis.

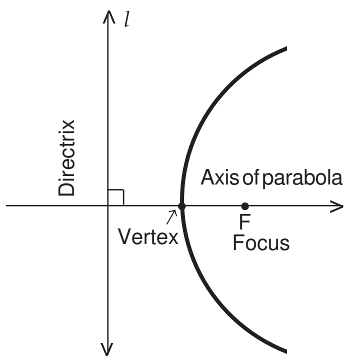

Fig. 11.36.

11.4.1 To find the equation of a parabola in the standard form $y^2 = 4ax$, $a > 0$

Let F be the focus, l be the directrix and Z be the foot of perpendicular from F to the line l.

Take ZF as x-axis with positive direction from Z to F. Let O be the mid-point of ZF, take O as origin, then the line through O and perpendicular to ZF becomes y-axis (shown in figure 11.37).

Let $ZF = 2a$

($a > 0$, because F does not lie on l), then

$ZO = OF = a.$

Since F lies to the right of O and Z lies to the left of O, co-ordinates of F, Z are $(a, 0)$, $(-a, 0)$ respectively. Therefore, the equation of the line l *i.e.* directrix is $x = -a$ *i.e.* $x + a = 0$.

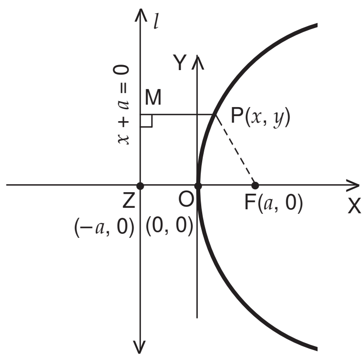

Fig. 11.37.

Let P (x, y) be any point in the plane of the line l and the point F, and MP be the perpendicular distance from P to the line l then P lies on ***parabola*** iff

$$\text{FP} = \text{MP}$$

$$\Leftrightarrow \quad \sqrt{(x-a)^2 + y^2} = \frac{|x+a|}{1}$$

$$\Leftrightarrow \quad (x-a)^2 + y^2 = (x+a)^2$$

$$\Leftrightarrow \quad x^2 + a^2 - 2ax + y^2 = x^2 + a^2 + 2ax$$

$$\Leftrightarrow \quad y^2 = 4ax.$$

Hence, the equation of a parabola in the standard form is $y^2 = 4ax$, $a > 0$, with focus F$(a, 0)$ and directrix $x + a = 0$.

Sometimes it is called a ***first standard form*** or a ***right hand parabola***.

11.4.2 Some facts about the parabola $y^2 = 4ax$, $a > 0$.

The equation of the parabola is

$$\text{F}(x, y) = y^2 - 4ax = 0 \qquad \ldots(i)$$

We note the following facts about the given parabola :

1. $\text{F}(x, -y) = (-y)^2 - 4ax = y^2 - 4ax$

$= \text{F}(x, y)$

$\Rightarrow$ the given parabola is symmetrical about x-axis.

This line is the **axis of the parabola**.

2. Focus is F $(a, 0)$ and directrix is $x + a = 0$.

3. The point O (0, 0) where the axis of parabola meets the parabola is the **vertex of the parabola,** it is mid-point of ZF where F $(a, 0)$ is the focus and Z is the foot of perpendicular from focus to the directrix.

4. If $x < 0$, then $y^2 = 4ax$ has no real solutions in y and so there is no point on the curve with negative x-coordinate *i.e.* on the left of y-axis.

When $x = 0$, we get $y^2 = 0 \Rightarrow y = 0$. Thus, (0, 0) is the only point of the y-axis which lies on it, therefore, the entire curve, except the origin, lies to the right of y-axis.

5. If P (x, y) is a point of the parabola in the first quadrant, then the equation (i) gives $y = 2\sqrt{ax}$ so that as x increases, y also increases and the curve is unbounded.

6. A chord passing through the focus F and perpendicular to the axis of parabola is called **latus-rectum** and its length is called **length of latus-rectum.**

7. Length of latus-rectum.

Let chord L′L be the latus-rectum of the parabola, then L′L passes through focus F $(a, 0)$ and is perpendicular to x-axis (as shown in fig. 11.38).

Let LF $= k$ $(k > 0)$, then the points L and L′ are (a, k) and $(a, -k)$ respectively.

As L (a, k) lies on the parabola $y^2 = 4ax$, we get

$$k^2 = 4a \times a \Rightarrow k = 2a.$$

$\therefore$ The points L, L′ are $(a, 2a)$, $(a, -2a)$

and length of latus-rectum $=$ L′L $= 2k = 4a$.

Thus, the length of latus rectum = 4a = 2 ZF = 4 OF.

The end points of the latus-rectum are L $(a, 2a)$, L′ $(a, -2a)$ and the equation of the latus-rectum is $x - a = 0$.

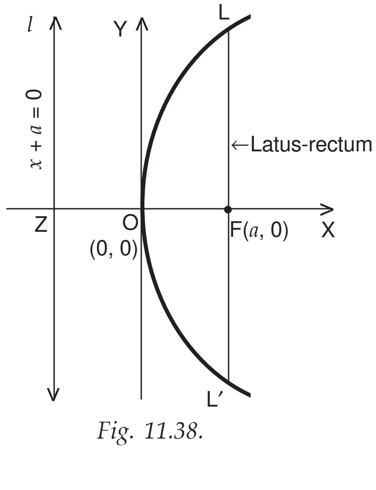

Fig. 11.38.

11.4.3 To find the equation of a parabola in other standard forms

Find the equation of a parabola with

(i) focus F (– a, 0), a > 0 and the line x – a = 0 as directrix.

(ii) focus F (0, a), a > 0 and the line y + a = 0 as directrix.

(iii) focus F (0, – a), a > 0 and the line y – a = 0 as directrix.

Solution. (*i*) Let P (x, y) be any point in the plane of directrix and focus, and MP be the perpendicular distance from P to the directrix, then P lies on parabola

$$\text{iff} \quad FP = MP$$

$$\Leftrightarrow \quad \sqrt{(x+a)^2 + y^2} = \frac{|x-a|}{1}$$

$$\Leftrightarrow \quad (x+a)^2 + y^2 = (x-a)^2$$

$$\Leftrightarrow \quad x^2 + a^2 + 2ax + y^2 = x^2 + a^2 - 2ax$$

$$\Leftrightarrow \quad \boldsymbol{y^2 = -4ax,\ a > 0.}$$

It is called ***2nd standard form*** or ***left hand parabola***.

(*ii*) Proceeding as above, it will be found that the equation of the parabola is

$$\boldsymbol{x^2 = 4ay,\ a > 0} \qquad \text{(Do it!)}$$

It is called ***3rd standard form*** or ***upward parabola*.**

(*iii*) Proceeding as in part (*i*), it will be found that the equation of the parabola is

$$\boldsymbol{x^2 = -4ay,\ a > 0} \qquad \text{(Do it!)}$$

It is called ***4th standard form*** *or* ***downward parabola***.

11.4.4 Four standard forms of the parabola

The students are advised to trace the following four standard forms of parabolas and to fix in their memory the positions (figures) of these parabolas with respect to co-ordinate axes.

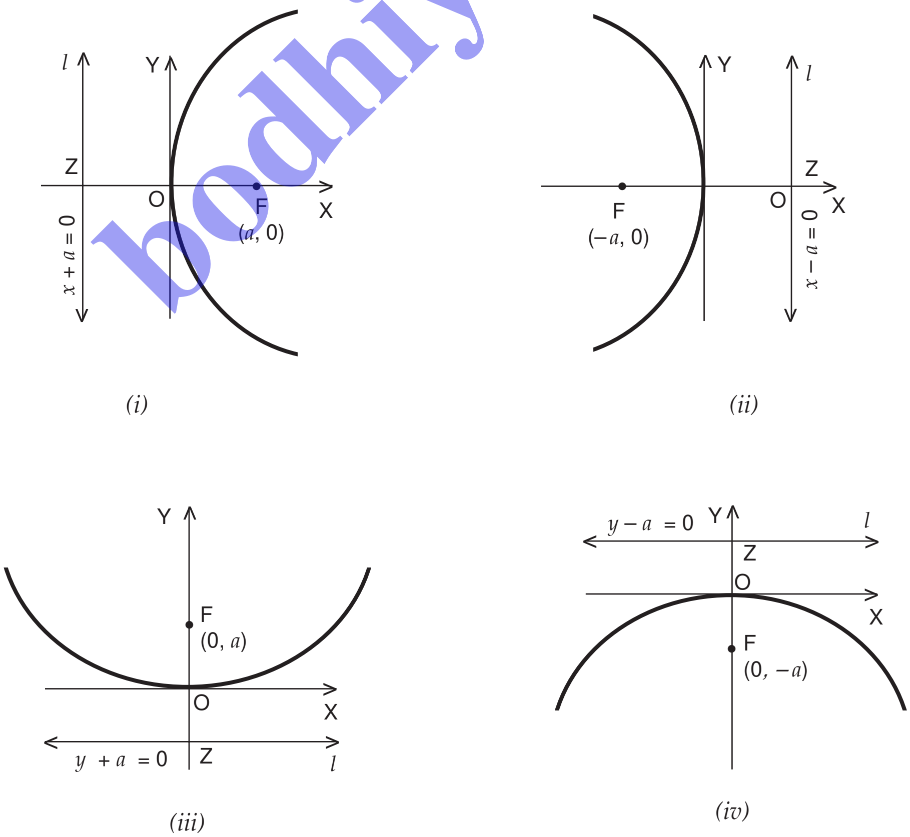

Fig. 11.39.

Main facts about the parabola

Equation	$y^2 = 4ax$ $(a > 0)$ *Right hand*	$y^2 = -4ax$ $a > 0$ *Left hand*	$x^2 = 4ay$ $a > 0$ *Upwards*	$x^2 = -4ay$ $a > 0$ *Downwards*
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Length of latus-rectum	$4a$	$4a$	$4a$	$4a$
Equation of latus-rectum	$x - a = 0$	$x + a = 0$	$y - a = 0$	$y + a = 0$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the co-ordinates of focus, the axis, the equation of the directrix and the length of latus-rectum of the parabola represented by the equation $3y^2 = 8x$.*

Solution. The given equation is $3y^2 = 8x$

i.e. $$y^2 = \frac{8}{3}x \qquad \ldots(i)$$

which is the same as $y^2 = 4ax$, so (*i*) represents a standard (right hand) parabola, and its axis lies along the x-axis. Hence x-axis itself is the axis of the given parabola.

Also $4a = \frac{8}{3} \Rightarrow a = \frac{2}{3}$, therefore, focus is $(a, 0)$ *i.e.* $\left(\frac{2}{3}, 0\right)$ and the equation of directrix is

$$x + \frac{2}{3} = 0 \qquad |\, x + a = 0$$

i.e. $3x + 2 = 0$.

Length of latus-rectum $= 4a = \frac{8}{3}$.

Example 2. *Find the co-ordinates of focus, the equation of directrix and the length of latus-rectum of the conic represented by the equation $x^2 = -16y$.* (NCERT)

Solution. The given equation is $x^2 = -16y$ $\ldots(i)$

which is comparable with $x^2 = -4ay$, so (*i*) represents a parabola of the fourth standard form.

Here $4a = 16 \;\Rightarrow\; a = 4$.

$\therefore$ The focus is $(0, -a)$ *i.e.* $(0, -4)$ and

the equation of the directrix is $y - 4 = 0$ $\qquad |\, y - a = 0$

The length of latus-rectum $= 4a = 16$.

Example 3. *Find the equation of the parabola with focus $(6, 0)$ and directrix $x = -6$. Also find the length of latus-rectum.* (NCERT)

Solution. The focus of the parabola is $F(6, 0)$ and its directrix is the line $x = -6$ *i.e.* $x + 6 = 0$.

Let $P(x, y)$ be any point in the plane of directrix and focus, and MP be the perpendicular distance from P to the directrix, then P lies on parabola iff FP $=$ MP

$$\Leftrightarrow \quad \sqrt{(x-6)^2 + (y-0)^2} = \frac{|x+6|}{1}$$

$\Leftrightarrow \quad x^2 - 12x + 36 + y^2 = x^2 + 12x + 36$

$\Leftrightarrow \quad y^2 = 24x$, which is the required equation of the parabola.

Comparing it with $y^2 = 4ax$, we get $4a = 24$.

$\therefore \quad$ Length of latus-rectum $= 4a = 24$.

Alternatively

Given the focus of the parabola is F (6, 0) which lies on x-axis, the x-axis itself is the axis of the parabola.

Also, the equation of the directrix is $x = -6$ *i.e.* $x + 6 = 0$, therefore, the parabola is of first standard form and its equation is

$$y^2 = 4ax, \text{ with } a = 6.$$

Hence the required equation of the parabola is $y^2 = 4 \times 6x$ *i.e.* $y^2 = 24\,x$.

Length of latus-rectum $= 4a = 24$.

Example 4. *Find the equation of a parabola with focus at $(-1, -2)$ and directrix $x - 2y + 3 = 0$.* (*NCERT Examplar Problems*)

Solution. The focus of the parabola is F$(-1, -2)$ and directrix is the line $x - 2y + 3 = 0$.

Let P(x, y) be any point in the plane of focus and directrix, and MP be the perpendicular distance from P to the directrix, then P lies on the parabola iff FP = MP

$$\Leftrightarrow \quad \sqrt{(x+1)^2 + (y+2)^2} = \frac{|\,x - 2y + 3\,|}{\sqrt{1^2 + (-2)^2}}$$

$\Leftrightarrow \quad 5[(x + 1)^2 + (y + 2)^2] = (x - 2y + 3)^2$

$\Leftrightarrow \quad 5(x^2 + 2x + 1 + y^2 + 4y + 4) = x^2 + 4y^2 + 9 - 4xy + 6x - 12y$

$\Leftrightarrow \quad 4x^2 + 4xy + y^2 + 4x + 32y + 16 = 0$, which is the required equation of the parabola.

Example 5. *Find the equation of the parabola with vertex at (0, 0) and focus at $(-2, 0)$.* (*NCERT*)

Solution. Since the focus of the parabola is F$(-2, 0)$ which lies on x-axis, the x-axis itself is the axis of the parabola.

Also, the vertex of the parabola is at O (0, 0), therefore, the parabola is of second standard form and its equation is

$$y^2 = -4ax, \text{ with } a = 2.$$

Hence the required equation of the parabola is

$$y^2 = -4 \times 2x \text{ i.e. } y^2 = -8x.$$

Example 6. *Find the equation of the parabola with vertex at origin and directrix the line $y + 3 = 0$. Also find its focus.*

Solution. The vertex of the parabola is at origin *i.e.* the point (0, 0). Let F be the focus of the parabola.

The directrix of the parabola is the line $y + 3 = 0$ *i.e.* $y = -3$, which is a straight line parallel to x-axis at a distance 3 units below origin.

If the directrix meets y-axis at Z, then OZ $= 3$.

Therefore, the given parabola is of the third standard form and its equation is $x^2 = 4ay$, where $a = $ OF $=$ OZ $= 3$.

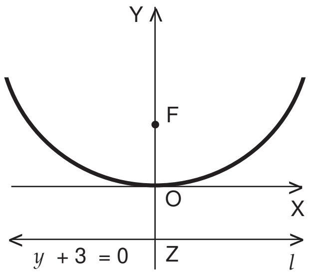

Fig. 11.40.

$\therefore$ The equation of the required parabola is $x^2 = 4 \times 3\,y$ *i.e.* $x^2 = 12\,y$.

The focus of the parabola is F $(0, a)$ *i.e.* F (0, 3).

Example 7. *Find the equation of the parabola with vertex at origin, symmetric with respect to y-axis and passing through (2, −3).* (NCERT)

Solution. The vertex of the parabola is at origin and it is symmetric with respect to y-axis *i.e.* axis of the parabola is the y-axis itself.

Also it passes through (2, − 3), a point in the fourth quadrant. The parabola is of the fourth standard form.

Let its equation be $x^2 = -4ay$ ...(i)

As it passes through (2, − 3), we get

$$2^2 = -4a.(-3) \Rightarrow a = \frac{1}{3}.$$

Substituting this value of a in (i), the equation of the parabola is

$$x^2 = -4.\frac{1}{3}y \text{ i.e. } 3x^2 = -4y.$$

Example 8. *If the points (0, 4) and (0, 2) are respectively the vertex and focus of a parabola, then find the equation of the parabola.* (*NCERT Examplar Problems*)

Solution. The vertex and the focus of the parabola are A(0, 4) and F(0, 2) respectively. AF = 2.

As points A and F lie on y-axis, so y-axis is the axis of the parabola. If the directrix meets the axis of parabola at point Z, then AZ = AF = 2.

∴ OZ = OF + FA + AZ = 2 + 2 + 2 = 6,

so the equation of directrix is $y = 6$

i.e. $y - 6 = 0$.

Let P(x, y) be any point in the plane of focus and directrix, and MP be the perpendicular distance from P to the directrix, then P lies on parabola iff FP = MP

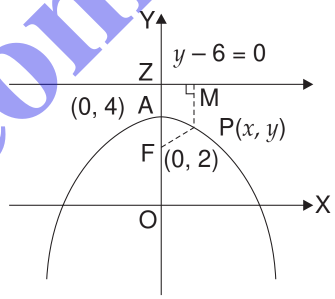

Fig. 11.41.

$$\Leftrightarrow \sqrt{(x-0)^2 + (y-2)^2} = \frac{|y-6|}{1}$$

$$\Leftrightarrow x^2 + (y-2)^2 = (y-6)^2$$

$$\Leftrightarrow x^2 + y^2 - 4y + 4 = y^2 - 12y + 36$$

$\Leftrightarrow x^2 + 8y = 32$, which is the required equation of the parabola.

Example 9. *Find the equations of the lines joining the vertex of the parabola $y^2 = 6x$ to the points which have abscissa 24.* (*NCERT Examplar Problems*)

Solution. The given parabola is $y^2 = 6x$ ...(i)

It is of the first standard form with vertex at O(0, 0).

Let P(24, k) be a point on parabola (i) whose abscissa is 24. As P lies on parabola,

$$k^2 = 6 \times 24 \Rightarrow k^2 = 144$$

$$\Rightarrow k = 12, -12$$

So, there are two points P(24, 12) and Q(24, − 12) on the parabola whose abscissa is 24.

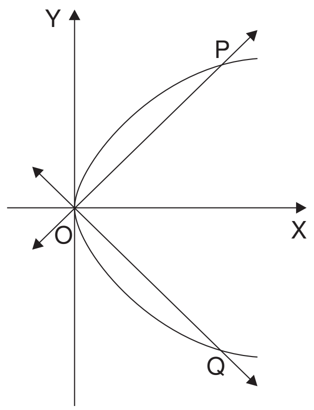

Fig. 11.42.

$$\text{Slope of OP} = \frac{12-0}{24-0} = \frac{1}{2},$$

$$\text{slope of OQ} = \frac{-12-0}{24-0} = \frac{-1}{2}.$$

ANSWERS

EXERCISE 11.1

1. (*i*) $x^2 + y^2 = 25$ (*ii*) $x^2 + y^2 - 4y = 0$ (*iii*) $x^2 + y^2 + 6x - 4y - 3 = 0$
(*iv*) $x^2 + y^2 - 2x - 2y = 0$ (*v*) $36x^2 + 36y^2 - 36x - 18y + 11 = 0$
(*vi*) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$

2. $x^2 + y^2 = 25$ **3.** $x^2 + y^2 - 16x + 12y + 75 = 0$

5. $x^2 + y^2 + 8x - 20 = 0$ **6.** $x^2 + y^2 + 6y - 7 = 0$ **7.** $x^2 + y^2 - 5x - 8y + 19 = 0$

8. $x^2 + y^2 + 2x - 2y - 3 = 0$ **9.** $x^2 + y^2 + 2x - 3y = 0$ **10.** $x^2 + y^2 - 3x + 2y = 0$

11. Inside the circle **12.** Outside the circle **13.** (*i*) and (*iv*)

14. degenerate (or point) circle

15. (*i*) $(-5, 3)$; 6 (*ii*) $(1, -2)$; $\sqrt{13}$ (*iii*) $(-4, -5)$; 7
(*iv*) $(2, 4)$; $\sqrt{65}$ (*v*) $\left(\frac{3}{4}, -\frac{5}{4}\right)$; $\frac{3}{4}\sqrt{10}$ (*vi*) $\left(\frac{a}{2}, \frac{b}{2}\right)$; $\frac{1}{2}\sqrt{a^2 + b^2}$

16. (*i*) circle ; $\left(-\frac{1}{2}, \frac{1}{2}\right)$; $\frac{1}{\sqrt{2}}$ (*ii*) empty set
(*iii*) circle ; $\left(\frac{5}{4}, \frac{7}{4}\right)$, $\frac{7}{4}\sqrt{2}$ (*iv*) point circle ; $(-1, -5)$, zero

17. -8 **18.** 3, -3 **19.** $x^2 + y^2 - 8x + 2y + 8 = 0$

20. $-\frac{1}{4}$ **21.** $x^2 + y^2 - 4x + 5y + 9 = 0$ **22.** 5

23. 10 **24.** $x^2 + y^2 - 130 = 0$ **25.** $x^2 + y^2 - 4x + 6y - 96 = 0$

26. $x^2 + y^2 - 2x + 6y - 40 = 0$ **27.** $x^2 + y^2 - 6x + 4y - 12 = 0$

28. $x^2 + y^2 - 2x + 2y - 47 = 0$

29. (*i*) $x^2 + y^2 - 12y + 11 = 0$ or $x^2 + y^2 + 4y - 21 = 0$
(*ii*) $x^2 + y^2 + 6y - 16 = 0$, $x^2 + y^2 - 6y - 16 = 0$

30. (*i*) $x^2 + y^2 - 2x - 2y + 1 = 0$ (*ii*) $x^2 + y^2 - 6x + 4y - 23 = 0$
(*iii*) 10, $-\frac{5}{2}$ (*iv*) $\frac{3}{4}$

31. $x^2 + y^2 - 5x - 8y + 19 = 0$; $6x + 4y = 31$

32. $(1, -1)$, $\sqrt{13}$, $x^2 + y^2 - 2x + 2y - 11 = 0$

33. (*i*) $x^2 + y^2 - 2x - 3y - 18 = 0$ (*ii*) $x^2 + y^2 - 13x - 11y + 68 = 0$

34. $x^2 + y^2 - 4x - 3y = 0$ **35.** $x^2 + y^2 - l(x + y) = 0$ **36.** $x^2 + y^2 - 3x - 2y - 21 = 0$

37. -3 ; $2x - 2y = 5$ **38.** $x^2 + y^2 + x - 2y - 41 = 0$ **39.** $(1, -6)$

40. (*i*) $x^2 + y^2 + 4x + 6y - 85 = 0$ (*ii*) $x^2 + y^2 - 4x - 6y - 87 = 0$

41. $x^2 + y^2 - 4x - 6y - 87 = 0$ **42.** $x^2 + y^2 + 4x - 8y - 84 = 0$

43. $4x^2 + 4y^2 + 16x + 20y - 23 = 0$

44. (*i*) $x^2 + y^2 - 4x + 6y - 51 = 0$ (*ii*) $x^2 + y^2 - 4x + 6y - 19 = 0$

47. $x^2 + y^2 - 6x - 8y + 15 = 0$ **48.** $x^2 + y^2 - 4x - 10y + 25 = 0$

49. $5x^2 + 5y^2 - 7x - 13y - 52 = 0$

50. (*i*) $x^2 + y^2 - 5x - y = 0$; $\left(\frac{5}{2}, \frac{1}{2}\right)$, $\frac{\sqrt{26}}{2}$ (*ii*) $x^2 + y^2 - 5x - y = 0$; $\left(\frac{5}{2}, \frac{1}{2}\right)$; $\frac{\sqrt{26}}{2}$
(*iii*) $x^2 + y^2 = 1$; $(0, 0)$, 1 (*iv*) $x^2 + y^2 - 22x - 4y + 25 = 0$; $(11, 2)$, 10
(*v*) $x^2 + y^2 - 14x - 6y - 111 = 0$; $(7, 3)$; 13

51. $x^2 + y^2 - 2x + 2y - 23 = 0$ **52.** $x^2 + y^2 - 4x - 2y - 20 = 0$, $(2, 1)$, 5