10

STRAIGHT LINES

INTRODUCTION

Co-ordinate geometry (or analytical geometry) is that branch of Mathematics which deals with the study of geometry by means of algebra. René Descartes, a French mathematician, realised around 1637 that a straight line or a curve in *a plane* can be represented by an algebraic equation. As a result, a new branch of Mathematics called ***Co-ordinate Geometry*** came into existence. In co-ordinate geometry, we represent a point in a plane by an ordered pair of real numbers, called *co-ordinates of the point*; and a straight line or a curve by an algebraic equation with real coefficients. Thus, we use algebra advantageously to the study of straight lines and geometric curves which reveal their nature and properties.

10.1 RECAPITULATION

Let us recapitulate some basic concepts of co-ordinate geometry studied in previous classes.

Co–ordinate system

When two numbered lines perpendicular to each other (usually horizontal and vertical) are placed together so that their origins (the points corresponding to zero) coincide, then the resuslting configuration is called a ***cartesian co-ordinate system*** *or a* ***co-ordinate plane****.*

Let X′OX and Y′OY be two numbered lines perpendicular to each other meeting at O (shown in fig. 10.1), then

(*i*) *X′OX is called* ***x-axis*** *(or* ***axis of x****).*

(*ii*) *Y′OY is called* ***y-axis*** *(or* ***axis of y****).*

(*iii*) *X′OX and Y′OY taken together are called* ***co-ordinate axes*** *or* ***rectangular axes*** *(rectangular because they are at right angles.)*

(*iv*) *The point O is called* ***origin****.*

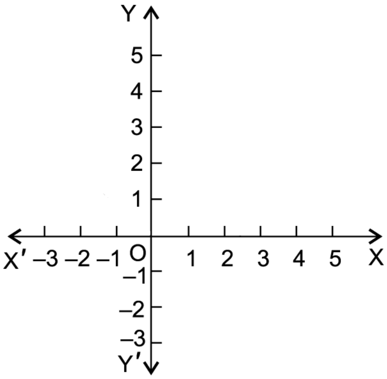


Fig. 10.1.

Co–ordinates of a point

Let P be any point in the co-ordinate plane. From P, draw PM perpendicular to X′OX (shown in fig. 10.2), then

(*i*) *OM is called the* ***x-coordinate*** *(or* ***abscissa****) of P and is usually denoted by x.*

(*ii*) *MP is called the* ***y-coordinate*** *(or* ***ordinate****) of P and is usually denoted by y.*

(iii) x *and* y *taken together are called the* ***cartesian co-ordinates*** *or simply* ***co-ordinates*** *of the point P and are denoted by* (x, y).

The position of each point of the plane is uniquely determined with reference to rectangular axes by means of an ordered pair of real numbers, called co-ordinates of the point; and conversely, corresponding to every ordered pair of real numbers we can find a unique point in the plane.

Thus, *there is a one-one correspondence between the set of points in a plane and the set of ordered pairs of real numbers.*

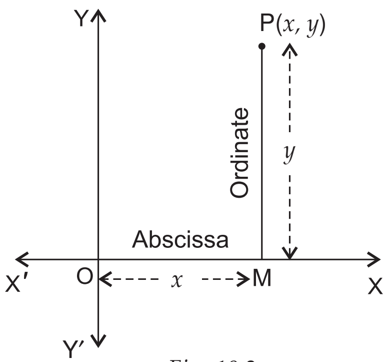


Fig. 10.2.

Distance between two points

The distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Section formula

The co-ordinates of the point which divides the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in the ratio $m : n$ are given by

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right).$$

The co-ordinates of the point which divides the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ externally in the ratio $m : n$ are given by

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right).$$

In particular, the co-ordinates of the mid-point of the line segment PQ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Centroid of a triangle

The co-ordinates of the centroid of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

Incentre of a triangle

Definition. *The point of the intersection of any two internal bisectors of the angles of a triangle is called the* ***incentre of the triangle.*** *It is usually denoted by I.*

NOTE

If the internal bisector of $\angle A$ of a Δ ABC meets the side BC at D, then $\frac{BD}{DC} = \frac{AB}{AC}$ (from Geometry).

Find the incentre of a triangle whose vertices are given. Hence prove that the internal bisectors of the angles of a triangle are concurrent.

Let A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) be the vertices of a triangle ABC, and a, b, c be the lengths of the sides of Δ ABC opposite to the vertices A, B, C respectively.

Let the internal bisector of $\angle A$ meet the side BC at D, then

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b} \qquad \ldots(i)$$

$\Rightarrow$ D divides segment BC internally in the ratio $c : b$.

$\therefore$ Co-ordinates of D are

$$\left(\frac{cx_3 + bx_2}{c+b}, \frac{cy_3 + by_2}{c+b}\right).$$

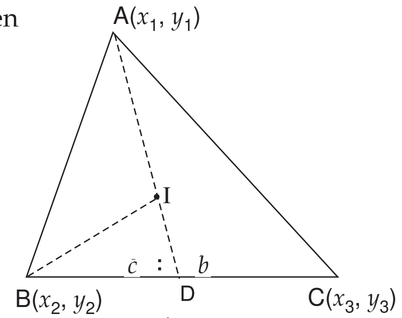


Fig. 10.3.

Let the internal bisector of $\angle B$ meet AD at I, so I is the incentre of Δ ABC.

In ΔABD, BI is internal bisector of $\angle$B,

$$\therefore \quad \frac{DI}{IA} = \frac{BD}{AB} = \frac{BD}{c} \qquad \ldots(ii)$$

From (i), $\frac{BD}{DC} = \frac{c}{b} \Rightarrow \frac{BD}{BD + DC} = \frac{c}{c+b} \Rightarrow \frac{BD}{BC} = \frac{c}{c+b}$

$$\Rightarrow \quad \frac{BD}{a} = \frac{c}{c+b} \Rightarrow BD = \frac{ac}{c+b}.$$

Substituting this value of BD in (ii), we get

$$\frac{DI}{IA} = \frac{\frac{ac}{c+b}}{c} = \frac{a}{c+b}$$

$\Rightarrow$ I divides segment DA internally in the ratio $a : (c + b)$,

therefore, by section formula, the co-ordinates of I are

$$\left(\frac{a\,.\,x_1 + (c+b)\,.\,\frac{cx_3 + bx_2}{c+b}}{a + (c+b)},\ \frac{a\,.\,y_1 + (c+b)\,.\,\frac{cy_3 + by_2}{c+b}}{a + (c+b)} \right)$$

i.e. $$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c},\ \frac{ay_1 + by_2 + cy_3}{a+b+c} \right).$$

The symmetry of the co-ordinates of I shows that it also lies on the internal bisector of $\angle$C. Hence the internal bisectors of the angles of a triangle are concurrent.

Area of a triangle

Find the area of a triangle whose vertices are given.

Let A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) be the vertices of a triangle ABC. Draw AM, BN and CL perpendiculars on x-axis, then

NM = OM – ON = $x_1 - x_2$,

ML = OL – OM = $x_3 - x_1$ and

NL = OL – ON = $x_3 - x_2$.

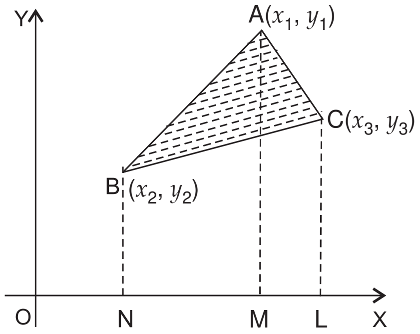


Fig. 10.4.

Now, area of ΔABC = area of trapezium ABNM
+ area of trapezium AMLC
– area of trapezium BNLC

$$= \frac{1}{2}(NB + MA)\,NM + \frac{1}{2}(MA + LC)\,ML - \frac{1}{2}(NB + LC)\,NL$$

$$\left(\because \text{ area of a trapezium } = \frac{1}{2}(\text{sum of the parallel sides}) \times \text{height}\right)$$

$$= \frac{1}{2}[(y_2 + y_1)(x_1 - x_2) + (y_1 + y_3)(x_3 - x_1) - (y_2 + y_3)(x_3 - x_2)]$$

$$= \frac{1}{2}[x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)].$$

Further, as the area of a triangle can never be negative, we must take the absolute value.

Therefore, area of ΔABC = $\frac{1}{2}\,|\,x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)\,|$.

Corollary. Condition for collinearity of three points.

Three points A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) will be *collinear* iff the area of Δ ABC = 0 *i.e.* iff

$$\frac{1}{2} \left| x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \right| = 0.$$

i.e. iff $x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) = 0.$

Locus of a point

*The **locus** of a point in a plane is the curve or the path traced out by the point moving under a given geometrical condition (or conditions).*

ILLUSTRATIVE EXAMPLES

Example 1. *The base of an equilateral triangle with side 2a lies along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle.* (NCERT)

Solution. Let ABC be the given equilateral triangle with side $2a$ whose base, say AB, lie along y-axis and the mid-point of the base is at origin.

Two triangles are possible which satisfy the given conditions. The third vertex C may lie either on the right of AB or on the left of AB.

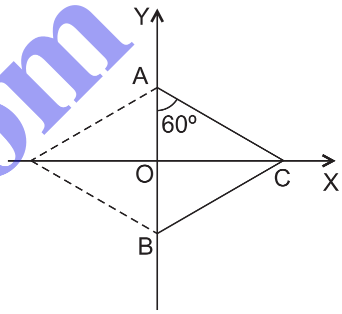


Fig. 10.5.

Since ABC is an equilateral triangle, $\angle BAC = 60°$.

Also OA $= \frac{1}{2} \times 2a = a.$

From ΔAOC, $\tan 60° = \frac{OC}{OA} \Rightarrow \sqrt{3} = \frac{OC}{a} \Rightarrow OC = \sqrt{3}a.$

$\therefore$ The vertices of triangle are A $(0, a)$, B $(0, -a)$, C $(\sqrt{3}\,a, 0)$ or C $(-\sqrt{3}\,a, 0)$.

Example 2. *Find the distance between* $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ *when*

(i) PQ is parallel to the y-axis *(ii) PQ is parallel to the x-axis.* (NCERT)

Solution. (*i*) When PQ is parallel to the y-axis, then $x_1 = x_2$.

$\therefore$ $PQ = \sqrt{(x_1 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|.$

(*ii*) When PQ is parallel to the x-axis, then $y_1 = y_2$.

$\therefore$ $PQ = \sqrt{(x_2 - x_1)^2 + (y_1 - y_1)^2} = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|.$

Example 3. *If the distance between the points (a, –2) and (5, 1) is 5 units, find the value(s) of a.*

Solution. The distance between the points $(a, -2)$ and $(5, 1)$

$$= \sqrt{(5 - a)^2 + (1 - (-2))^2} = \sqrt{(5 - a)^2 + 3^2}.$$

According to given, $\sqrt{(5 - a)^2 + 3^2} = 5$

$\Rightarrow$ $(5 - a)^2 + 9 = 25 \Rightarrow (5 - a)^2 = 16$

$\Rightarrow$ $5 - a = \pm 4 \Rightarrow a = 1, 9.$

$\therefore$ The required values of a are 1, 9.

Example 4. *Show that the points (7, 10), (–2, 5) and (3, – 4) are the vertices of an isosceles right angled triangle.*

Solution. Let the points be A (7, 10), B (–2, 5) and C (3, – 4), then

$$AB = \sqrt{(-2-7)^2 + (5-10)^2} = \sqrt{81+25} = \sqrt{106},$$

$$BC = \sqrt{(3+2)^2 + (-4-5)^2} = \sqrt{25+81} = \sqrt{106} \text{ and}$$

$$CA = \sqrt{(7-3)^2 + (10-(-4))^2} = \sqrt{16+196} = \sqrt{212}$$

$\Rightarrow$ $AB^2 = 106$, $BC^2 = 106$ and $CA^2 = 212$,

$\therefore$ $AB^2 + BC^2 = 106 + 106 = 212 = CA^2$

$\Rightarrow$ ΔABC is right angled and it is right angled at B.

Also $AB = \sqrt{106} = BC \Rightarrow \Delta ABC$ is isosceles.

Example 5. *Find the point on the x-axis which is equidistant from the points (7, 6) and (3, 4).* *(NCERT)*

Solution. Let P(x, 0) be any point on the x-axis.
Let the given points be A(7, 6) and B(3, 4).
According to given, AP = BP

$$\Rightarrow \quad \sqrt{(x-7)^2 + (0-6)^2} = \sqrt{(x-3)^2 + (0-4)^2}$$

$$\Rightarrow \quad (x-7)^2 + 36 = (x-3)^2 + 16$$

$$\Rightarrow \quad x^2 - 14x + 49 + 36 = x^2 - 6x + 9 + 16$$

$$\Rightarrow \quad -8x = -60 \Rightarrow x = \frac{15}{2}.$$

Hence, the required point is $\left(\frac{15}{2}, 0\right)$

Example 6. *If two vertices of an equilateral triangle are (0, 0) and (0, $2\sqrt{3}$), find the third vertex.*

Solution. Let O(0, 0), A(0, $2\sqrt{3}$) and B(x, y) be the vertices of the equilateral triangle OAB. Then

$$OA = \sqrt{(0-0)^2 + (2\sqrt{3}-0)^2} = \sqrt{12},$$

$$OB = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

$$\text{and } AB = \sqrt{(x-0)^2 + (y-2\sqrt{3})^2} = \sqrt{x^2 + y^2 - 4\sqrt{3}y + 12}.$$

Since ΔOAB is equilateral, OA = OB = AB

$$\Rightarrow \quad OA^2 = OB^2 = AB^2$$

$$\Rightarrow \quad 12 = x^2 + y^2 = x^2 + y^2 - 4\sqrt{3}\,y + 12$$

$$\Rightarrow \quad x^2 + y^2 = 12 \qquad \ldots(i)$$

$$\text{and} \quad x^2 + y^2 = x^2 + y^2 - 4\sqrt{3}\,y + 12$$

$$\Rightarrow \quad 4\sqrt{3}\,y = 12 \quad \Rightarrow y = \sqrt{3} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$x^2 + (\sqrt{3})^2 = 12 \quad \Rightarrow x^2 + 3 = 12$$

$$\Rightarrow \quad x^2 = 9 \quad \Rightarrow x = 3, -3.$$

Hence, the third vertex of the triangle is $(3, \sqrt{3})$ or $(-3, \sqrt{3})$.

Example 7. *Find the ratio in which the point P whose abscissa is 3 divides the join of A (6, 5) and B (–1, 4) and hence find the co-ordinates of P.*

Solution. Let the point P divide the segment AB in the ratio $k : 1$, by section formula, co-ordinates of P are $\left(\frac{-k+6}{k+1}, \frac{4k+5}{k+1}\right)$.

But abscissa of point P is 3 (given)

$$\Rightarrow \quad \frac{-k+6}{k+1} = 3 \Rightarrow 3k + 3 = -k + 6$$

$$\Rightarrow \quad 4k = 3 \Rightarrow k = \frac{3}{4}.$$

$\therefore$ The required ratio is $\frac{3}{4} : 1$ *i.e.* 3 : 4 internally.

Co-ordinates of P are $\left(3, \frac{4 . \frac{3}{4} + 5}{\frac{3}{4} + 1}\right)$ *i.e.* $\left(3, \frac{32}{7}\right)$.

Example 8. *The centre of a circle is C (–2, 5) and one end of a diameter is A (3, –7), find the co-ordinates of the other end.*

Solution. Let the other end of the diameter of the given circle be B (α, β) whose one end is A (3, –7).

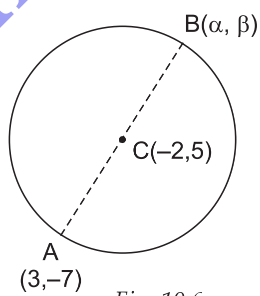


Fig. 10.6.

Mid-point of AB is $\left(\frac{\alpha+3}{2}, \frac{\beta-7}{2}\right)$.

Since the centre C (– 2, 5) of the circle is the mid-point of AB,

$$\frac{\alpha+3}{2} = -2 \text{ and } \frac{\beta-7}{2} = 5$$

$$\Rightarrow \quad \alpha = -7 \text{ and } \beta = 17.$$

$\therefore$ The co-ordinates of the other end of the diameter are (–7, 17).

Example 9. *If three consecutive vertices of a parallelogram are (–2, –1), (1, 0) and (4, 3), find the fourth vertex.*

Solution. Let A (–2, –1), B (1, 0), C (4, 3) and D (x, y) be the vertices of the parallelogram ABCD. Then mid-point of AC is $\left(\frac{-2+4}{2}, \frac{-1+3}{2}\right)$ *i.e.* (1, 1).

Also mid-point of BD is $\left(\frac{1+x}{2}, \frac{0+y}{2}\right)$.

Since the diagonals of a parallelogram bisect each other, the mid-points of AC and BD are same

$$\Rightarrow \quad \frac{1+x}{2} = 1 \text{ and } \frac{0+y}{2} = 1 \Rightarrow x = 1 \text{ and } y = 2.$$

Hence, the fourth vertex of the parallelogram is (1, 2).

Example 10. *Find the co-ordinates of the incentre of the triangle whose vertices are (–2, 4), (5, 5) and (4, –2).*

Solution. Let A (–2, 4), B (5, 5) and C (4, –2) be the vertices of the given triangle ABC, then

$$a = |\,BC\,| = \sqrt{(4-5)^2 + (-2-5)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2},$$

$$b = |\,CA\,| = \sqrt{(4+2)^2 + (-2-4)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2} \text{ and}$$

$$c = |\,AB\,| = \sqrt{(5+2)^2 + (5-4)^2} = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}.$$

$\therefore$ The co-ordinates of the incentre of Δ ABC are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

i.e. $$\left(\frac{5\sqrt{2}\,.(-2) + 6\sqrt{2}\,.5 + 5\sqrt{2}\,.4}{5\sqrt{2} + 6\sqrt{2} + 5\sqrt{2}}, \frac{5\sqrt{2}\,.4 + 6\sqrt{2}\,.5 + 5\sqrt{2}\,.(-2)}{5\sqrt{2} + 6\sqrt{2} + 5\sqrt{2}}\right)$$

i.e. $$\left(\frac{40\sqrt{2}}{16\sqrt{2}}, \frac{40\sqrt{2}}{16\sqrt{2}}\right) \text{ i.e. } \left(\frac{5}{2}, \frac{5}{2}\right).$$

Example 11. *Find the area of the triangle whose vertices are (10, – 6), (2, 5) and (– 1, 3).*

Solution. The area of the triangle whose vertices are (10, – 6), (2, 5) and (– 1, 3)

$$= \frac{1}{2}\,|\,10\,(5-3) + 2\,(3+6) + (-1)\,(-6-5)\,|\text{ sq. units}$$

$$= \frac{1}{2}\,|\,20 + 18 + 11\,|\text{ sq. units}$$

$$= \frac{1}{2}\,|\,49\,|\text{ sq. units} = \frac{49}{2}\text{ sq. units.}$$

Example 12. *Draw the quadrilateral in the cartesian plane, whose vertices are (–4, 5), (0, 7), (5, –5) and (–4, –2). Also, find its area.* *(NCERT)*

Solution. Plot the points A(–4, 5), B(0, 7), C(5, –5) and D(–4, –2) in the cartesian plane as shown in fig. 10.7. Then join AB, BC, CD and DA to get the quadrilateral ABCD.

Area of ΔABC

$$= \frac{1}{2}\,|-4\,(7+5) + 0\,(-5-5) + 5\,(5-7)|\text{ sq. units}$$

$$= \frac{1}{2}\,|-48 + 0 - 10|\text{ sq. units}$$

$$= \frac{1}{2}\,|-58|\text{ sq. units} = 29\text{ sq. units.}$$

Area of ΔACD

$$= \frac{1}{2}\,|-4\,(-5+2) + 5\,(-2-5) + (-4)\,(5+5)|\text{ sq. units}$$

$$= \frac{1}{2}\,|12 - 35 - 40|\text{ sq. units}$$

$$= \frac{1}{2}\,|-63|\text{ sq. units}$$

$$= \frac{63}{2}\text{ sq. units.}$$

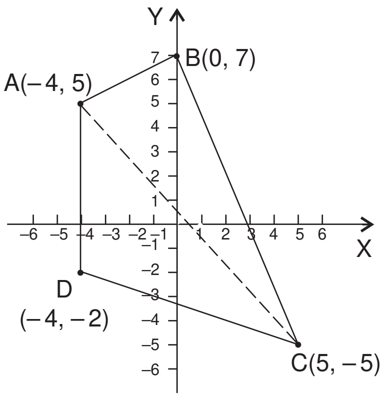


Fig. 10.7.

$\therefore$ Area of quadrilateral ABCD $= 29$ sq. units $+ \dfrac{63}{2}$ sq. units

$$= 60\frac{1}{2}\text{ sq. units}$$

Example 13. *For what value of x are the points (1, 5), (x, 1) and (4, 11) collinear?*

Solution. Area of the triangle formed by given points

$$= \frac{1}{2}\,|\,1\,(1-11) + x\,(11-5) + 4\,(5-1)\,|$$

$$= \frac{1}{2}\,|\,6x + 6\,| = 3\,|\,x + 1\,|.$$

The given points are collinear iff area of triangle = 0

i.e. iff $3\,|\,x + 1\,| = 0$ *i.e.* $|\,x + 1\,| = 0$

i.e. $x + 1 = 0$ $\qquad (\because \,|\,x\,| = 0 \text{ iff } x = 0)$

i.e. iff $x = -1$.

Hence, the required value of x is -1.

Example 14. *If P (x, y) is any point on the line joining the poins A (a, 0) and B (0, b), then show that* $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Solution. As P (x, y) lies on the line joining the points A $(a, 0)$ and B $(0, b)$, so the points P, A, B are collinear

$\Rightarrow$ area of Δ PAB = 0

$$\Rightarrow \quad \frac{1}{2}\,|\,x\,(0 - b) + a\,(b - y) + 0\,(y - 0)\,| = 0$$

$$\Rightarrow \quad |\,ab - ay - bx\,| = 0 \Rightarrow ab - ay - bx = 0$$

$$\Rightarrow \quad bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1. \qquad \text{(Dividing by } ab\text{)}$$

Example 15. *The co-ordinates of points P, Q, R and S are (–3, 5), (4, –2), (p, 3p) and (6, 3) respectively. If the areas of Δs PQR and QRS are in the ratio 2 : 3, find p.*

Solution. Area of ΔPQR $= \dfrac{1}{2}\,|\,-3\,(-2 - 3p) + 4\,(3p - 5) + p\,(5 + 2)\,|$

$$= \frac{1}{2}\,|\,28p - 14\,| = 7\,|\,2p - 1\,|.$$

Area of ΔQRS $= \dfrac{1}{2}\,|\,4\,(3p - 3) + p\,(3 + 2) + 6\,(-2 - 3p)\,|$

$$= \frac{1}{2}\,|\,-p - 24\,|.$$

Given that area of ΔPQR : area of ΔQRS = 2 : 3

$$\Rightarrow \quad \frac{7\,|\,2p - 1\,|}{\frac{1}{2}\,|\,-p - 24\,|} = \frac{2}{3} \Rightarrow 14\left|\frac{2p - 1}{-p - 24}\right| = \frac{2}{3}$$

$$\left|\frac{2p - 1}{-p - 24}\right| = \frac{1}{21} \Rightarrow \frac{2p - 1}{-p - 24} = \frac{1}{21} \text{ or } \frac{2p - 1}{-p - 24} = -\frac{1}{21}$$

$$\Rightarrow \quad 42p - 21 = -p - 24 \text{ or } 42p - 21 = p + 24$$

$$\Rightarrow \quad 43p = -3 \text{ or } 41p = 45$$

$$\Rightarrow \quad p = -\frac{3}{43} \text{ or } p = \frac{45}{41}.$$

Example 16. *If the co-ordinates of two points A, B are (1, 2), (3, 8) respectively, find a point P such that | PA | = | PB | and area of ΔPAB = 10.*

Solution. Let co-ordinates of P be (α, β), then

$$|\,PA\,| = |\,PB\,| \Rightarrow PA^2 = PB^2$$

$$\Rightarrow \quad (\alpha - 1)^2 + (\beta - 2)^2 = (\alpha - 3)^2 + (\beta - 8)^2$$

$$\Rightarrow \quad -2\alpha + 1 - 4\beta + 4 = -6\alpha + 9 - 16\beta + 64$$

$$\Rightarrow \quad 4\alpha + 12\beta = 68 \Rightarrow \alpha + 3\beta = 17 \qquad \ldots(i)$$

Also area of ΔPAB = 10

$$\Rightarrow \quad \frac{1}{2}\,|\,\alpha\,(2 - 8) + 1\,(8 - \beta) + 3\,(\beta - 2)\,| = 10$$

$\Rightarrow \quad |-6\alpha + 2\beta + 2| = 20 \Rightarrow |-3\alpha + \beta + 1| = 10$

$\Rightarrow \quad -3\alpha + \beta + 1 = 10 \text{ or } -10$

$\Rightarrow \quad -3\alpha + \beta = 9 \quad \ldots(ii) \quad \text{or} \quad -3\alpha + \beta = -11 \quad \ldots(iii)$

Solving (*i*) and (*ii*), we get

$\alpha = -1, \beta = 6$

Solving (*i*) and (*iii*), we get

$\alpha = 5, \beta = 4.$

Hence, the point is either (–1, 6) or (5, 4).

EXERCISE 10.1

Very short answer type questions (1 to 20) :

1. What is the distance of the point P(x, y) from x-axis?

2. A is a point on y-axis whose ordinate is 5 and B is the point (–3, 1). Compute the length of AB.

3. The distance between A (1, 3) and B (x, 7) is 5. Find the values of x.

4. Find the abscissa of points whose ordinate is 4 and which are at a distance of 5 units from (5, 0).

5. How many points are there on the x-axis whose distance from the point (2, 3) is less than 3 units?

6. (*i*) Find the point on x-axis which is equidistant from (3, 2) and (– 5, – 2).

(*ii*) What point on the y-axis is equidistant from (3, 2) and (– 5, – 2) ?

7. The points A (0, 3), B (–2, a) and C (–1, 4) are the vertices of a right angled triangle at A, find the value of a.

8. The centre of a circle is $(2\alpha - 1, 3\alpha + 1)$ and it passes through the point (–3, –1). Find the value (s) of α if a diameter of the circle is of length 20 units.

9. In what ratio does the point P $\left(\frac{1}{2}, 6\right)$ divide the line segment joining the points A (3, 5) and B (–7, 9) ?

10. Point C (– 4, 1) divides the line segment joining the points A (2, –2) and B in the ratio 3 : 5. Find the point B.

11. The mid-point of the line segment joining ($2a$, 4) and (–2, $3b$) is (1, $2a + 1$). Find the values of a and b.

12. If the points A (–2, –1), B (1, 0), C (a, 3) and D (1, b) form a parallelogram ABCD, find the values of a and b.

13. The three vertices of a parallelogram taken in order are (–1, 1), (3, 1) and (2, 2) respectively. Find the coordinates of the fourth vertex.

14. If the middle points of the sides of a triangle are (1, 1), (2, –3) and (3, 2), find the centroid of the triangle.

15. If (4, –3) and (–9, 7) are two vertices of a triangle whose centroid is (1, 4), then find the third vertex.

16. The vertices of a triangle are A (–5, 3), B (p, –1) and C (6, q). Find the values of p and q if the centroid of the triangle ABC is the point (1, –1).

17. If the point (–3, a) is the image of the point (1, a + 4) in the point (b, 1), find the values of a and b.

18. Show that the points (3, –2), (5, 2) and (8, 8) are collinear.

19. For what value of k are the points (8, 1), (k, –4) and (2, –5) collinear?

20. Find the equation of the locus of a point which is equidistant from the points (1, 3) and (–2, 1).

21. Show that the points :

(i) (2, –2), (8, 4), (5, 7), (–1, 1) are the vertices of a rectangle.

(ii) (3, 2), (0, 5), (–3, 2), (0, –1) are the vertices of a square.

22. Two vertices of an isosceles triangle are (2, 0) and (2, 5). Find the third vertex if the length of equal sides is 3 units.

23. Find the co-ordinates of the point which is at a distance of 2 units from (5, 4) and 10 units from (11, –2).

24. Using distance formula, show that (3, 3) is the centre of the circle passing through the points (6, 2), (0, 4) and (4, 6). Find the radius of the circle.

25. In what ratio is the line joining the points (3, 4) and (–2, 1) divided by the y-axis ? Also find the co-ordinates of the point of division.

26. Find the ratio in which the point P (k, 6) divides the line segment joining the points A (–4, 3) and B (2, 8). Also find the value of k.

27. If the points A (0, 4), B (1, 2) and C (3, 3) are three corners of a square, find

(i) the co-ordinates of the point at which the diagonals intersect.

(ii) the co-ordinates of D, the fourth corner of the square.

28. If A (–1, 3), B (1, –1) and C (5, 1) are three vertices of a triangle ABC, find the length of the median through A.

29. If two vertices of a parallelogram are (3, 2), (– 1, 0) and its diagonals meet at (2, – 5), find the other two vertices of the parallelogram.

30. (i) If a vertex of a triangle is (–2, 3) and the middle points of the sides through it are (0, – 1) and (5, 4), find the other vertices.

(ii) Find the third vertex of a triangle if its two vertices are (– 1, 4) and (5, 2) and mid-point of one side is (0, 3).

31. Find the co-ordinates of the centre of the circle inscribed in a triangle whose angular points are (– 36, 7), (20, 7) and (0, – 8).

32. If the vertices of a triangle are (1, k), (4, –3) and (–9, 7) and its area is 15 sq. units, find the value(s) of k.

33. Find the area of the triangle formed by the points (p + 1, 1), (2 p + 1, 3), (2 p + 2, 2 p) and show that these points are collinear if $p = 2$ or $-\dfrac{1}{2}$.

34. Find the area of the quadrilateral, the co-ordinates of whose vertices are (2, 1), (6, 0), (5, –2) and (– 3, – 1).

35. The co-ordinates of A, B and C are (6, 3), (–3, 5) and (4, –2) respectively, and P is any point (x, y). Show that the ratio of the area of the triangles PBC and ABC is $\left|\dfrac{x+y-2}{7}\right|$.

36. A, B are two points (3, 4) and (5, –2), find a point P such that | PA | = | PB | and area of ΔPAB = 10.

10.2 SHIFTING OF ORIGIN

Sometimes a problem with a given set of axes can be solved more easily by shifting the origin to a new point, the axes remaining parallel to the original axes. Shifting the origin to a new point without changing the direction of the axes is known as **translation of axes**. We shall see how the co-ordinates of a point and the equation of a curve are affected by translation of axes, and how we can shift back to the original axes.

10.2.1 To shift the origin to a new point, without changing the direction of axes

Let OX, OY be the original axes and O′ be the new origin. Let co-ordinates of O′referred to original axes *i.e.* OX, OY be (h, k).

Let O′X′ and O′Y′ be drawn parallel to and in the same direction as OX and OY respectively. Let P be any point in the plane whose co-ordinates referred to original axes OX, OY be (x, y) and referred to new axes O′X′, O′Y′ be (X, Y).

Draw PM′M parallel to OY meeting O′X′ in M′ and OX in M (shown in figure 10.8). Produce Y′O′ to meet OX in N. Then

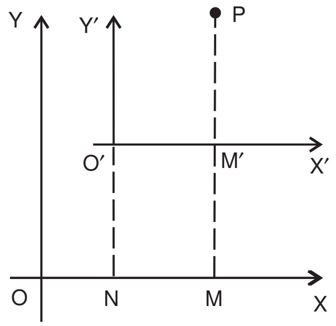


Fig. 10.8.

$$x = OM = ON + NM = ON + O'M' = h + X = X + h$$

and $$y = MP = MM' + M'P = NO' + M'P = k + Y = Y + k.$$

Thus, we get

$$\boldsymbol{x = X + h,\ y = Y + k}$$

$$\Rightarrow \quad \boldsymbol{X = x - h,\ Y = y - k.}$$

Therefore, we have

(i) The point whose co-ordinaes were (x, y) has now the co-ordinates $(x - h, y - k)$.

(ii) The co-ordinates of the old origin referred to the new axes are $(-h, -k)$.

(iii) The curve whose equation was $f(x, y) = 0$ is now represented by $f(X + h, Y + k) = 0$.

An obvious fact

A translation of axes cannot affect magnitudes like the distance between two points, the distance of a point from a line, the area of the figure etc. Hence, while dealing with problems of this type, there is no need of shifting back to original axes.

ILLUSTRATIVE EXAMPLES

Example 1. *Transform $2x - 3y + 5 = 0$ to the parallel axes through the point $(2, -3)$.*

Solution. Let the co-ordinates (x, y) of any point on the given line change to (X, Y) on shifting the origin to the point $(2, -3)$, the new axes remaining parallel to the original axes. Then, we get

$$x = X + 2 \text{ and } y = Y - 3.$$

Substituting for x and y in the given equation $2x - 3y + 5 = 0$, the transformed equation is

$$2(X + 2) - 3(Y - 3) + 5 = 0$$

or $$2X - 3Y + 18 = 0.$$

Example 2. *Transform $x^2 + y^2 - 6x + 10y - 2 = 0$ to the parallel axes through $(3, -5)$.*

Solution. Let the co-ordinates (x, y) of any point on the given curve change to (X, Y) on shifting the origin to the point $(3, -5)$, the new axes remaining parallel to the original axes. Then, we get

$$x = X + 3 \text{ and } y = Y - 5.$$

The given equation is $x^2 + y^2 - 6x + 10y - 2 = 0$ …(*i*)

Substituting for x and y in the given equation (*i*), the transformed equation is

$$(X + 3)^2 + (Y - 5)^2 - 6(X + 3) + 10(Y - 5) - 2 = 0$$

or $$X^2 + Y^2 = 36.$$

REMARK

The given equation can be written as $(x - 3)^2 + (y + 5)^2 = 36$, it represents a circle with centre $(3, -5)$. By transferring to the parallel axes through the point $(3, -5)$, we have infact shifted the origin to the centre of the circle and the equation has changed from general form to the simplest form $X^2 + Y^2 = 36$.

This is the advantage of translation of axes.

Example 3. *Find the point to which the origin should be shifted so that the equation $y^2 - 6y - 4x + 13 = 0$ will not contain term in y and the constant term.*

Solution. Let the origin be shifted to the point (h, k), axes remaining parallel to the original axes. If the co-ordinates (x, y) of any point on the given curve change to (X, Y), then

$x = X + h$ and $y = Y + k$.

Substituting for x and y in the given equation, the transformed equation is

$$(Y + k)^2 - 6 (Y + k) - 4 (X + h) + 13 = 0$$

i.e. $\quad Y^2 + 2 (k - 3) Y - 4X + (k^2 - 6k - 4h + 13) = 0.$

For this equation to be free from the term containing Y and the constant term, we must have

$2(k - 3) = 0$ and $k^2 - 6k - 4h + 13 = 0$

$\Rightarrow \quad k = 3$ and $3^2 - 6 \times 3 - 4h + 13 = 0$

$\Rightarrow \quad k = 3$ and $4 - 4h = 0$ *i.e.* $h = 1$.

Hence, the origin is shifted to the point $(1, 3)$.

Example 4. *Reduce the equation $x^2 + 4y^2 - 6x - 8y - 12 = 0$ to the form $AX^2 + BY^2 = K$ by shifting the origin to a suitable point.*

Solution. Let the origin be shifted to the point (h, k), axes remaining parallel to the original axes. If the co-ordinates (x, y) of any point on the given curve change to (X, Y), then

$$x = X + h \text{ and } y = Y + k.$$

Substituting for x and y in the given equation, the transformed equation is

$$(X + h)^2 + 4 (Y + k)^2 - 6 (X + h) - 8 (Y + k) - 12 = 0$$

i.e. $\quad X^2 + 4Y^2 + (2h - 6) X + (8k - 8) Y + h^2 + 4k^2 - 6h - 8k - 12 = 0 \quad \ldots(i)$

Since the given equation is to be reduced to the form $AX^2 + BY^2 = K$, the coefficients of X and Y are zero

$\Rightarrow \quad 2h - 6 = 0$ and $8k - 8 = 0 \Rightarrow h = 3, k = 1$, and then

$$h^2 + 4k^2 - 6h - 8k - 12 = 9 + 4 . 1 - 6 . 3 - 8 . 1 - 12 = -25.$$

Hence, on shifting the origin to the point $(3, 1)$, the given equation reduces to $X^2 + 4Y^2 - 25 = 0$ *i.e.* $X^2 + 4Y^2 = 25$.

REMARK

The idea behind the above method is the following :

The given equation can be written as

$$(x^2 - 6x + 9) + 4(y^2 - 2y + 1) = 25$$

i.e. $\quad (x - 3)^2 + 4(y - 1)^2 = 25.$

Now, it is clear to which point the origin should be shifted to simplify the given equation. On shifting the origin to the point $(3, 1)$ the given equation reduces to $X^2 + 4Y^2 = 25$.

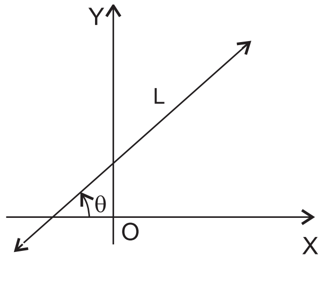
EXERCISE 10.2

1. Shift the origin to the point $(-2, 3)$, axes remaining parallel to the original axes. What will be the new co-ordinates of the point $(1, -2)$.
2. Transform the line $x + y - 1 = 0$ to the parallel axes through the point $(1, 0)$.
3. Find the transformed equation of the curve $x^2 + y^2 + 4x - 6y = 0$ when the origin is shifted to the point $(-2, 3)$.

4. What does the equation $(x - a)^2 + (y - b)^2 = r^2$ become when the axes are transferrd to parallel axes through the point $(a - c, b)$?
5. Reduce the equation $x^2 + y^2 + 8x - 6y - 25 = 0$ to the form $AX^2 + BY^2 = K^2$ by shifting the origin to a suitable point.
6. Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ may not contain term in y and the constant term.
7. Find the point to which the origin should be shifted so that the equation $x^2 + xy - 3x - y + 2 = 0$ may not contain any first degree terms in x and y.

10.3 SLOPE OF A STRAIGHT LINE

Inclination of a straight line



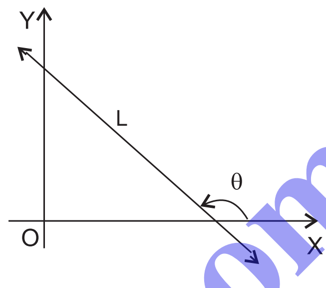


Fig. 10.9.

The angle which a straight line, say L, makes with the positive direction of x-axis measured in the anticlockwise direction to the part of the line above the x-axis (shown in fig. 10.9) is called the ***inclination*** *(or* ***angle of inclination****) of the line L. The inclination is usually denoted by* θ. *Evidently,* $0° \le \theta < 180°$.

In particular :

(*i*) The inclination of a line parallel to x-axis or the x-axis itself is 0°.

(*ii*) The inclination of a line parallel to y-axis or the y-axis itself is 90°.

Horizontal, vertical and oblique lines

(*i*) Any line parallel to x-axis or the x-axis itself is called a *horizontal line*.

(*ii*) Any line parallel to y-axis or the y-axis itself is called a *vertical line*.

(*iii*) A line which is neither horizontal nor vertical is called an *oblique line*.

Slope (or gradient) of a straight line

If θ $(\neq 90°)$ *is the inclination of a straight line, then* $\tan\theta$ *is called its* ***slope*** *(or* ***gradient****). The slope of a line is usually denoted by m.*

Thus, if θ $(\neq 90°)$ is the inclination of a line, then $m = \tan\theta$.

REMARKS

1. Since $\tan\theta$ is not defined when $\theta = 90°$, therefore, the slope of a vertical line is not defined.
2. Slope of y-axis is not defined.
3. Since the inclination of every line parallel to x-axis is 0°, so its slope = $\tan 0° = 0$. Therefore, *the slope of every horizontal line is zero*.
4. Slope of x-axis is zero.

10.3.1 Slope of a straight line joining two points

To find the slope of a non-vertical line passing through two given points.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points on a non-vertical line l. Since line l is non-vertical, $x_2 \neq x_1$.

Let θ be the inclination of the line l. The inclination θ may be acute or obtuse. We shall consider two cases. From P, Q draw perpendiculars PM, QN on x-axis and PL ⊥ NQ as shown in figs. 10.10 and 10.11.

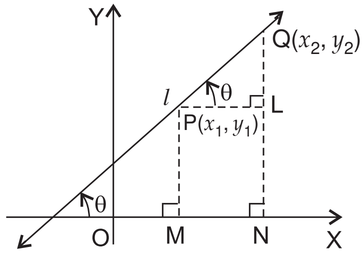


Fig. 10.10.

Case I. *When angle θ is acute :*

From fig. 10.10, we have

PL = MN = ON – OM = $x_2 - x_1$ and

LQ = NQ – NL = NQ – MP = $y_2 - y_1$.

In right triangle QPL, ∠QPL = θ.

$$\therefore \quad \tan\theta = \frac{LQ}{PL} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \ldots(i)$$

Case II. *When angle θ is obtuse* :

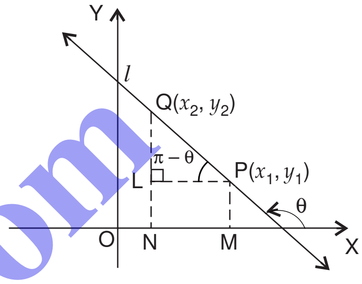


Fig. 10.11.

From fig. 10.11, we have

LP = NM = OM – ON = $x_1 - x_2$ and

LQ = NQ – NL = NQ – MP = $y_2 - y_1$.

In right triangle QPL, ∠LPQ = π – θ.

$$\therefore \quad \tan(\pi - \theta) = \frac{LQ}{LP} = \frac{y_2 - y_1}{x_1 - x_2}$$

$$\Rightarrow \quad -\tan\theta = \frac{y_2 - y_1}{x_1 - x_2}$$

$$\Rightarrow \quad \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}.$$

∴ In both cases, slope of the line l = m = tan θ = $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Hence, the slope m of a non-vertical line passing through the point P(x_1, y_1) and Q(x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the slope of the line passing through the points :*

(i) (3, –2) and (–1, 4) *(ii) (3, –2) and (7, –2)* *(iii) (3, –2) and (3, 4)* *(NCERT)*

Solution. (*i*) Slope of the line passing through the points (3, –2) and (–1, 4)

$$= \frac{4-(-2)}{-1-3} \qquad \Big| \text{ slope } = \frac{y_2 - y_1}{x_2 - x_2}$$

$$= \frac{6}{-4} = -\frac{3}{2}.$$

(*ii*) Slope of the line passing through the points (3, –2) and (7, –2)

$$= \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0.$$

Alternatively, we note that the given points (3, –2) and (7, –2) have same y co-ordinates, therefore, the line is horizontal and hence its slope = 0.

(*iii*) We note that the given points (3, –2) and (3, 4) have same x-coordinates, therefore, the line is vertical and hence its slope is not defined.

ANSWERS

EXERCISE 10.1

1. $|y|$ **2.** 5 units **3.** 4 or -2 **4.** 2 or 8 **5.** None

6. (*i*) $(-1, 0)$ (*ii*) $(0, -2)$ **7.** 1 **8.** $2, -\frac{46}{13}$ **9.** 1 : 3 internally

10. $(-14, 6)$ **11.** $a = 2, b = 2$ **12.** $a = 4, b = 2$

13. $(-2, 2)$ **14.** $(2, 0)$ **15.** $(8, 8)$ **16.** $p = 2, q = -5$

17. $a = -1, b = -1$ **19.** 3 **20.** $6x + 4y - 5 = 0$

22. $\left(2+\frac{\sqrt{11}}{2}, \frac{5}{2}\right)$ or $\left(2-\frac{\sqrt{11}}{2}, \frac{5}{2}\right)$ **23.** $(5, 6)$ or $(3, 4)$ **24.** $\sqrt{10}$ units

25. 3 : 2 internally; $\left(0, \frac{11}{5}\right)$

26. 3 : 2 internally; $-\frac{2}{5}$ **27.** (*i*) $\left(\frac{3}{2}, \frac{7}{2}\right)$ (*ii*) $(2, 5)$ **28.** 5 units

29. $(1, -12), (5, -10)$ **30.** (*i*) $(2, -5), (12, 5)$ (*ii*) $(-5, 4)$ or $(1, 2)$

31. $(-1, 0)$ **32.** $-3, \frac{21}{13}$ **33.** $\frac{1}{2}|2p^2 - 3p - 2|$ sq. units **34.** 15 sq. units

36. $(7, 2)$ or $(1, 0)$

EXERCISE 10.2

1. $(3, -5)$ **2.** $X + Y = 0$ **3.** $X^2 + Y^2 - 13 = 0$

4. $(X - c)^2 + Y^2 = r^2$ **5.** $X^2 + Y^2 = 50$ **6.** $\left(\frac{3}{4}, -2\right)$ **7.** $(1, 1)$

EXERCISE 10.3

1. (*i*) $-\frac{12}{7}$ (*ii*) -1 (*iii*) 0 (*iv*) not defined

2. (*i*) 0 (*ii*) not defined

3. (*i*) $\frac{1}{\sqrt{3}}$ (*ii*) $-\sqrt{3}$ (*iii*) -1 (*iv*) not defined **4.** $-\frac{1}{\sqrt{3}}$

5. (*i*) 30° (*ii*) 45° (*iii*) 135° (*iv*) 60° (*v*) 120° (*vi*) 0°

6. 135° **7.** No **8.** 1 **9.** $-\frac{7}{3}$ **10.** $-\frac{3}{5}$

EXERCISE 10.4

2. (*i*) Perpendicular (*ii*) parallel (*iii*) neither (*iv*) perpendicular

3. $\frac{10}{3}$ **4.** 9 **5.** $-\frac{4}{3}$ **6.** (*ii*), (*iv*) **7.** 2

8. The acute angle between the lines is given by $\tan\theta = \frac{11}{23}$

9. The acute angle between the lines AB and BC is given by $\tan\theta = \frac{2}{3}$

11. (*i*) $-\frac{1}{8}$ (*ii*) 8 (*iii*) $-\frac{1}{8}$ **15.** (*i*) $\frac{7}{3}$ (*ii*) $(5, 8)$ **17.** $D = 2(T + 1)$