5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

INTRODUCTION

We know that $x^2 \geq 0$ for all $x \in \mathbf{R}$ *i.e.* the square of a real number (whether positive, negative or zero) is non-negative. Hence the equations $x^2 = -1$, $x^2 = -5$, $x^2 + 7 = 0$ etc. are not solvable *in real number system*. Thus, there is a need to extend the real number system to a larger system so that we can have solutions of such equations. In fact, our main objective is to solve the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbf{R}$ and the discriminant $= b^2 - 4ac < 0$, which is not possible in real number system. In this chapter, we shall extend the real number system to a larger system called **complex number system** so that the solutions of quadratic equations $ax^2 + bx + c = 0$, where a, b, c are real numbers are possible. We shall also solve quadratic equations with complex coefficients.

5.1 COMPLEX NUMBERS

We know that the equation $x^2 + 1 = 0$ is not solvable in the real number system *i.e.* it has no real roots. Many mathematicians indicated the square roots of negative numbers, but **Euler** was the first to introduce the symbol i (*read* '*iota*') to represent $\sqrt{-1}$, and he defined $i^2 = -1$.

If follows that i is a solution of the equation $x^2 + 1 = 0$. Also $(-i)^2 = i^2 = -1$. Thus the equation $x^2 + 1 = 0$ has two solutions, $x = \pm i$, where $i = \sqrt{-1}$.

The number i is called an ***imaginary number***. In general, the square roots of all negative real numbers are called ***imaginary numbers***. Thus $\sqrt{-1}$, $\sqrt{-5}$, $\sqrt{-\frac{9}{4}}$ etc. are all imaginary numbers.

Complex number

A number of the form a + ib, where a and b are real numbers, is called a ***complex number****.*

For example, $3 + 5i$, $-2 + 3i$, $-2 + i\sqrt{5}$, $7 + i\left(-\frac{2}{3}\right)$ are all complex numbers.

The system of numbers $\mathbf{C} = \{z \,;\, z = a + ib \,;\, a, b \in \mathbf{R}\}$ *is called the set of* ***complex numbers.***

Standard form of a complex number

If a complex number is expressed in the form $a + ib$ *where* $a, b \in \mathbf{R}$ *and* $i = \sqrt{-1}$, *then it is said to be in the* ***standard form****.*

For example, the complex numbers $2 + 5i$, $-3 + \sqrt{2}\,i$, $-\frac{2}{3} - 7i$ are all in the standard form.

Real and imaginary parts of a complex number

If $z = a + ib$ $(a, b \in \mathbf{R})$ is a complex number, then a is called the **real part,** denoted by $\text{Re}(z)$ and b is called **imaginary part**, denoted by $\text{Im}(z)$.

For example :

(*i*) If $z = 2 + 3i$, then $Re(z) = 2$ and $Im(z) = 3$.

(*ii*) If $z = -3 + \sqrt{5}\,i$, then $Re(z) = -3$ and $Im(z) = \sqrt{5}$.

(*iii*) If $z = 7$, then $z = 7 + 0\,i$, so that $Re(z) = 7$ and $Im(z) = 0$.

(*iv*) If $z = -5i$, then $z = 0 + (-5)i$, so that $Re(z) = 0$ and $Im(z) = -5$.

Note that imaginary part of a complex number is a real number.

In $z = a + ib$ ($a, b \in \mathbf{R}$), if $b = 0$ then $z = a$, which is a **real number.** If $a = 0$ and $b \neq 0$, then $z = ib$, which is called **purely imaginary number**. If $b \neq 0$, then $z = a + ib$ is **non-real complex number**. Since every real number a can be written as $a + 0\,i$, we see that $\mathbf{R} \subset \mathbf{C}$ *i.e.* the set of real numbers **R** is a **proper subset** of **C**, the set of complex numbers.

Note that $\sqrt{3}$, 0, 2, π are real numbers; $3 + 2\,i$, $3 - 2\,i$ etc. are non-real complex numbers; $2\,i$, $-\sqrt{2}\,i$ etc. are purely imaginary numbers.

Equality of two complex numbers

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are called ***equal****, written as $z_1 = z_2$, if and only if $a = c$ and $b = d$.*

For example, if the complex numbers $z_1 = a + ib$ and $z_2 = -3 + 5i$ are equal, then $a = -3$ and $b = 5$.

5.1.1 Algebra of complex numbers

In this section, we shall define the usual mathematical operations — addition, subtraction, multiplication, division, square, power etc. on complex numbers and will develop the algebra of complex numbers.

Addition of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then their sum $z_1 + z_2$ is defined as

$z_1 + z_2 = (a + c) + i\,(b + d)$.

For example, let $z_1 = 2 + 3i$ and $z_2 = -5 + 4i$, then

$z_1 + z_2 = (2 + (-5)) + (3 + 4)\,i = -3 + 7i$.

Properties of addition of complex numbers

(*i*) ***Closure property***

The sum of two complex numbers is a complex number *i.e.* if z_1 and z_2 are any two complex numbers, then $z_1 + z_2$ is always a complex number.

(*ii*) ***Addition of complex numbers is commutative***

If z_1 and z_2 are any two complex numbers, then $z_1 + z_2 = z_2 + z_1$.

(*iii*) ***Addition of complex numbers is associative***

If z_1, z_2 and z_3 are any three complex numbers, then

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

(*iv*) ***The existence of additive identity***

Let $z = x + iy$, $x, y \in \mathbf{R}$, be any complex number, then

$(x + iy) + (0 + i\,0) = (x + 0) + i\,(y + 0) = x + iy$ and

$(0 + i\,0) + (x + iy) = (0 + x) + i\,(0 + y) = x + iy$

$\Rightarrow\ (x + iy) + (0 + i\,0) = x + iy = (0 + i\,0) + (x + iy)$.

Therefore, $0 + i\,0$ acts as the additive identity. It is simply written as 0.

Thus, $z + 0 = z = 0 + z$ for all complex numbers z.

(*v*) ***The existence of additive inverse***

For a complex number $z = a + ib$, its negative is defined as

$-z = (-a) + i(-b) = -a - ib$.

Note that $z + (-z) = (a - a) + i(b - b) = 0 + i0 = 0$.

Thus $-z$ acts as additive inverse of z.

Subtraction of complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then the subtraction of z_2 from z_1 is defined as

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (a + ib) + (-c - id) \\ &= (a - c) + i(b - d). \end{aligned}$$

For example, let $z_1 = 2 + 3i$ and $z_2 = -1 + 4i$, then

$$\begin{aligned} z_1 - z_2 &= (2 + 3i) - (-1 + 4i) \\ &= (2 + 3i) + (1 - 4i) \\ &= (2 + 1) + (3 - 4)\, i \ = 3 - i. \end{aligned}$$

and

$$\begin{aligned} z_2 - z_1 &= (-1 + 4i) - (2 + 3i) \\ &= (-1 + 4i) + (-2 - 3i) \\ &= (-1 - 2) + (4 - 3)\, i \ = -3 + i. \end{aligned}$$

Multiplication of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then their product $z_1 z_2$ is defined as

$$z_1 z_2 = (ac - bd) + i(ad + bc).$$

Note that intuitively,

$(a + ib)(c + id) = ac + ibc + iad + i^2 bd$; now put $i^2 = -1$, thus

$(a + ib)(c + id) = ac + i(bc + ad) - bd = (ac - bd) + i(ad + bc)$.

For example, let $z_1 = 3 + 7i$ and $z_2 = -2 + 5i$, then

$$\begin{aligned} z_1 z_2 &= (3 + 7i)(-2 + 5i) \\ &= (3 \times (-2) - 7 \times 5) + i(3 \times 5 + 7 \times (-2)) \\ &= -41 + i. \end{aligned}$$

Properties of multiplication of complex numbers

(*i*) ***Closure property***

The product of two complex numbers is a complex number *i.e.* if z_1 and z_2 are any two complex numbers, then $z_1 z_2$ is always a complex number.

(*ii*) ***Multiplication of complex numbers is commutative***

If z_1 and z_2 are any two complex numbers, then $z_1 z_2 = z_2 z_1$.

(*iii*) ***Multiplication of complex numbers is associative***

If z_1, z_2 and z_3 are any three complex numbers, then $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

(*iv*) ***The existence of multiplicative identity***

Let $z = x + iy$, $x, y \in \mathbf{R}$, be any complex number, then

$(x + iy)(1 + i0) = (x.1 - y.0) + i(x.0 + y.1) = x + iy$ and

$(1 + i0)(x + iy) = (1.x - 0.y) + i(1.y + 0.x) = x + iy$

$\Rightarrow (x + iy)(1 + i0) = x + iy = (1 + i0)(x + iy)$.

Therefore, $1 + i0$ acts as the multiplicative identity. It is simply written as 1.

Thus $z.1 = z = 1.z$ for all complex numbers z.

(*v*) ***Existence of multiplicative inverse***

For every non-zero complex number $z = a + ib$, we have the complex number

$\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$ (denoted by z^{-1} or $\frac{1}{z}$) such that

$$z \cdot \frac{1}{z} = 1 = \frac{1}{z} \cdot z \qquad \text{(check it)}$$

$\frac{1}{z}$ is called the multiplicative inverse of z.

Note that intuitively, $\frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$.

(*vi*) ***Multiplication of complex numbers is distributive over addition of complex numbers***

If z_1, z_2 and z_3 are any three complex numbers, then

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

and $$(z_1 + z_2)z_3 = z_1z_3 + z_2z_3.$$

These results are known as ***distributive laws***.

Division of complex numbers

Division of a complex number $z_1 = a + ib$ by $z_2 = c + id \neq 0$ is defined as

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 . z_2^{-1} = (a+ib)\cdot\left(\frac{c}{c^2+d^2} - i\frac{d}{c^2+d^2}\right) = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}.$$

Note that intuitively,

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}.$$

For example, if $z_1 = 3 + 4i$ and $z_2 = 5 - 6i$, then

$$\frac{z_1}{z_2} = \frac{3+4i}{5-6i} = \frac{3+4i}{5-6i} \times \frac{5+6i}{5+6i} = \frac{(3\times5-4\times6)+(3\times6+4\times5)i}{5^2-6^2\times i^2}$$

$$= \frac{-9+38i}{25+36} = -\frac{9}{61} + \frac{38}{61}i.$$

Integral powers of a complex number

If z is any complex number, then positive integral powers of z are defined as

$z^1 = z$, $z^2 = z.z$, $z^3 = z^2.z$, $z^4 = z^3.z$ and so on.

If z is any non-zero complex number, then negative integral powers of z are defined as :

$$z^{-1} = \frac{1}{z},\ z^{-2} = \frac{1}{z^2},\ z^{-3} = \frac{1}{z^3} \text{ etc.}$$

If $z \neq 0$, then $z^0 = 1$.

5.1.2 Powers of i

Integral power of i are defined as :

$i^0 = 1$, $i^1 = i$, $i^2 = -1$,

$i^3 = i^2.i = (-1)\ i = -i$,

$i^4 = (i^2)^2 = (-1)^2 = 1$,

$i^5 = i^4.i = 1.i = i$,

$i^6 = i^4.i^2 = 1.(-1) = -1$, and so on.

$$i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$$

Remember that $\frac{1}{i} = -i$

$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$

$i^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i$

$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$, and so on.

Note that $i^4 = 1$ ***and*** $i^{-4} = 1$***. It follows that for any integer k,***

$$i^{4k} = 1,\ i^{4k+1} = i,\ i^{4k+2} = i^2 = -1,\ i^{4k+3} = i^3 = -i.$$

Also, we note that $i^2 = -1$ and $(-i)^2 = i^2 = -1$.

Therefore, i and $-i$ are both square roots of -1. However, by the symbol $\sqrt{-1}$, we shall mean i only *i.e.* $\sqrt{-1} = i$.

We observe that i and $-i$ are both the solutions of the equation $x^2 + 1 = 0$.

Similarly, $\left(\sqrt{5}i\right)^2 = \left(\sqrt{5}\right)^2 i^2 = 5\,(-1) = -5$,

and $\left(-\sqrt{5}i\right)^2 = \left(-\sqrt{5}\right)^2 i^2 = 5(-1) = -5$.

Therefore, $\sqrt{5}\, i$ and $-\sqrt{5}\, i$ are both square roots of -5. However, by the symbol $\sqrt{-5}$, we shall mean $\sqrt{5}\, i$ only *i.e.* $\sqrt{-5} = \sqrt{5}\, i$.

In general, if a is any positive real number, then $\sqrt{-a} = \sqrt{a}\,i$.

We already know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all positive real numbers a and b. This result is also true when either $a > 0, b < 0$ or $a < 0, b > 0$. But what if $a < 0, b < 0$? Let us examine :

we note that $i^2 = i \times i = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)}$ (by assuming $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all real numbers) $= \sqrt{1} = 1$. Thus, we get $i^2 = 1$ which is contrary to the fact that $i^2 = -1$.

Therefore, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is **not true** when a and b are both negative real numbers.

Further, if any of a and b is zero, then $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$.

5.1.3 Identities

If z_1 and z_2 are any two complex numbers, then the following results hold :

(*i*) $(z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$

(*ii*) $(z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$

(*iii*) $(z_1 + z_2)(z_1 - z_2) = z_1^2 - z_2^2$

(*iv*) $(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$

(*v*) $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$.

Proof. (*i*) $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2)$

$= (z_1 + z_2)z_1 + (z_1 + z_2)z_2$ (Distributive law)

$= z_1^2 + z_2z_1 + z_1z_2 + z_2^2$ (Distributive law)

$= z_1^2 + z_1z_2 + z_1z_2 + z_2^2$ (Commutative law)

$= z_1^2 + 2z_1z_2 + z_2^2$.

We leave the proofs of the other results for the reader.

5.1.4 Modulus of a complex number

Modulus of a complex number $z = a + ib$, denoted by mod(z) or $|z|$, is defined as

$|z| = \sqrt{a^2 + b^2}$, where $a = \text{Re}(z)$, $b = \text{Im}(z)$.

Sometimes, $|z|$ is called **absolute value** of z. Note that $|z| \geq 0$.

For example :

(*i*) If $z = -3 + 5i$, then $|z| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$.

(*ii*) If $z = 3 - \sqrt{7}\,i$, then $|z| = \sqrt{3^2 + (-\sqrt{7})^2} = \sqrt{9 + 7} = 4$.

Properties of modulus of a complex number

If z, z_1 and z_2 are complex numbers, then

(*i*) $|-z| = |z|$

(*ii*) $|z| = 0$ if and only if $z = 0$

(*iii*) $|z_1 z_2| = |z_1|\,|z_2|$

(*iv*) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$.

Proof. (*i*) Let $z = a + ib$, where $a, b \in \mathbf{R}$, then $-z = -a - ib$.

$\therefore \quad |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$.

(*ii*) Let $z = a + ib$, then $|z| = \sqrt{a^2 + b^2}$.

Now $|z| = 0$ iff $\sqrt{a^2 + b^2} = 0$

i.e. iff $a^2 + b^2 = 0$ *i.e.* iff $a^2 = 0$ and $b^2 = 0$

i.e. iff $a = 0$ and $b = 0$ *i.e.* iff $z = 0 + i\,0$

i.e. iff $z = 0$.

(*iii*) Let $z_1 = a + ib$, and $z_2 = c + id$, then

$z_1 z_2 = (ac - bd) + i\,(ad + bc)$.

$$\therefore |z_1 z_2| = \sqrt{(ac - bd)^2 + (ad + bc)^2}$$

$$= \sqrt{a^2c^2 + b^2d^2 - 2abcd + a^2d^2 + b^2c^2 + 2abcd}$$

$$= \sqrt{(a^2 + b^2)(c^2 + d^2)}$$

$$= \sqrt{a^2 + b^2}\,\sqrt{c^2 + d^2} \qquad (\because a^2 + b^2 \geq 0,\ c^2 + d^2 \geq 0)$$

$$= |z_1|\,|z_2|.$$

(*iv*) Here $z_2 \neq 0 \Rightarrow |z_2| \neq 0$.

Let $\dfrac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow |z_1| = |z_2 z_3|$

$\Rightarrow |z_1| = |z_2|\,|z_3|$ (using part (*iii*))

$\Rightarrow \dfrac{|z_1|}{|z_2|} = |z_3| \Rightarrow \dfrac{|z_1|}{|z_2|} = \left|\dfrac{z_1}{z_2}\right| \qquad \left(\because z_3 = \dfrac{z_1}{z_2}\right)$

REMARK

From (*iii*), on replacing both z_1 and z_2 by z, we get

$|z\,z| = |z|\,|z|$ *i.e.* $|z^2| = |z|^2$.

Similary, $|z^3| = |z^2 z| = |z^2|\,|z| = |z|^2\,|z| = |z|^3$ etc.

5.1.5 Conjugate of a complex number

Conjugate of a complex number $z = a + ib$, denoted by $\bar{z}$, is defined as

$\bar{z} = a - ib$ i.e. $\overline{a+ib} = a - ib$.

For example :

(*i*) $\overline{2+5i} = 2 - 5i$, $\overline{2-5i} = 2 + 5i$

(*ii*) $\overline{-3-7i} = -3 + 7i$, $\overline{-3+7i} = -3 - 7i$.

Properties of conjugate of a complex number

If z, z_1 and z_2 are complex numbers, then

(*i*) $\overline{(\bar{z})} = z$ (*ii*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (*iii*) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(*iv*) $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$ (*v*) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, provided $z_2 \neq 0$

(*vi*) $|\bar{z}| = |z|$ (*vii*) $z\,\bar{z} = |z|^2$ (*viii*) $z^{-1} = \dfrac{\bar{z}}{|z|^2}$, provided $z \neq 0$.

Proof. (*i*) Let $z = a + ib$, where $a, b \in \mathbf{R}$, so that $\bar{z} = a - ib$.

$\therefore \ \overline{(\bar{z})} = \overline{a-ib} = a + ib = z$.

(*ii*) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\overline{z_1 + z_2} = \overline{(a+ib)+(c+id)} = \overline{(a+c)+i(b+d)}$$

$$= (a + c) - i(b + d) = (a - ib) + (c - id) = \bar{z}_1 + \bar{z}_2.$$

(*iii*) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\overline{z_1 - z_2} = \overline{(a+ib)-(c+id)} = \overline{(a-c)+i(b-d)}$$

$$= (a - c) - i(b - d) = (a - ib) - (c - id)$$

$$= \bar{z}_1 - \bar{z}_2.$$

(*iv*) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\overline{z_1 z_2} = \overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(ad+bc)}$$

$$= (ac - bd) - i(ad + bc).$$

Also $\bar{z}_1 \bar{z}_2 = (a - ib)(c - id) = (ac - bd) - i(ad + bc)$.

Hence $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$.

(*v*) Here $z_2 \neq 0 \Rightarrow \bar{z}_2 \neq 0$.

Let $\dfrac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow \bar{z}_1 = \overline{z_2 z_3}$

$\Rightarrow \bar{z}_1 = \bar{z}_2 \, \bar{z}_3$ (using part (*iv*))

$\Rightarrow \dfrac{\bar{z}_1}{\bar{z}_2} = \bar{z}_3 \Rightarrow \dfrac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\dfrac{z_1}{z_2}\right)}$ $\left(\because z_3 = \dfrac{z_1}{z_2}\right)$

(*vi*) Let $z = a + ib$, then $\bar{z} = a - ib$.

$\therefore |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$.

(*vii*) Let $z = a + ib$, then $\bar{z} = a - ib$.

$$\therefore\ z\,\bar{z} = (a + ib)\,(a - ib)$$
$$= (aa - b(-b)) + i(a(-b) + ba) \qquad \text{(Def. of multiplication)}$$
$$= (a^2 + b^2) + i\,.\,0$$
$$= a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2.$$

Remember that (a + ib) (a – ib) = $a^2 + b^2$.

(*viii*) Let $z = a + ib \neq 0$, then $|z| \neq 0$.

$$\therefore\ z\,\bar{z} = (a + ib)\,(a - ib) = a^2 + b^2 = |z|^2$$

$$\Rightarrow \frac{z\,\bar{z}}{|z|^2} = 1 \Rightarrow \frac{\bar{z}}{|z|^2} = \frac{1}{z} = z^{-1}$$

Thus, $z^{-1} = \dfrac{\bar{z}}{|z|^2}$, provided $z \neq 0$.

REMARK

From (*iv*), on replacing both z_1 and z_2 by z, we get

$\overline{z\,z} = \bar{z}\,\bar{z}$ *i.e.* $\overline{z^2} = (\bar{z})^2$.

Similarly, $\overline{(z^3)} = \overline{(z^2 z)} = \overline{(z^2)}\,\bar{z} = (\bar{z})^2\,\bar{z} = (\bar{z})^3$ etc.

NOTE

The order relations '*greater than*' and '*less than*' are not defined for complex numbers *i.e.* the inequalities $2 + 3i > -2 + 5i$, $4i \geq 1 - 2i$, $-1 + 3i < 5$ etc. are meaningless.

ILLUSTRATIVE EXAMPLES

Example 1. *A student says*

$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i\,.\,i = i^2 = -1$. *Thus* $1 = -1$.

Where is the fault ?

Solution. $1 = \sqrt{1} = \sqrt{(-1)(-1)}$ is true, but $\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$ is wrong.

Because if both a, b are negative real numbers, then $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is not true.

Example 2. *If* $z = \sqrt{37} + \sqrt{-19}$, *find* Re($z$), Im ($z$), $\bar{z}$ *and* $|z|$.

Solution. Given $z = \sqrt{37} + \sqrt{-19} = \sqrt{37} + i\sqrt{19}$.

$\therefore$ $\quad$ Re $(z) = \sqrt{37}$ and Im$(z) = \sqrt{19}$.

$$\bar{z} = \overline{\sqrt{37} + i\sqrt{19}} = \sqrt{37} - i\sqrt{19}.$$

$$|z| = \sqrt{(\sqrt{37})^2 + (\sqrt{19})^2} = \sqrt{37 + 19} = \sqrt{56} = 2\sqrt{14}.$$

Example 3. *If* $4x + i(3x - y) = 3 - 6i$ *and* x, y *are real numbers, then find the values of* x *and* y. *(NCERT)*

Solution. Given $4x + i\,(3x - y) = 3 - 6i$

$\Rightarrow \quad 4x + i\,(3x - y) = 3 + i\,(-6)$.

Equating real and imaginary parts on both sides, we get

$$4x = 3 \text{ and } 3x - y = -6$$

$\Rightarrow \quad x = \frac{3}{4}$ and $3 \times \frac{3}{4} - y = -6$

$\Rightarrow \quad x = \frac{3}{4}$ and $y = 6 + \frac{9}{4} = \frac{33}{4}$.

Hence $x = \frac{3}{4}$ and $y = \frac{33}{4}$.

Example 4. *For what real values of x and y are the following numbers equal*

(i) $(1 + i)\, y^2 + (6 + i)$ *and* $(2 + i)\, x$ *(ii)* $x^2 - 7x + 9yi$ *and* $y^2 i + 20i - 12$?

Solution. (*i*) Given $(1 + i)\, y^2 + (6 + i) = (2 + i)\, x$

$\Rightarrow \quad (y^2 + 6) + i(y^2 + 1) = 2x + ix$

$\Rightarrow \quad y^2 + 6 = 2x$ and $y^2 + 1 = x$

$\Rightarrow \quad x = 5$ and $y^2 = 4 \Rightarrow x = 5$ and $y = \pm 2$.

Hence, the required values of x and y are

$x = 5, y = 2; x = 5, y = -2$.

(*ii*) Given $x^2 - 7x + 9yi = y^2 i + 20i - 12$

$\Rightarrow \quad (x^2 - 7x) + i(9y) = (-12) + i(y^2 + 20)$

$\Rightarrow \quad x^2 - 7x = -12$ and $9y = y^2 + 20$

$\Rightarrow \quad x^2 - 7x + 12 = 0$ and $y^2 - 9y + 20 = 0$

$\Rightarrow \quad (x - 4)(x - 3) = 0$ and $(y - 5)(y - 4) = 0$

$\Rightarrow \quad x = 4, 3$ and $y = 5, 4$.

Hence, the required values of x and y are

$$x = 4, y = 5;\ x = 4, y = 4;\ x = 3, y = 5;\ x = 3, y = 4.$$

Example 5. *Express each of the following in the standard form a + ib :*

(i) $\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right) - \left(-\frac{4}{3} + i\right)$ *(NCERT)* *(ii)* $3(7 + i7) + i(7 + i7)$ *(NCERT)*

(iii) $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3})$ *(iv)* $\frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$. *(NCERT)*

Solution. (*i*) $\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right) - \left(-\frac{4}{3} + i\right)$

$$= \left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right) + \left(\frac{4}{3} - i\right)$$

$$= \left(\frac{1}{3} + 4 + \frac{4}{3}\right) + i\left(\frac{7}{3} + \frac{1}{3} - 1\right) = \frac{17}{3} + \frac{5}{3}i.$$

(*ii*) $3(7 + i7) + i(7 + i7) = (21 + 21i) + (7i + 7i^2)$

$= 21 + 21i + 7i + 7(-1) = (21 - 7) + (21 + 7)i$

$= 14 + 28i.$

(*iii*) $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = (-2 + \sqrt{3}\, i)(-3 + 2\sqrt{3}\, i)$

$= (6 - 2\sqrt{3}\sqrt{3}) + (-3\sqrt{3} - 4\sqrt{3})i$

$= 0 - 7\sqrt{3}\, i.$

(iv) $\dfrac{(3+i\sqrt{5})\,(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})} = \dfrac{(3)^2+(\sqrt{5})^2}{\sqrt{2}i+\sqrt{2}i}$ $\quad((a+ib)\,(a-ib) = a^2+b^2)$

$= \dfrac{9+5}{2\sqrt{2}i} = \dfrac{14}{2\sqrt{2}}\cdot\dfrac{1}{i} = \dfrac{7}{\sqrt{2}}(-i)$ $\quad\left(\because \dfrac{1}{i} = -i\right)$

$= 0 - \dfrac{7}{\sqrt{2}}i.$

Example 6. *Express the following in the form a + ib :*

(i) $(-i)\,(2i)\left(-\dfrac{1}{8}i\right)^3$ *(NCERT)* *(ii)* i^{102} *(iii)* i^{-39} *(NCERT)*

(iv) $\left(-\sqrt{-1}\right)^{31}$ *(v)* $i^9 + i^{19}$ *(NCERT)* *(vi)* $i^{35} + \dfrac{1}{i^{35}}$.

Solution. (*i*) $(-i)\,(2i)\left(-\dfrac{1}{8}i\right)^3 = (-1)^4 \times 2 \times \left(\dfrac{1}{8}\right)^3 \times i^5$

$= 1 \times 2 \times \dfrac{1}{512} \times i^4 \times i$

$= \dfrac{1}{256} \times 1 \times i = 0 + \dfrac{1}{256}i.$

(*ii*) $i^{102} = i^{4\times25+2} = i^2$ $\quad(\because i^{4k+2} = i^2,\ k \in \mathbf{I})$

$= -1 = -1 + i0.$

(*iii*) $i^{-39} = i^{4\times(-10)+1} = i$ $\quad(\because i^{4k+1} = i,\ k \in \mathbf{I})$

$= 0 + i.$

(*iv*) $\left(-\sqrt{-1}\right)^{31} = (-i)^{31} = (-1)^{31}\ i^{31}$

$= -i^{4\times7+3} = -i^3$ $\quad(\because i^{4k+3} = i^3,\ k \in \mathbf{I})$

$= -i^2\,.\,i = -(-1)\,i = i = 0 + i.$

(*v*) $i^9 + i^{19} = i^{2\times4+1} + i^{4\times4+3} = i + i^3$

$= i + i^2\,.\,i = i + (-1)\,i = 0 = 0 + i0.$

(*vi*) $i^{35} + \dfrac{1}{i^{35}} = i^{35} + i^{-35} = i^{4\times8+3} + i^{4\times(-9)+1}$

$= i^3 + i = i^2\,i + i = (-1)i + i$

$= 0 = 0 + i0.$

Example 7. *Express each of the following in the standard form a + ib :*

(i) $(1-i)^4$ *(NCERT)* *(ii)* $\left(-2-\dfrac{1}{3}i\right)^3$ *(NCERT)*

(iii) $(2i-i^2)^2 + (1-3i)^3$ *(iv)* $\left(i^{18}+\left(\dfrac{1}{i}\right)^{25}\right)^3$ *(NCERT)*

(v) $(1+i)^6 + (1-i)^3$ *(NCERT Examplar Problems)*

Solution. (*i*) $(1-i)^4 = ((1-i)^2)^2 = (1+i^2-2i)^2$

$= (1+(-1)-2i)^2 = (-2i)^2 = 4i^2$

$= 4\,(-1) = -4 = -4 + i0.$

(*ii*) $\left(-2-\frac{1}{3}i\right)^3 = (-1)^3\left(2+\frac{1}{3}i\right)^3$

$= -\left[2^3 + 3 \times 2^2 \times \frac{1}{3}i + 3 \times 2 \times \left(\frac{1}{3}i\right)^2 + \left(\frac{1}{3}i\right)^3\right]$

$= -\left[8 + 4i + \frac{2}{3}i^2 + \frac{1}{27}i^3\right]$

$= -\left[8+4i+\frac{2}{3}(-1)+\frac{1}{27}(-i)\right]$

$= -\left[\frac{22}{3} + \frac{107}{27}i\right] = -\frac{22}{3} - \frac{107}{27}i.$

(*iii*) $(2i - i^2)^2 + (1-3i)^3 = (2i+1)^2 + (1-3i)^3$

$= (4i^2 + 4i + 1) + (1 - 9i + 27i^2 - 27i^3)$

$= -4 + 4i + 1 + 1 - 9i - 27 + 27i = -29 + 22i.$

(*iv*) $\left(i^{18}+\left(\frac{1}{i}\right)^{25}\right)^3 = \left(i^{4\times4+2}+(-i)^{25}\right)^3$ $\left(\because \frac{1}{i} = -i\right)$

$= \left(i^2+(-1)^{25}i^{25}\right)^3 = (-1 - i^{4\times6+1})^3$

$= (-1-i)^3 = (-1)^3(1+i)^3$

$= -\left[1+3i+3i^2+i^3\right]$

$= -\left[1+3i-3-i\right] = -(-2+2i)$

$= 2 - 2i.$

(*v*) $(1+i)^6 = ((1+i)^2)^3 = (1+i^2+2i)^3 = (1-1+2i)^3 = (2i)^3$

$= 8i^3 = 8(-i) = -8i$

and $(1-i)^3 = 1 - i^3 - 3i + 3i^2 = 1-(-i) - 3i + 3(-1)$

$= -2 - 2i$

$\therefore (1+i)^6 + (1-i)^3 = -8i + (-2-2i) = -2 - 10i.$

Example 8. *Find the multiplicative inverse of* $\sqrt{5} + 3i$. (NCERT)

Solution. Let $z = \sqrt{5} + 3i$,

then $\bar{z} = \sqrt{5} - 3i$ and $|z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14.$

We know that the multiplicative inverse of z is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5}-3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14}i.$$

Alternatively

$$z^{-1} = \frac{1}{z} = \frac{1}{\sqrt{5}+3i} = \frac{1}{\sqrt{5}+3i} \times \frac{\sqrt{5}-3i}{\sqrt{5}-3i}$$

$$= \frac{\sqrt{5}-3i}{(\sqrt{5})^2-(3i)^2} = \frac{\sqrt{5}-3i}{5-9(-1)} = \frac{\sqrt{5}-3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14}i.$$

Example 9. *Express the following in the form a + ib :*

(*i*) $\frac{i}{1+i}$ (*ii*) $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$ (NCERT)

(*iii*) $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ (NCERT) (*iv*) $\frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i}.$

Solution. (*i*) $\frac{i}{1+i} = \frac{i}{1+i} \times \frac{1-i}{1-i} = \frac{i-i^2}{1-i^2} = \frac{i-(-1)}{1-(-1)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i.$

(*ii*) $$\frac{5+\sqrt{2}i}{1-\sqrt{2}i} = \frac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \frac{1+\sqrt{2}i}{1+\sqrt{2}i} = \frac{5+5\sqrt{2}i+\sqrt{2}i+2i^2}{1^2-(\sqrt{2}i)^2}$$

$$= \frac{5+6\sqrt{2}i-2}{1-2(-1)} = \frac{3+6\sqrt{2}i}{3} = 1+2\sqrt{2}i.$$

(*iii*) $$\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \frac{1+i-2+8i}{(1-4i)(1+i)} \times \frac{3-4i}{5+i}$$

$$= \frac{-1+9i}{1+i-4i+4} \times \frac{3-4i}{5+i} = \frac{(-1+9i)(3-4i)}{(5-3i)(5+i)}$$

$$= \frac{-3+4i+27i+36}{25+5i-15i+3} = \frac{33+31i}{28-10i}$$

$$= \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i} = \frac{33\times 28+330i+31\times 28i-310}{(28)^2-(10i)^2}$$

$$= \frac{924-310+(330+868)i}{784-100(-1)} = \frac{614+1198i}{884} = \frac{307}{442} + \frac{599}{442}i.$$

(*iv*) $$\frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i} = \frac{(1+i)(3+i)(3+i)-(1-i)(3-i)(3-i)}{(3-i)(3+i)}$$

$$= \frac{(1+i)(8+6i)-(1-i)(8-6i)}{9-i^2}$$

$$= \frac{(2+14i)-(2-14i)}{9+1} = \frac{28i}{10} = 0 + \frac{14}{5}i.$$

Example 10. (*i*) *If* $\frac{(1+i)^2}{2-i} = x+iy$*, then find the value of* $x+y$. (*NCERT Examplar Problems*)

(*ii*) *If* $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x+iy$*, then find* (x, y). (*NCERT Examplar Problems*)

(*iii*) *If* $\left(\frac{1-i}{1+i}\right)^{100} = a+ib$*, then find* (a, b). (*NCERT Examplar Problems*)

Solution. (*i*) $x + iy = \frac{(1+i)^2}{2-i} = \frac{1+i^2+2i}{2-i} = \frac{1-1+2i}{2-i} = \frac{2i}{2-i}$

$$= \frac{2i}{2-i} \times \frac{2+i}{2+i} = \frac{4i+2i^2}{2^2-i^2} = \frac{4i+2(-1)}{4-(-1)}$$

$$= \frac{-2+4i}{5} = -\frac{2}{5} + \frac{4}{5}i.$$

$$\Rightarrow x = -\frac{2}{5} \text{ and } y = \frac{4}{5}.$$

$$\therefore x + y = -\frac{2}{5} + \frac{4}{5} = \frac{2}{5}.$$

(*ii*) We have, $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1^2-i^2}$

$$= \frac{1+i^2+2i}{1-(-1)} = \frac{1-1+2i}{2} = \frac{2i}{2} = i$$

$\therefore \quad \frac{1-i}{1+i} = \frac{1}{i} = -i$ $\left(\because \frac{1}{i} = -i\right)$

Given $x + iy = \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = i^3 - (-i)^3 = i^3 + i^3$

$= 2i^3 = 2(-i) = 0 - 2i$

$\Rightarrow \quad x = 0$ and $y = -2$.

Hence, the orders pair (x, y) is $(0, -2)$.

(*iii*) Given $a + ib = \left(\frac{1-i}{1+i}\right)^{100} = (-i)^{100}$ (see part (*ii*))

$= (-1)^{100}\ (i)^{4 \times 25} = 1 \times 1 = 1 = 1 + 0i$

$\Rightarrow \quad a = 1$ and $b = 0$.

Hence, the ordered pair (a, b) is $(1, 0)$.

Example 11. *(i) If $(1 + i)z = (1 - i)\bar{z}$, then show that $z = -i\bar{z}$.* (*NCERT Examplar Problems*)

(ii) If $z_1 = 2 - i$ and $z_2 = -2 + i$, find Re $\left(\frac{z_1 z_2}{\bar{z}_1}\right)$. (*NCERT*)

Solution. (*i*) Given $(1 + i)z = (1 - i)\bar{z}$

$$\Rightarrow \quad \frac{z}{\bar{z}} = \frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1^2 - i^2} = \frac{1 + i^2 - 2i}{1-(-1)}$$

$$\Rightarrow \quad \frac{z}{\bar{z}} = \frac{1 - 1 - 2i}{2} = -\frac{2i}{2} = -i$$

$\Rightarrow \quad z = -i\bar{z}$.

(*ii*)

$$\frac{z_1 z_2}{\bar{z}_1} = \frac{(2-i)(-2+i)}{\overline{2-i}} = \frac{-4 + 2i + 2i - i^2}{2+i} = \frac{-4 + 4i - (-1)}{2+i}$$

$$= \frac{-3 + 4i}{2+i} = \frac{-3+4i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{-6 + 3i + 8i - 4i^2}{2^2 - i^2} = \frac{-6 + 11i - 4(-1)}{4-(-1)}$$

$$= \frac{-2 + 11i}{5} = -\frac{2}{5} + \frac{11}{5}i.$$

$$\therefore \quad \text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = -\frac{2}{5}.$$

Example 12. *(i) Find the conjugate of $\frac{(3 - 2i)(2 + 3i)}{(1 + 2i)(2 - i)}$* (*NCERT*)

(ii) Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$ (*NCERT*)

(iii) Find the modulus of $\frac{(2-3i)^2}{-1+5i}$

Solution. (*i*) Let $z = \frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{6 + 9i - 4i + 6}{2 - i + 4i + 2}$

$$= \frac{12 + 5i}{4 + 3i} = \frac{12 + 5i}{4 + 3i} \times \frac{4 - 3i}{4 - 3i} = \frac{48 - 36i + 20i + 15}{4^2 - (3i)^2}$$

$$= \frac{63 - 16i}{16 - 9(-1)} = \frac{63 - 16i}{25} = \frac{63}{25} - \frac{16}{25}i.$$

$$\therefore \text{ Conjugate of } z = \frac{63}{25} + \frac{16}{25}i.$$

Example 32. *If α and β are different complex numbers with $|\beta| = 1$, then find* $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$. *(NCERT)*

Solution. We have $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{(\beta - \alpha)\bar{\beta}}{(1 - \bar{\alpha}\beta)\bar{\beta}}\right|$ (Note this step)

$= \left|\frac{(\beta - \alpha)\bar{\beta}}{\bar{\beta} - \bar{\alpha}\beta\bar{\beta}}\right| = \left|\frac{(\beta - \alpha)\bar{\beta}}{\bar{\beta} - \bar{\alpha}}\right|$ (Given $|\beta| = 1 \Rightarrow |\beta|^2 = 1 \Rightarrow \beta\bar{\beta} = 1$)

$= \frac{|(\beta - \alpha)\bar{\beta}|}{|\bar{\beta} - \bar{\alpha}|}$ $\left(\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right)$

$= \frac{|\beta - \alpha|\,|\bar{\beta}|}{|\overline{\beta - \alpha}|}$ $(\because \bar{z}_1 - \bar{z}_2 = \overline{z_1 - z_2})$

$= \frac{|\beta - \alpha|\,|\beta|}{|\beta - \alpha|}$ $(\because |\bar{z}| = |z|)$

$= |\beta| = 1$ ($|\beta| = 1$, given)

Hence, $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = 1.$

Example 33. *If* $|z_1| = |z_2| = ... = |z_n| = 1$, *prove that*

$$|z_1 + z_2 + ... + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right|.$$ (*NCERT Examplar Problems*)

Solution. Given $|z_1| = |z_2| = \ldots = |z_n| = 1$

$\Rightarrow \quad |z_1|^2 = |z_2|^2 = \ldots = |z_n|^2 = 1$

$\Rightarrow \quad z_1\bar{z}_1 = 1,\ z_2\bar{z}_2 = 1,\ \ldots,\ z_n\bar{z}_n = 1$

$\Rightarrow \quad \frac{1}{z_1} = \bar{z}_1,\ \frac{1}{z_2} = \bar{z}_2,\ \ldots,\ \frac{1}{z_n} = \bar{z}_n.$

$\therefore \quad \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right| = |\bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_n|$

$= |\overline{z_1 + z_2 + \ldots + z_n}| = |z_1 + z_2 + \ldots + z_n|.$ $(\because |\bar{z}| = |z|)$

EXERCISE 5.1

Very short answer type questions (1 to 31) :

1. Evaluate the following :

(*i*) $\sqrt{-9} \times \sqrt{-4}$ (*ii*) $\sqrt{(-9)(-4)}$ (*iii*) $\sqrt{-25} \times \sqrt{16}$

(*iv*) $3\sqrt{-16}\sqrt{-25}$ (*v*) $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$.

2. If $z = -3 - i$, find $Re(z)$, $Im(z)$, $\bar{z}$ and $|z|$.

3. If $z^2 = -i$, then is it true that $z = \pm\frac{1}{\sqrt{2}}(1 - i)$?

4. If $z^2 = -3 + 4i$, then is it true that $z = \pm(1 + 2i)$?

5. If $i = \sqrt{-1}$, then show that $(x + 1 + i)(x + 1 - i) = x^2 + 2x + 2$.

6. Find real values of x and y if

(*i*) $2y + (3x - y)i = 5 - 2i$ (*ii*) $(3x - 1) + (\sqrt{3} + 2y)i = 5$

(*iii*) $(3y - 2) + i(7 - 2x) = 0$.

7. If $x, y \in \mathbf{R}$ and $(5y - 2) + i(3x - y) = 3 - 7i$, find the values of x and y.

8. If x, y are reals and $(3y + 2) + i(x + 3y) = 0$, find the values of x and y.

9. If x, y are reals and $(1 - i)x + (1 + i)y = 1 - 3i$, find the values of x and y.

10. For any two complex numbers z_1 and z_2, prove that

$$\text{Re}\,(z_1 z_2) = \text{Re}\,(z_1)\,\text{Re}(z_2) - \text{Im}\,(z_1)\,\text{Im}\,(z_2).$$ (*NCERT*)

11. For any complex number z, prove that

$$\text{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \text{Im}(z) = \frac{z - \bar{z}}{2i}.$$

12. If z is a complex number, show that $\dfrac{z - \bar{z}}{2i}$ is real.

Express the following (13 to 20) complex numbers in the standard form a + ib :

13. (*i*) $(-5i)\left(\frac{1}{8}i\right)$ (*NCERT*) (*ii*) $(5i)\left(-\frac{3}{5}i\right)$. (*NCERT*)

14. (*i*) $(1 - i) - (-1 + 6i)$ (*NCERT*) (*ii*) $\left(\frac{1}{5} + i\,\frac{2}{5}\right) - \left(4 + i\,\frac{5}{2}\right)$. (*NCERT*)

15. (*i*) $(-2 + 3i) + 3\left(-\frac{1}{2}i + 1\right) - (2i)$ (*ii*) $(7 + i5)(7 - i5)$.

16. (*i*) $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i)$ (*NCERT*) (*ii*) $(-5 + 3i)^2$.

17. (*i*) $\left(\frac{1}{2} + 2i\right)^3$ (*ii*) $(5 - 3i)^3$. (*NCERT*)

18. (*i*) $\left(\frac{1}{3} + 3i\right)^3$ (*NCERT*) (*ii*) $(\sqrt{5} + 7i)(\sqrt{5} - 7i)^2$.

19. (*i*) i^{99} (*ii*) i^{-35}. (*NCERT*)

20. (*i*) $(-\sqrt{-4})^3$ (*ii*) $i + i^2 + i^3 + i^4$.

21. Find the value of $(-1 + \sqrt{-3})^2 + (-1 - \sqrt{-3})^2$.

22. If n is any integer, then find the value of

(*i*) $(-\sqrt{-1})^{4n+3}$ (*ii*) $\dfrac{i^{4n+1} - i^{4n-1}}{2}$. (*NCERT Examplar Problems*)

23. Find the multiplicative inverse of $-i$. (*NCERT*)

24. Express the following numbers in the form $a + ib$, $a, b \in \mathbf{R}$:

(*i*) $\dfrac{i}{1+i}$ (*ii*) $\dfrac{1-i}{1+i}$.

25. If $(a + ib)(c + id) = A + iB$, then show that $(a^2 + b^2)(c^2 + d^2) = A^2 + B^2$.

26. Find the modulus of the following complex numbers :

(*i*) $(3 - 4i)(-5 + 12i)$ (*ii*) $\dfrac{5 - 12i}{-3 + 4i}$.

27. Find the modulus of the following :

(*i*) $\dfrac{(2 - 3i)^2}{4 + 3i}$ (*ii*) $(\sqrt{7} - 3i)^3$.

28. (*i*) If $z = 3 - \sqrt{7}\,i$, then find $|z^{-1}|$.

(*ii*) If $z = x + iy$, $x, y \in \mathbf{R}$, then find $|iz|$.

29. Find the conjugate of i^7.

30. Write the conjugate of $(2 + 3i)(1 - 2i)$ in the form $a + ib$, $a, b \in \mathbf{R}$.

31. Solve for x : $|1 + i|^x = 2$.

Express the following (32 and 33) complex numbers in the standard form a + ib :

32. (*i*) $i^{55} + i^{60} + i^{65} + i^{70}$ (*ii*) $\dfrac{i + i^2 + i^4}{1 + i^2 + i^4}$.

44. If $a + ib = \dfrac{(x+i)^2}{2x^2+1}$, then prove that $a^2 + b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$.

45. If $z = 1 + 2i$, then find the value of $z^3 + 7z^2 - z + 16$.

46. Show that if $\left|\dfrac{z-5i}{z+5i}\right| = 1$, then z is a real number.

47. (*i*) If $z = x + iy$ and $\left|\dfrac{z-2}{z-3}\right| = 2$, show that $3(x^2 + y^2) - 20x + 32 = 0$.

(*NCERT Examplar Problems*)

(*ii*) If $z = x + iy$ and $\dfrac{|z-1-i|+4}{3|z-1-i|-2} = 1$, show that $x^2 + y^2 - 2x - 2y = 7$.

48. Find the least positive integral value of n for which $\left(\dfrac{1+i}{1-i}\right)^n$ is a real number.

49. Find the real value of θ such that $\dfrac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number.

50. If z is a complex number such that $|z| = 1$, prove that $\dfrac{z-1}{z+1}$ $(z \neq -1)$ is purely imaginary. What is the exception? (*NCERT Examplar Problems*)

51. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$, then find the value of $|z_1 + z_2 + z_3|$. (*NCERT Examplar Problems*)

5.2 ARGAND PLANE

We know that corresponding to every real number there exists a unique point on the number line (called real axis) and conversely corresponding to every point on the line there exists a unique real number *i.e.* there is a one-one correspondence between the set **R** of real numbers and the points on the real axis.

In a similar way, corresponding to every ordered pair (x, y) of real numbers there exists a unique point P in the co-ordinate plane with x as **abscissa** and y as **ordinate** of the point P and conversely corresponding to every point P in the plane there exists a unique ordered pair of real numbers. Thus, there is a one-one correspondence between the set of ordered pairs $\{(x, y) ; x, y \in \mathbf{R}\}$ and the points in the co-ordinate plane.

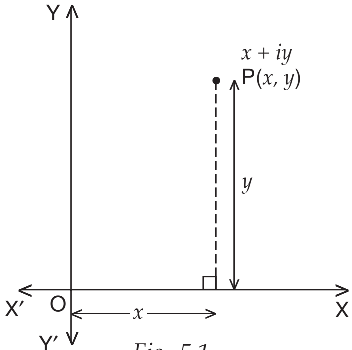


Fig. 5.1.

The point P with co-ordinates (x, y) is said to represent the complex number $z = x + iy$.

It follows that the complex number $z = x + iy$ can be uniquely represented by the point $P(x, y)$ in the co-ordinate plane and conversely corresponding to the point $P(x, y)$ in the plane there exists a unique complex number $z = x + iy$. The co-ordinate plane that represents the complex numbers is called the **complex plane** or **Argand plane**.

The complex numbers 3, −2, 2 *i*, −2 *i*, 2 + 3 *i*, 2 − 3 *i*, −2 + 3 *i* and −3 − 3 *i* which correspond to the ordered pairs (3, 0), (−2, 0), (0, 2), (0, −2), (2, 3), (2, −3), (−2, 3) and (−3, −3) respectively have been represented geometrically in the co-ordinate plane by the points A, B, C, D, E, F, G and H respectively in fig. 5.2.

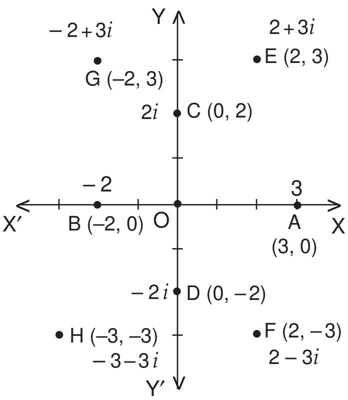


Fig. 5.2.

Note that every real number $x = x + 0i$ is represented by point $(x, 0)$ lying on x-axis, and every purely imaginary number iy is represented by point $(0, y)$ lying on y-axis. Consequently, x-axis is called the **real axis** and y-axis is called the **imaginary axis.**

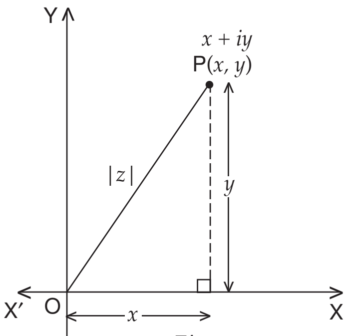


Fig. 5.3.

If the point $P(x, y)$ represents the complex number $z = x + iy$, then the distance between the points P and the origin O $(0, 0) = \sqrt{x^2 + y^2} = |z|$. Thus, the modulus of z *i.e.* $|z|$ is the distance between points P and O (see fig. 5.3).

Geometric representation of $-z$, $\bar{z}$ ***and*** $-\bar{z}$. If $z = x + iy$, $x, y \in \mathbf{R}$ is represented by the point $P(x, y)$ in the complex plane, then the complex numbers $-z$, $\bar{z}$, $-\bar{z}$ are represented by the points $P'(-x, -y)$, $Q(x, -y)$ and $Q'(-x, y)$ respectively in the complex plane (see fig. 5.4.).

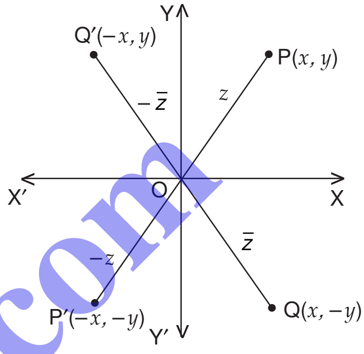


Fig. 5.4.

Geometrically, the point $Q(x, -y)$ is the mirror image of the point $P(x, y)$ in the real axis. Thus, conjugate of z *i.e.* $\bar{z}$ is the mirror image of z in the x-axis.

5.3 POLAR REPRESENTATION OF COMPLEX NUMBERS

Let the point $P(x, y)$ represent the non-zero complex number $z = x + iy$ in the Argand plane. Let the directed line segment OP be of length r (> 0) and θ be the radian measure of the angle which OP makes with the positive direction of x-axis (shown in fig. 5.5). Then $r = \sqrt{x^2 + y^2} = |z|$ and is called ***modulus*** of z; and θ is called ***amplitude*** or ***argument*** of z and is written as ***amp(z)*** or ***arg(z)***.

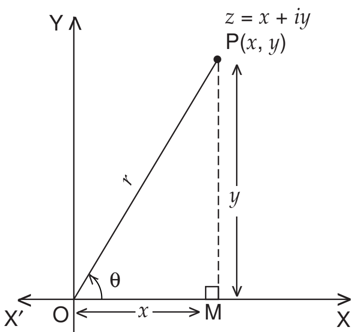


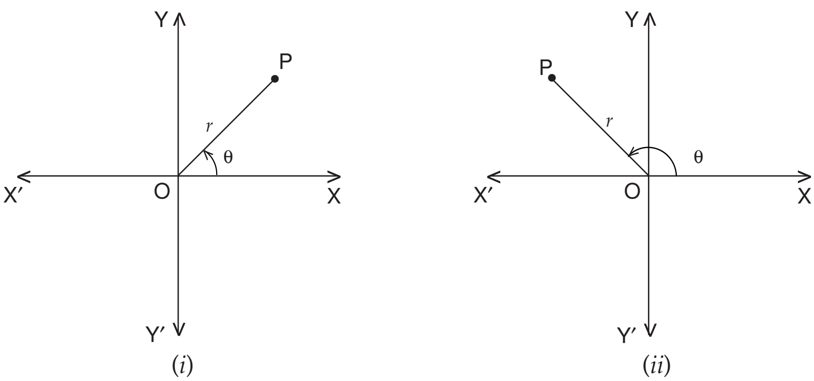
Fig. 5.5.

From figure 5.5, we see that

$x = r\cos\theta$ and $y = r\sin\theta$

$\therefore\ z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$.

Thus, $z = r(\cos\theta + i\sin\theta)$. This form of z is called ***polar form*** of the complex number z.



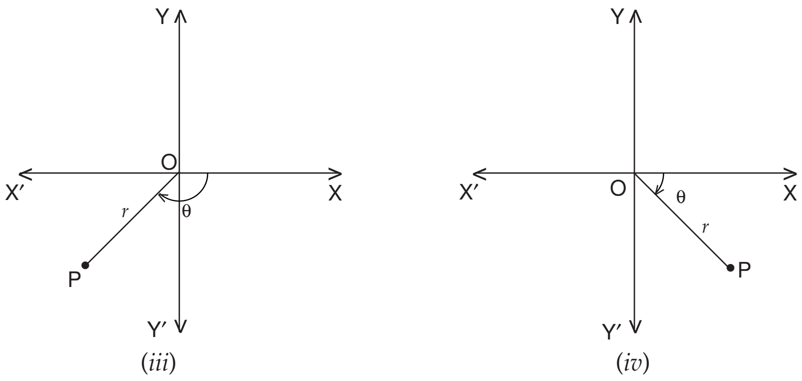


Fig. 5.6.

For any non-zero complex number z, there corresponds only one value of θ in $-\pi < \theta \le \pi$ (see fig. 5.6). *The unique value of θ such that $-\pi < \theta \le \pi$ is called* ***principal value*** *of* ***amplitude*** or ***argument***.

Thus every (non-zero) complex number $z = x + iy$ can be uniquely expressed as $z = r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < \theta \le \pi$ and conversely, for every $r > 0$ and θ such that $-\pi < \theta \le \pi$, we get a unique (non-zero) complex number $z = r(\cos\theta + i\sin\theta) = x + iy$.

Note that the complex number zero cannot be put into the form $r(\cos\theta + i\sin\theta)$ and so, the argument of zero complex number does not exist.

REMARK

If we take origin as the pole and the positive direction of the x-axis as the initial line, then the point P is uniquely determined by the ordered pair of real numbers (r, θ), called the **polar co-ordinates** of the point P (see fig. 5.6).

ILLUSTRATIVE EXAMPLES

Example 1. *Convert the following complex numbers in the polar form and represent them in Argand plane :*

(i) $\sqrt{3} + i$ (*NCERT*) *(ii)* $-\sqrt{3} + i$ (*NCERT*) *(iii)* $-1 - i\sqrt{3}$ (*NCERT*)

(iv) $2 - 2i$ *(v)* -3 (*NCERT*) *(vi)* $-5i$.

Solution. (*i*) Let $z = \sqrt{3} + i = r(\cos\theta + i\sin\theta)$.

Then $r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$.

On squaring and adding, we get

$r^2(\cos^2\theta + \sin^2\theta) = (\sqrt{3})^2 + 1^2$

$\Rightarrow \quad r^2 = 4 \Rightarrow r = 2.$

$\therefore \quad \cos\theta = \dfrac{\sqrt{3}}{2}$ and $\sin\theta = \dfrac{1}{2}$.

The value of θ such that $-\pi < \theta \le \pi$ and satisfying both the above equations is given by $\theta = \dfrac{\pi}{6}$.

Hence, $z = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$, which is the required polar form.

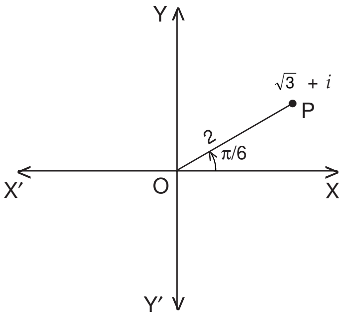


Fig. 5.7.

The complex number $z = \sqrt{3} + i$ is represented in fig. 5.7.

ANSWERS

EXERCISE 5.1

1. (*i*) -6 (*ii*) 6 (*iii*) $20i$ (*iv*) -60 (*v*) 0

2. $-3;\ -1;\ -3+i;\ \sqrt{10}$. **3.** Yes **4.** Yes

6. (*i*) $x = \frac{1}{6}, y = \frac{5}{2}$ (*ii*) $x = 2, y = -\frac{\sqrt{3}}{2}$ (*iii*) $x = \frac{7}{2}, y = \frac{2}{3}$

7. $x = -2, y = 1$ **8.** $x = 2, y = -\frac{2}{3}$ **9.** $x = 2, y = -1$

13. (*i*) $\frac{5}{8} + i0$ (*ii*) $3 + i0$ **14.** (*i*) $2 - 7i$ (*ii*) $-\frac{19}{5} - \frac{21}{10}i$

15. (*i*) $1 - \frac{1}{2}i$ (*ii*) $74 + i0$ **16.** (*i*) $(-6+\sqrt{2}) + \sqrt{3}(1+2\sqrt{2})i$ (*ii*) $16 - 30i$

17. (*i*) $-\frac{47}{8} - \frac{13}{2}i$ (*ii*) $-10 - 198i$ **18.** (*i*) $-\frac{242}{27} - 26i$ (*ii*) $54\sqrt{5} - 378i$

19. (*i*) $0 - i$ (*ii*) $0 + i$ **20.** (*i*) $0 + 8i$ (*ii*) $0 + i0$

21. -4 **22.** (*i*) i (*ii*) i **23.** i

24. (*i*) $\frac{1}{2} + \frac{1}{2}i$ (*ii*) $0 - i$ **26.** (*i*) 65 (*ii*) $\frac{13}{5}$

27. (*i*) $\frac{13}{5}$ (*ii*) 64 **28.** (*i*) $\frac{1}{4}$ (*ii*) $\sqrt{x^2+y^2}$

29. i **30.** $8 + i$ **31.** 2

32. (*i*) $0 + i0$ (*ii*) $0 + i$ **33.** (*i*) $1 - i$ (*ii*) $16 + i0$

34. (*i*) $\frac{2}{13} + \frac{3}{13}i$ (*ii*) $\frac{4}{25} + \frac{3}{25}i$ (*iii*) $\frac{3}{16} - \frac{\sqrt{7}}{16}i$

35. (*i*) $\frac{21}{25} - \frac{47}{25}i$ (*ii*) $-\frac{1}{4} - \frac{\sqrt{3}}{4}i$ (*iii*) $\frac{2}{5} + \frac{29}{5}i$

(*iv*) $\frac{8}{65} + \frac{1}{65}i$ (*v*) $\frac{1}{2} + \frac{1}{2}i$ (*vi*) $\frac{63}{25} - \frac{16}{25}i$

36. (*i*) $\frac{40}{41} - \frac{9}{41}i;\ \frac{40}{41} + \frac{9}{41}i;\ 1$ (*ii*) $1 + i;\ 1 - i;\ \sqrt{2}$

(*iii*) $-1 + i;\ -1 - i;\ \sqrt{2}$ (*iv*) $-9 + 46i;\ -9 - 46i;\ \sqrt{2197}$

37. (*i*) $0 + \frac{1}{2}i$ (*iii*) 1 (*iv*) 2^n **38.** (*i*) $\frac{11}{5}$ (*ii*) $\frac{1}{5}$ (*iii*) 0

39. (*i*) $x = \frac{5}{13}, y = \frac{14}{13}$ (*ii*) $x = 6, y = 1$ (*iii*) $x = \frac{2}{21}, y = -\frac{8}{21}$

40. (*i*) $\frac{3}{2} - 2i$ is the only solution (*ii*) all purely imaginary numbers

45. $-17 + 24i$ **48.** 2

49. $(2n+1)\frac{\pi}{2}, n \in \mathbf{I}$ **50.** Exception is $z = 1$ **51.** 1

EXERCISE 5.2

1. (*i*) True (*ii*) True (*iii*) True (*iv*) True (*v*) True

2. 0 **3.** $-\theta$

4. (*i*) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (*ii*) $1 - i$ (*iii*) $0 - 3i$ (*iv*) $\frac{5}{2} + \frac{5\sqrt{3}}{2}i$ **5.** $-2\sqrt{3} + 2i$