3

TRIGONOMETRIC FUNCTIONS

INTRODUCTION

In general, there are two approaches to trigonometry. One approach centres around the study of triangles to which you have already been introduced in high school. Other one is the *unit circle* approach in which we use radian measure of an angle to define trigonometric functions of real numbers. It meets the requirements of calculus and modern mathematics.

3.1 ANGLE AND ITS MEASUREMENT

3.1.1 What is an angle ?

An angle is made up of two rays with a common end point. This end point is the *vertex* of the angle. The rays are the *sides* of the angle. In fig. 3.1, the angle may be named $\angle ABC$ or $\angle B$.

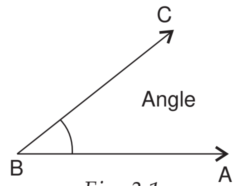


Fig. 3.1.

Signs of angles

The above definition is useful in geometry. In *trigonometry*, we need broader definition of an angle.

Let a revolving ray starting from OX, rotate about O in a plane and stop at position OP. Then it is said to trace out an angle XOP. OX is called *initial side*, OP is *final or terminal side* and O is the *vertex* of the angle. If rotation is *anticlockwise*, the angle is *positive*, if rotation is clockwise, the angle is negative.

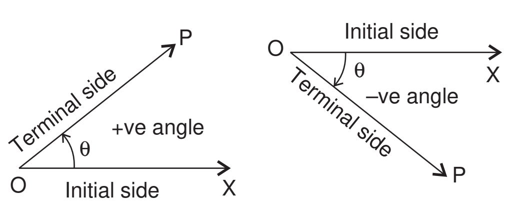


Fig. 3.2.

An angle is said to lie in a particular quadrant if the terminal side of the angle lies in that quadrant.

Two angles are called *coterminal* if they have the same position of initial side and terminal side. Thus, keeping the initial side fixed (as OX), there are an *unlimited* number of angles corresponding to each ray.

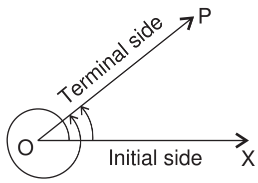


Fig. 3.3.

3.1.2 Measuring angles

The measure of an angle is the amount of rotation made to get the terminal side from its initial side. There are several units for measuring angles.

One unit of measuring angle is one complete rotation (or revolution) as shown in fig. 3.4.

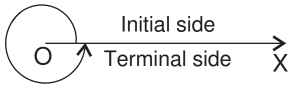


Fig. 3.4.

This unit is convenient for large angles. For example, we can say that a rapidly spinning wheel of a machine is making an angle of 20 revolutions per second. The most commonly used units of measurements are :

1. Degree measure

In this system an angle is measured in degrees, minutes and seconds. A complete rotation describes 360° *i.e.* $1° = \frac{1}{360}$th of a complete rotation.

$\therefore$ **1 right angle = 90°** (Since right angle is $\frac{1}{4}$th of full rotation).

A degree is further subdivided as

1 degree = 60 minutes, written as $1° = 60'$

and 1 minute = 60 seconds, written as $1' = 60''$.

2. Radian measure

In this system an angle is measured in radians.

A radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.

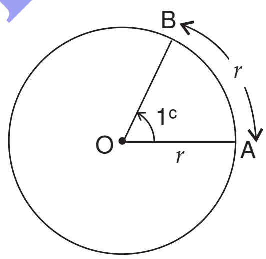


Fig. 3.5.

In fig. 3.5, let AB be an arc of a circle with centre O and of radius r such that length of arc AB = r, then $\angle AOB$ = 1 radian (written at 1^c).

Theorem. ***A radian is a constant angle.***

Proof. In fig. 3.5, let AB be an arc of a circle with centre O and of radius r such that length of arc AB = r. Then, by definition, $\angle AOB$ = 1 radian.

We shall use our knowledge from geometry that *angles at the centre of a circle are in the ratio of subtending arcs.*

As the full circumference subtends an angle of 360° at the centre,

$$\therefore \quad \frac{\angle AOB}{360°} = \frac{\text{arc AB}}{\text{circumference}} = \frac{r}{2\pi r}$$

$$\Rightarrow \quad \angle AOB = \frac{360°}{2\pi}$$

$$\Rightarrow \quad 1 \text{ radian} = \frac{180°}{\pi}.$$

Since the right hand side is independent of radius r, we find that **a radian is a constant angle.**

Radian (circular) measure of an angle

The radian (circular) measure of an angle is the number of radians it contains.

Corollary. ***π radians = 180° = 2 right angles.***

3.1.3 Relation between degree and radian

From the above theorem, we know that ***π radians = 180°.***

This gives us formula of *conversion* from one system to other.

Taking $\pi \simeq \frac{355}{113}$, we get 1 radian = $\frac{2}{\pi}$ right angles

$$= \frac{2}{\pi} \times 90° = 180° \times \frac{113}{355} = \frac{4068°}{71} = \left(57 + \frac{21}{71}\right)°$$

$$= 57° + \frac{21}{71} \times 60' = 57° + \left(17 + \frac{53}{71}\right)' = 57°\,17' + \frac{53}{71} \times 60''$$

i.e. 1 radian = 57° 17′ 45″ nearly.

Also $1° = \frac{\pi}{180}$ radians $= \frac{355}{113} \times \frac{1}{180}$ radians = 0.017453 radians nearly.

3.1.4 Notational convention

If an angle is given without mentioning units, it is assumed to be in radians.

Thus, whenever we write angle θ°, we mean the angle whose degree measure is θ and whenever we write angle x, we mean the angle whose radian measure is x. Hence $\pi = 180°$ and $\frac{\pi}{3} = 60°$ are written with the assumption that π and $\frac{\pi}{3}$ are radian measures.

The relation between degree measures and radian measures of some standard angles are given below :

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

3.1.5 Length of an arc of a circle

Theorem. *If an arc of length l subtends an angle **θ** radians at the centre of a circle of radius r, then l = r**θ**.*

Proof. Let arc AP of length l subtend an angle θ radians at the centre. Mark point B on circumference such that ∠AOB = 1 radian. Thus length of arc AB = r.

Now $\frac{\angle AOP}{\angle AOB} = \frac{\text{length of arc AP}}{\text{length of arc AB}}$

$\Rightarrow \quad \frac{\theta \text{ radians}}{1 \text{ radian}} = \frac{l}{r} \Rightarrow l = r\theta.$

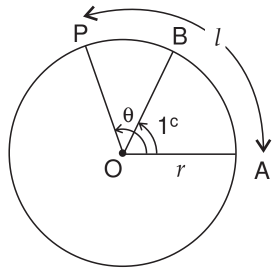


Fig. 3.6.

NOTE

It is assumed that l and r have same linear units.

ILLUSTRATIVE EXAMPLES

Example 1. *Draw diagrams for the following angles. In which quadrant do they lie ?*

(i) 135° *(ii) – 740°.*

Solution. Diagrams are given below for the two angles. OX is initial side and OP is terminal side. From the diagram, we see that 135° lies in second quadrant and – 740° = – 2 × 360° – 20° lies in fourth quadrant.

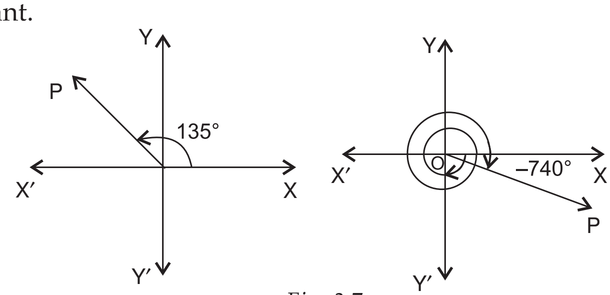


Fig. 3.7.

Example 2. *Convert the following into radian measures :*

(i) 25° (NCERT) *(ii) –47° 30′ (NCERT)* *(iii) 5° 37′ 30″.*

Solution. We know that 180° = π radians, therefore, $1° = \frac{\pi}{180}$ radians.

$$(i)\ 25° = \left(25 \times \frac{\pi}{180}\right) \text{ radians} = \frac{5\pi}{36} \text{ radians.}$$

$$(ii)\ -47°\ 30' = -\left(47 + \frac{30}{60}\right)^{\circ} = -\left(47\frac{1}{2}\right)^{\circ}$$

$$= -\left(\frac{95}{2} \times \frac{\pi}{180}\right) \text{ radians} = -\frac{19\pi}{72} \text{ radians.}$$

$$(iii)\ 5°\ 37'\ 30'' = 5° + \left(37 + \frac{30}{60}\right)' = 5° + \left(37\frac{1}{2}\right)' = 5° + \left(\frac{75}{2}\right)'$$

$$= 5° + \left(\frac{75}{2} \times \frac{1}{60}\right)^{\circ} = 5° + \left(\frac{5}{8}\right)^{\circ} = \left(5\frac{5}{8}\right)^{\circ} = \left(\frac{45}{8}\right)^{\circ}$$

$$= \left(\frac{45}{8} \times \frac{\pi}{180}\right) \text{ radians} = \frac{\pi}{32} \text{ radians.}$$

Example 3. *Convert the following radian measures into degree measures* $\left(\textit{use } \pi = \frac{22}{7}\right)$ *:*

(i) $\frac{11}{16}$ *(ii) –4* *(iii)* $\frac{7\pi}{6}$*. (NCERT)*

Solution. We know that π radians = 180°, therefore, 1 radian $= \left(\frac{180}{\pi}\right)^{\circ}$.

$$(i)\ \frac{11}{16} \text{ radians} = \left(\frac{11}{16} \times \frac{180}{\pi}\right)^{\circ} = \left(\frac{11}{16} \times 180 \times \frac{7}{22}\right)^{\circ} = \left(\frac{315}{8}\right)^{\circ}$$

$$= \left(39\frac{3}{8}\right)^{\circ} = 39° + \left(\frac{3}{8} \times 60\right)' = 39° + \left(22 + \frac{1}{2}\right)'$$

$$= 39° + 22' + \left(\frac{1}{2} \times 60\right)'' = 39° + 22' + 30''$$

$$= 39°\ 22'\ 30''.$$

$$(ii)\ 4 \text{ radians} = \left(4 \times \frac{180}{\pi}\right)^{\circ} = \left(4 \times 180 \times \frac{7}{22}\right)^{\circ} = \left(\frac{2520}{11}\right)^{\circ} = \left(229 + \frac{1}{11}\right)^{\circ}$$

$$= 229° + \left(\frac{1}{11} \times 60\right)' = 229° + \left(5 + \frac{5}{11}\right)' = 229° + 5' + \left(\frac{5}{11} \times 60\right)''$$

$$= 229° + 5' + 27'' \text{ (approximately)}$$

$$= 229°\ 5'\ 27'' \text{ (approximately).}$$

$$(iii)\ \frac{7\pi}{6} \text{ radians} = \left(\frac{7\pi}{6} \times \frac{180}{\pi}\right)^{\circ} = 210°.$$

Example 4. *Express in radians the fourth angle of a quadrilateral which has three angles 46° 30′ 10″, 75° 44′ 45″ and 123° 9′ 35″. Take* $\pi = \frac{355}{113}$.

Solution. The sum of three given angles

$$= 46°\ 30'\ 10'' + 75°\ 44'\ 45'' + 123°\ 9'\ 35''$$

$$= 245°\ 24'\ 30'' \qquad (\because 90'' = 1'\ 30'' \text{ and } 84' = 1°\ 24')$$

As the sum of all four angles of a quadrilateral is 360°,

$\therefore$ the fourth angle = 360° – (245° 24′ 30″)

= 114° 35′ 30″ $\quad (\because 360° = 359° \; 59' \; 60'')$

To convert it into radians :

$$114° \; 35' \; 30'' = 114° + \left(35 + \frac{30}{60}\right)' = 114° + \left(\frac{71}{2}\right)' = 114° + \left(\frac{71}{2} \cdot \frac{1}{60}\right)^\circ$$

$$= \left(114 + \frac{71}{120}\right)^\circ = \left(\frac{13751}{120}\right)^\circ$$

$$= \left(\frac{13751}{120} \times \frac{\pi}{180}\right) \text{ radians} \qquad (\because 180° = \pi \text{ radians})$$

$$= \left(\frac{13751}{120} \times \frac{1}{180} \times \frac{355}{113}\right) \text{ radians}$$

= 2 radians nearly.

Example 5. *Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length 21 cm.* (NCERT)

Solution. The pendulum describes a circle of radius 75 cm and its tip describes an arc of length 21 cm. Let θ radians be the angle through which the pendulum swings.

Here r = 75 cm and l = 21 cm

$$\therefore \quad \theta = \frac{l}{r} = \frac{21}{75} = \frac{7}{25}.$$

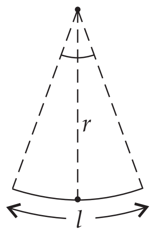


Fig. 3.8.

Example 6. *Find the radius of the circle in which a central angle of 60° intercepts an arc of 37.4 cm length* $\left(use\ \pi = \frac{22}{7}\right)$. (NCERT)

Solution. Here l = 37.4 cm and $\theta = 60° = \left(60 \times \frac{\pi}{180}\right)$ radians $= \frac{\pi}{3}$ radians, so the radian measure of θ is $\frac{\pi}{3}$.

Let r cm be the radius of the circle.

We know that $\theta = \frac{l}{r}$

$$\Rightarrow \quad r = \frac{l}{\theta} = \frac{37.4}{\frac{\pi}{3}} = 37.4 \times 3 \times \frac{7}{22} = 35.7.$$

Hence, the radius of the circle = 35.7 cm.

Example 7. *Find the degree measure of the angle subtended at the centre of a circle of diameter 200 cm by an arc of length 22 cm* $\left(use\ \pi = \frac{22}{7}\right)$. (NCERT)

Solution. Here radius of circle $r = \frac{1}{2}$ diameter $= \frac{1}{2} \times 200$ cm = 100 cm,

length of arc l = 22 cm.

$\therefore \quad \theta = \frac{l}{r}$ radians $= \frac{22}{100}$ radians $= \frac{11}{50} \times \frac{180}{\pi}$ degrees

$$= \frac{11}{50} \times 180 \times \frac{7}{22} \text{ degrees} = \left(\frac{63}{5}\right)^{o} = 12^\circ + \left(\frac{3}{5} \times 60\right)' = 12^\circ\ 36'.$$

Example 8. *In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of the minor arc of the circle.* *(NCERT)*

Solution. Here radius of circle $r = \frac{1}{2} \times 40$ cm $= 20$ cm.

Let O be the centre of circle and AB be a chord of length 20 cm.

Since OA = OB = 20 cm and AB = 20 cm,

ΔOAB is equilateral, therefore,

$\angle AOB = 60^\circ = \frac{\pi}{180} \times 60$ radians $= \frac{\pi}{3}$ radians.

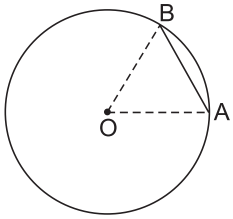


Fig. 3.9.

Let the length of the minor arc AB be l, then

$$l = r\,\theta = 20 \times \frac{\pi}{3} \text{ cm} = \frac{20}{3}\pi \text{ cm}.$$

Example 9. *If the arcs of the same length in two circles subtend angles of 60° and 75° at their respective centres, find the ratio of their radii.* *(NCERT)*

Solution. Ler r_1 and r_2 be the radii of the two given circles and let their arcs of the same length, say l, subtend angles of 60° and 75° at respective centres.

$$60^\circ = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c,\ 75^\circ = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c.$$

Using the formula $l = r\,\theta$, we get

$$l = r_1 \times \frac{\pi}{3} = r_2 \times \frac{5\pi}{12} \Rightarrow \frac{r_1}{r_2} = \frac{5\pi}{12} \times \frac{3}{\pi} = \frac{5}{4}.$$

Hence $r_1 : r_2 = 5 : 4$.

Example 10. *The large hand of a big clock is 35 cm long. How many cm does its tip move in 9 minutes ?*

Solution. The angle traced by the large hand in 60 minutes = 360°

$= 2\,\pi$ radians $\qquad (\because 180^\circ = \pi^c)$

$\therefore$ The angle traced by the large hand in 9 minutes

$$= \frac{2\,\pi}{60} \times 9 \text{ radians} = \frac{3\,\pi}{10} \text{ radians.}$$

Let l be the length of the arc moved by the tip of the minutes hand, then

$$l = r\theta = 35 \times \frac{3\,\pi}{10} \text{ cm} = 35 \times \frac{3}{10} \times \frac{22}{7} \text{ cm} = 33 \text{ cm.}$$

Example 11. *A wheel of a motor is rotating at 1200 r.p.m. If the radius of the wheel is 35 cm, what linear distance does a point of its rim traverse in 30 seconds?*

What steps should be taken to discourage reckless driving? ***(Value Based)***

Solution. Radius of the wheel = 35 cm,

$\therefore$ circumference of the wheel $= 2\pi r = \left(2.\frac{22}{7}.35\right)$ cm $= 220$ cm.

Hence, the linear distance travelled by a point of the rim in one revolution = 220 cm.

Now, the speed of the wheel is 1200 revolutions per minute = $\frac{1200}{60}$ *i.e.* 20 revolutions per second.

$\therefore$ The number of revolutions in 30 seconds = 20.30 = 600.

$\therefore$ The linear distance travelled by a point of the rim in 30 seconds

= (600 × 220) cm = 132000 cm = 1.32 km.

Speed limits should be fixed and monitored properly. There should be fines and imprisonment for reckless driving. Licences of drivers involved in reckless driving should be cancelled or suspended. Conducting proper training of drivers should be mandutory to teach them about the risks associated.

Example 12. *In a right angled triangle, the difference between two acute angles is* $\frac{\pi}{18}$ *in radian measure. Express the angles in degrees.*

Solution. Since the triangle is right angled, sum of two acute angles is 90°.

Let the two acute angles be x and y, $x > y$.

Then $\quad x + y = 90°$ ...(*i*)

Also $\quad x - y = \frac{\pi}{18}$ radians $= \left(\frac{\pi}{18} \times \frac{180}{\pi}\right)^{\circ}$ ($\because$ π radians = 180°)

i.e. $\quad x - y = 10°$ ...(*ii*)

Solving (*i*) and (*ii*) simultaneously, we get

$$x = 50°,\ y = 40°.$$

Example 13. *If the angles of a triangle are in the ratio 3 : 4 : 5, find the smallest angle in degrees and the greatest angle in radians.*

Solution. Let the three angles be $3x$, $4x$ and $5x$ degrees, then

$$3x + 4x + 5x = 180$$

$$\Rightarrow 12x = 180 \Rightarrow x = 15.$$

$\therefore$ The smallest angle = $3x$ degrees = 3 × 15 degrees = 45° and the greatest angle = $5x$ degrees = 5 × 15 degrees = 75°

$$= \left(75 \times \frac{\pi}{180}\right) \text{ radians } = \frac{5\pi}{12} \text{ radians.}$$

Example 14. *The angles of a triangle are in A.P. and the number of degrees in the least to the number of radians in the greatest is 60 : π. Find the angles in degrees and radians.*

Solution. Let the angles be $(a - d)°$, $a°$, $(a + d)°$, where $d > 0$.

Then $(a - d) + a + (a + d) = 180 \Rightarrow 3a = 180 \Rightarrow a = 60$.

Hence the angles are $(60 - d)°$, $60°$, $(60° + d)°$.

Least angle = $(60 - d)°$.

Greatest angle = $(60 + d)° = (60 + d).\ \frac{\pi}{180}$ radians

$\left(\text{As } 180° = \pi \text{ radians} \Rightarrow 1° = \frac{\pi}{180} \text{ radians}\right)$

By given condition, $(60 - d) : (60 + d)\ \frac{\pi}{180} = 60 : \pi$

$$\Rightarrow \quad \frac{(60-d).180}{(60+d).\pi} = \frac{60}{\pi} \quad \Rightarrow \quad \frac{3(60-d)}{60+d} = 1$$

$$\Rightarrow \quad 60 + d = 180 - 3d \Rightarrow 4d = 120 \Rightarrow d = 30.$$

Thus the angles are (60 – 30)°, 60°, (60 + 30)° *i.e.* 30°, 60°, 90°.

In radians, the angles are $30 \cdot \frac{\pi}{180}, 60 \cdot \frac{\pi}{180}, 90 \cdot \frac{\pi}{180}$ *i.e.* $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ radians.

Example 15. *Taking the moon's distance from the earth as 360000 km and the angle subtended by the moon at any point O on the earth as half a degree, estimate the diameter of the moon. (Use $\pi = 3.1416$)*

Solution. As arc AB is a part of very large circle (of radius 360000 km), the diameter AB of the moon is approximately equal to the length of the arc AB.

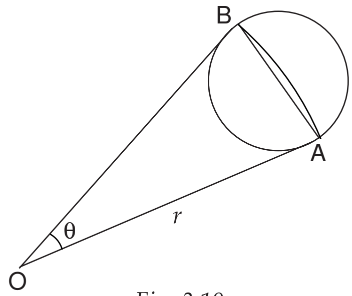


Fig. 3.10.

Now, angle $\theta = \left(\frac{1}{2}\right)^\circ = \frac{1}{2} \cdot \frac{\pi}{180}$ radians

$= \frac{\pi}{360}$ radians.

$\therefore$ AB $= r\theta = 360000 \times \frac{\pi}{360}$ km $= 1000\pi$ km

$= 1000 \times 3.1416$ km $= 3141.6$ km.

EXERCISE 3.1

Very short answer type questions (1 to 4) :

1. Draw diagrams for the following angles :

(*i*) $-135°$ (*ii*) $740°$.

In which quadrant do they lie ?

(*iii*) Find another positive angle whose initial and final sides are same as that of $-135°$, and indicate on the same diagram.

2. If θ lies in second quadrant, in which quadrant the following will lie ?

(*i*) $\frac{\theta}{2}$ (*ii*) 2θ (*iii*) $-\theta$.

3. Express the following angles in radian measure :

(*i*) 240° (*ii*) $-315°$ (*iii*) 570°.

4. Express the following angles in degree measure :

(*i*) $\frac{5\pi}{3}$ (*ii*) $\frac{13\pi}{4}$ (*iii*) $-\frac{24\pi}{5}$.

5. Express the following angles in radian measure :

(*i*) 35° (*ii*) 520° (*iii*) 40° 20′ (*iv*) $-37°$ 30′.

6. Find the degree measures corresponding to the following radian measures :

(*i*) 6 (*ii*) $\frac{3}{4}$ (*iii*) -3.

7. A wheel makes 360 revolutions in a minute. Through how many radians does it turn in one second ? (*NCERT*)

8. Find the angle in radians through which a pendulum swings if its length is 75 cm and the tip describes an arc of length :

(*i*) 10 cm (*ii*) 15 cm. (*NCERT*)

9. Find the radius of the circle in which a central angle of 45° makes an arc of length 187 cm. $\left(\text{use } \pi = \frac{22}{7}\right)$.

10. Find the length of an arc of a circle of diameter 20 cm which subtends an angle of 45° at the centre.

11. An engine is travelling along a circular railway track of radius 1500 metres with a speed of 60 km/hr. Find the angle in degrees turned by the engine in 10 seconds.

What role does railways play in India's transportation system especially for goods?

(*Value Based*)

12. If the arcs of the same length in two circles subtend angles of 65° and 110° at their respective centres, find the ratio of their radii.

13. Large hand of a clock is 21 cm long. How much distance does its extremity move in 20 minutes ?

14. The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? Use $\pi = 3.14$.

15. Find the angles in degrees through which a pendulum swings if its length is 50 cm and the tip describes an arc of length :

(*i*) 10 cm (*ii*) 16 cm (*iii*) 26 cm $\left(\text{use } \pi = \frac{22}{7}\right)$.

16. Find the length of an arc of a circle of radius 75 cm that spans a central angle of measure 126°. Take $\pi = 3.1416$.

17. The circular measures of two angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$. Find the third angle in degree measure. Take $\pi = \frac{22}{7}$.

18. The difference between two acute angles of a right angled triangle is $\frac{\pi}{5}$ in radian measure. Find these angles in degrees.

19. The angles of a triangle are in A.P. and the greatest angle is double the least. Find all the angles in circular measure.

20. Estimate the diameter of the sun supposing that it subtends an angle of 32′ at the eye of an observer. Given that the distance of the sun is 91×10^6 km. Take $\pi = \frac{22}{7}$.

3.2 TRIGONOMETRIC FUNCTIONS OF A REAL NUMBER

In calculus and in many applications of mathematics, we need the trigonometric functions of real numbers rather than angles. By making a small but crucial change in our viewpoint, we can define trigonometric functions of real numbers.

In the previous section of this chapter, we found that the radian measure of an angle is given by the equation $\theta = \frac{l}{r}$. In this result, we assumed that l and r have same linear units and therefore the ratio $\frac{l}{r}$ is a real number with no units.

In particular, in the equation $\theta = \frac{l}{r}$, if we take $r = 1$ then we get $\theta = \frac{l}{1} = l$ (a real number).

Consider the unit circle *i.e.* a circle of radius 1 unit (in length) with centre O. Let A be any point on the circle. Consider OA as the initial side of the angle AOP, then the radian measure of the angle AOP is equal to the length of the arc AP (see fig. 3.11).

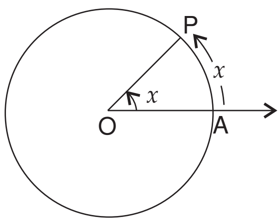
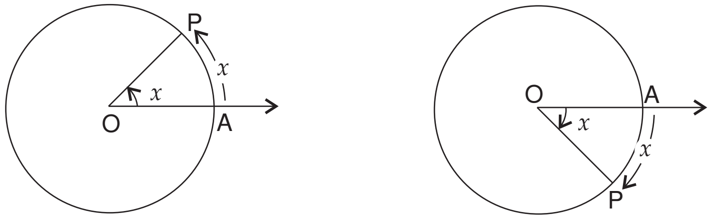


Fig. 3.11.

Here, we have used the letter x, rather than our usual θ, to emphasize the fact that both the radian measure of the angle and the measure of the arc AP are given by the same real number.

The conventions regarding the measure of arc length on the unit circle are same as those for angles. In fig. 3.12, we measure arc length (or radian measure of the angle) from the point A and take positive in anticlockwise direction and negative in the clockwise direction.



(*i*) x is positive (*ii*) x is negative

Fig. 3.12.

You may think of x as either the measure of an arc length or the radian measure of an angle. But in both cases, x is a real number.

3.2.1 Trigonometric (or circular) functions of a real number

Let O be the centre of a circle of ***unit radius***. Choose the axes as shown in fig. 3.13. Let A be the point (1, 0) and P(a, b) a point on the unit circle such that the length of arc AP is equal to x, or equivalently, let P (a, b) be the point where the terminal side of the angle AOP with radian measure x meets the unit circle, then the two basic trigonometric (or circular) functions of the real number x are defined as

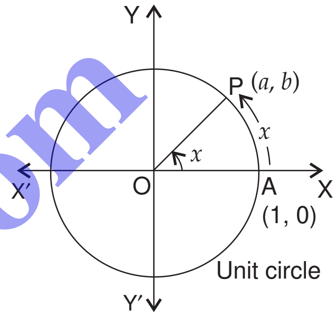


Fig. 3.13.

(i) sin x = b, for all $x \in R$

(ii) cos x = a, for all $x \in R$.

REMARK

1. Note that

$$\sin \angle AOP = \frac{MP}{OP} = \frac{b}{1} = \sin x \text{ etc.}$$

Hence we do not distinguish between trigonometric ratios of an angle AOP whose radian measure is x and the trigonometric function of a real number x.

2. From the above definitions it follows that if P is a point on the unit circle such that length of arc AP is x or equivalently P is a point where the terminal side of the angle with radian measure x meets the unit circle, then the co-ordinates of the point P are (cos x, sin x).

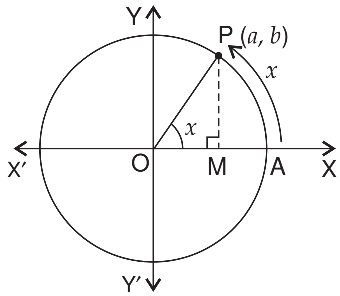


Fig. 3.14.

3.2.2 Values of sin x and cos x at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

We know that in unit circle, the length of circumference is 2π.

If we start from A and move in the anticlockwise direction then at the points A, B, A′, B′ and A, the arc lengths travelled are 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π.

Also the co-ordinates of the points A, B, A′, B′ and A are (1, 0), (0, 1), (–1, 0), (0, –1) and (1, 0) respectively. Therefore,

(*i*) $\sin 0 = 0$ (*ii*) $\cos 0 = 1$

(*iii*) $\sin \frac{\pi}{2} = 1$ (*iv*) $\cos \frac{\pi}{2} = 0$

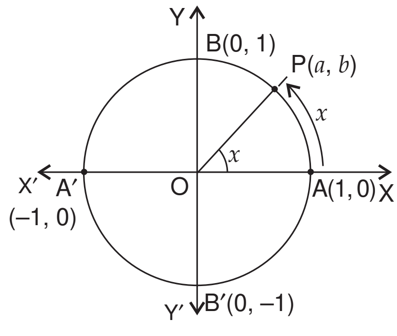


Fig. 3.15.

(*v*) $\sin \pi = 0$ (*vi*) $\cos \pi = -1$

(*vii*) $\sin \frac{3\pi}{2} = -1$ (*viii*) $\cos \frac{3\pi}{2} = 0$

(*ix*) $\sin 2\pi = 0$ (*x*) $\cos 2\pi = 1$.

Further, $\sin x = 0$ when the point P on the unit circle coincides with the points A or A′ *i.e.* when $x = 0, \pi, 2\pi, 3\pi, \ldots$ or $-\pi, -2\pi, -3\pi, \ldots$

i.e. when $x = 0, \pm\pi, \pm 2\pi, \ldots$ *i.e.* when x is an integral multiple of π

i.e. when $x = n\pi$ where n is any integer.

Also $\cos x = 0$ when the point P on the unit circle coincides with the points B or B′ *i.e.* when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, or $-\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, \ldots$ *i.e.* when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ *i.e.* when x is an odd multiple of $\frac{\pi}{2}$ *i.e.* when $x = (2n+1)\frac{\pi}{2}$ where n is any integer.

Thus, $\sin x = 0$ ***when*** $x = n\pi$, n ***is any integer***

and $\cos x = 0$ ***when*** $x = (2n+1)\frac{\pi}{2}$, n ***is any integer.***

3.2.3 Other trigonometric functions

The other trigonometric functions of the real number x are defined in terms of sine and cosine functions as follows :

$$\operatorname{cosec} x = \frac{1}{\sin x},\ x \neq n\pi,\ n \text{ is any integer}$$

$$\sec x = \frac{1}{\cos x},\ x \neq (2n+1)\frac{\pi}{2},\ n \text{ is any integer}$$

$$\tan x = \frac{\sin x}{\cos x},\ x \neq (2n+1)\frac{\pi}{2},\ n \text{ is any integer}$$

$$\cot x = \frac{\cos x}{\sin x},\ x \neq n\pi,\ n \text{ is any integer.}$$

3.2.4 Relations between trigonometric functions of real numbers

The following identities are the immediate consequences of the above definitions of trigonometric functions :

Reciprocal relations

(*i*) $\sin x = \frac{1}{\operatorname{cosec} x}$ and $\operatorname{cosec} x = \frac{1}{\sin x}$ (*ii*) $\cos x = \frac{1}{\sec x}$ and $\sec x = \frac{1}{\cos x}$

(*iii*) $\tan x = \frac{1}{\cot x}$ and $\cot x = \frac{1}{\tan x}$.

From these results, it follows that :

(*i*) $\sin x . \operatorname{cosec} x = 1$ (*ii*) $\cos x . \sec x = 1$ (*iii*) $\tan x . \cot x = 1$.

Quotient relations

(*i*) $\tan x = \frac{\sin x}{\cos x}$ (*ii*) $\cot x = \frac{\cos x}{\sin x}$.

3.2.5 Fundamental identity $\sin^2 x + \cos^2 x = 1$ for all $x \in \mathbf{R}$

Proof. Since the point P(a, b) lies on the unit circle (see fig. 3.13), with centre O (0, 0), we have OP = 1

$$\Rightarrow \quad \sqrt{(a-0)^2 + (b-0)^2} = 1$$

$$\Rightarrow \quad a^2 + b^2 = 1.$$

Now replacing a by cos x and b by sin x, we get

$$\cos^2 x + \sin^2 x = 1.$$

Thus, $\sin^2 x + \cos^2 x = 1$ for all $x \in \mathbf{R}$.

Two other ways of writing this identity are :

$1 - \sin^2 x = \cos^2 x$ and $1 - \cos^2 x = \sin^2 x$.

Other fundamental identities

(i) $\boldsymbol{1 + \tan^2 x = \sec^2 x,\ x \neq (2n+1)\frac{\pi}{2}}$***, n is any integer***

(ii) $\boldsymbol{1 + \cot^2 x = \operatorname{cosec}^2 x,\ x \neq n\pi}$***, n is any integer.***

Proof. We know that $\sin^2 x + \cos^2 x = 1$, for all $x \in \mathbf{R}$.

(i) Dividing both sides of the identity $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$, we get

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}, \text{ assuming that } \cos x \neq 0$$

$$\Rightarrow \quad \tan^2 x + 1 = \sec^2 x,\ x \neq (2n+1)\frac{\pi}{2},\ n \text{ is any integer.}$$

Two other ways of writing this identity are :

$\sec^2 x - 1 = \tan^2 x$ and $\sec^2 x - \tan^2 x = 1$.

(ii) Dividing both sides of the identity $\sin^2 x + \cos^2 x = 1$ by $\sin^2 x$, we get

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}, \text{ assuming that } \sin x \neq 0$$

$$\Rightarrow \quad 1 + \cot^2 x = \operatorname{cosec}^2 x,\ x \neq n\pi,\ n \text{ is any integer.}$$

Two other ways of writing this identity are :

$\operatorname{cosec}^2 x - 1 = \cot^2 x$ and $\operatorname{cosec}^2 x - \cot^2 x = 1$.

3.2.6 Opposite Real Number Identities

(i) $\sin(-x) = -\sin x$ ***(ii)*** $\cos(-x) = \cos x$

(iii) $\tan(-x) = -\tan x$ ***(iv)*** $\cot(-x) = -\cot x$

(v) $\sec(-x) = \sec x$ ***(vi)*** $\operatorname{cosec}(-x) = -\operatorname{cosec} x$.

Proof. Let O be the centre of a unit circle and A be the point (1, 0). Let P be a point on the unit circle such that length of arc AP is equal to x (or equivalently, let P be the point where the terminal side of the angle with radian measure x meets the unit circle), then the co-ordinates of the point P are (cos x, sin x).

On the other hand, if we start from A and move on the unit circle in the clockwise direction to the point Q such that arc length AQ $= -x$, the co-ordinates of the point Q are (cos $(-x)$, sin $(-x)$).

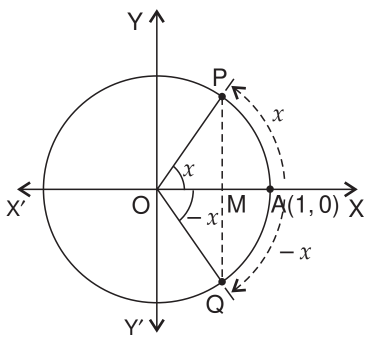


Fig. 3.16.

Let PQ meet OA at M. In Δs OPM and OQM,

OP = OQ, OM = OM

and $\angle$POM = $\angle$QOM (∵ length of arc AP = length of arc AQ, so these subtend equal angles at the centre of the circle)

$\therefore$ ΔOPM $\cong$ ΔOQM

$\Rightarrow$ MP = MQ and $\angle$OMP = 90°.

Hence, the x-coordinates of the points P and Q are same, while the y-coordinates are negatives of each other. Thus, we have

$$\cos(-x) = \cos x \text{ and } \sin(-x) = -\sin x.$$

To establish the third identity, we have

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x.$$

We leave the proofs of the other three identities for the reader.

REMARK

In the above figure, the arc length terminates in the first quadrant. However, the argument used will work no matter where the arc length terminates.

3.2.7 Periodic functions

Definition. *A function f is said to be* ***periodic*** *iff there exists a constant real quantity p such that $f(x + p) = f(x)$ for all $x \in D_f$.*

There may exist more than one value of p satisfying the above relation. The least positive value of p satisfying above relation is called the **period** of f.

Periodicity of sine and cosine functions

Let O be the centre of a unit circle and A be the point (1, 0). Let P be a point on the unit circle such that length of arc AP is equal to x. We know that the circumference of the unit circle is 2π.

($\because$ circumference $= 2\pi r$, here $r = 1$)

Thus, if we begin from any point P on the unit circle and travel a distance of 2π along the perimeter, we return to the same point P. In other words, the arc lengths x and $x + 2\pi$ (measured from A, as usual) yield the same terminal point P on the unit circle. Since the trigonometric functions are defined in terms of the co-ordinates of the point P, we have

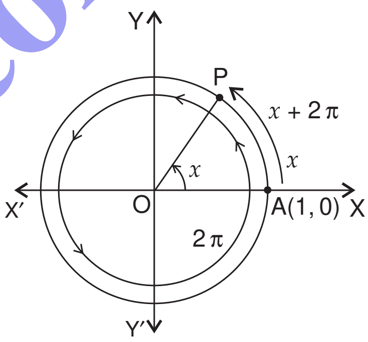


Fig. 3.17.

$$\mathbf{\sin(x + 2\pi) = \sin x,}$$

$$\mathbf{\cos(x + 2\pi) = \cos x.}$$

These results are true for all real values of x. They provide us useful information about their graphs; the graphs of both functions repeat themselves at intervals of 2π.

Further, as these functions do not change on changing x to $x + 2\pi$, therefore, sine and cosine functions are periodic with period 2π.

Similar results hold for other trigonometric functions in their respective domains :

$$\tan(x + 2\pi) = \tan x, \ \cot(x + 2\pi) = \cot x$$

$$\sec(x + 2\pi) = \sec x, \ \operatorname{cosec}(x + 2\pi) = \operatorname{cosec} x.$$

As these functions do not change on changing x to $x + 2\pi$, therefore, all these functions are periodic. The period of secant and cosecant functions is 2π; and the period of tangent and cotangent functions is π (see article 3.4.12).

The above results can be generalised. If we begin from any point P on the unit circle and make two complete anticlockwise revolutions, the arc length travelled is $2(2\pi)$ *i.e.* 4π. For three complete revolutions, the arc length travelled is $3(2\pi)$ *i.e.* 6π. In general, if n is any integer, the arc length travelled for n complete revolutions is $2n\pi$ (when n is positive, the revolution are anticlockwise; when n is negative, the revolutions are clockwise). Consequently, we get the following results :

For any real number x and any integer n, we have

$$\boldsymbol{\sin(x + 2n\pi) = \sin x,}$$

$$\boldsymbol{\cos(x + 2n\pi) = \cos x.}$$

Similar results hold for other trigonometric functions in their respective domains :

$$\tan (x + 2n\pi) = \tan x, \ \cot (x + 2n\pi) = \cot x,$$
$$\sec (x + 2n\pi) = \sec x, \ \ \text{cosec}\, (x + 2n\pi) = \text{cosec}\, x.$$

3.2.8 Values of trigonometric functions for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π

In our earlier classes, we found the values of trigonometric ratios for 30°, 45° and 60°. The values of trigonometric functions for $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ are same as that of trigonometric ratios for 30°, 45° and 60° respectively.

The values of trigonometric functions for $x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π can be memorised with the help of following table :

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin x	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	n.d.	0	n.d.	0
cot x	n.d.	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	n.d.	0	n.d.
sec x	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	n.d.	-1	n.d.	1
cosec x	n.d.	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	n.d.	-1	n.d.

('n.d.' stands for 'not defined')

3.2.9 Signs of trigonometric functions

Let O be the centre of a unit circle and A be the point (1, 0). Let P(a, b) be a point on the unit circle such that length of arc AP = x or equivalently, let P(a, b) be the point where the terminal side of the angle AOP with radian measure x meets the unit circle, then the six trigonometric functions of the real number x are defined as

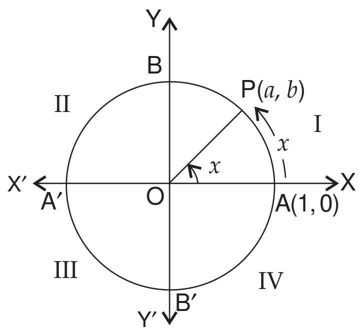


Fig. 3.18.

(*i*) $\sin x = b$, for all $x \in \mathbf{R}$

(*ii*) $\cos x = a$, for all $x \in \mathbf{R}$

(*iii*) $\tan x = \frac{b}{a}$, $x \neq (2n + 1)\frac{\pi}{2}$, n is any integer

(*iv*) $\cot x = \frac{a}{b}$, $x \neq n\pi$, n is any integer

(*v*) $\sec x = \frac{1}{a}$, $x \neq (2n + 1)\frac{\pi}{2}$, n is any integer

(*vi*) $\text{cosec}\, x = \frac{1}{b}$, $x \neq n\pi$, n is any integer.

We know that the circumference of the unit circle is 2π.

Note that in the unit circle, $-1 \leq a \leq 1$ and $-1 \leq b \leq 1$.

Also $a > 0$, $b > 0$ in I quadrant,

$a < 0$, $b > 0$ in II quadrant,

$a < 0$, $b < 0$ in III quadrant,

$a > 0$, $b < 0$ in IV quadrant.

Signs of trigonometric functions of x (a real number)

In the first quadrant, $a > 0$, $b > 0$. Consequently, all the six trigonometrical functions are +ve.

In the second quadrant, $a < 0$, $b > 0$. So $\sin x = b$ and $\text{cosec } x = \frac{1}{b}$ are positive and all other four trigonometric functions *i.e.* $\cos x = a$, $\tan x = \frac{b}{a}$, $\cot x = \frac{a}{b}$ and $\sec x = \frac{1}{a}$ are negative.

In the third quadrant, $a < 0$, $b < 0$. So $\tan x = \frac{b}{a}$ and $\cot x = \frac{a}{b}$ are positive and all other four trigonometric functions *i.e.* $\sin x = b$, $\cos x = a$, $\sec x = \frac{1}{a}$ and $\text{cosec } x = \frac{1}{b}$ are negative.

In the fourth quadrant, $a > 0$, $b < 0$. So $\cos x = a$ and $\sec x = \frac{1}{a}$ are positive and all other four trigonometric functions *i.e.* $\sin x = b$, $\tan x = \frac{b}{a}$, $\cot x = \frac{a}{b}$ and $\text{cosec } x = \frac{1}{b}$ are negative.

This can be summarised as :

Quadrant →	I	II	III	IV
t-functions which are +ve	All	$\sin x$ $\text{cosec } x$	$\tan x$ $\cot x$	$\cos x$ $\sec x$

This table can be memorised with the help of phrase :

Add	**Sugar**	**To**	**Coffee**
↓	↓	↓	↓
All	sin	tan	cos
(I)	(II)	(III)	(IV)

3.2.10 Domain and range of trigonometric functions

Domain of trigonometric functions

We know that $\sin x$ and $\cos x$ are defined for all real values of x; $\tan x$ and $\sec x$ are defined for all real values of x except when $x = (2n + 1)\frac{\pi}{2}$, where n is an integer; $\cot x$ and $\text{cosec } x$ are defined for all real values of x except when $x = n\pi$, where n is an integer.

This can be summarised as :

Function	Domain
sin, cos	all real numbers
tan, sec	all real numbers other than $(2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}$
cot, cosec	all real numbers other than $n\pi$, $n \in \mathbf{Z}$

Range of trigonometric functions

As $-1 \le a \le 1$ and $-1 \le b \le 1$ in unit circle,

$-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$.

Thus, the maximum and minimum values of $\sin x$ and $\cos x$ are 1 and -1 respectively.

Since $\tan x = \frac{b}{a}$ and $\cot x = \frac{a}{b}$, and any of a and b (see fig. 3.16) can be greater than the other, $\tan x$ and $\cot x$ can take any real value.

Now $-1 \le a \le 1,\ a \ne 0 \ \Rightarrow \ \frac{1}{a} \ge 1$ or $\frac{1}{a} \le -1$

$\Rightarrow$ sec $x \ge 1$ or sec $x \le -1$.

Also $-1 \le b \le 1,\ b \ne 0 \ \Rightarrow \ \frac{1}{b} \ge 1$ or $\frac{1}{b} \le -1$

$\Rightarrow$ cosec $x \ge 1$ or cosec $x \le -1$.

This information can be summarised as :

Function	Range
sin, cos	[–1, 1]
tan, cot	any real value
sec, cosec	any real value except (–1, 1)

ILLUSTRATIVE EXAMPLES

Example 1. *Which of the six trigonometric functions are positive for* $x = -\frac{10\pi}{3}$?

Solution. Given $x = -\frac{10\pi}{3}$. We know that terminal position of $x + 2n\pi$, where $n \in \mathbf{Z}$, is the same as that of x.

Here, $-\frac{10\pi}{3} + 2 \times 2\pi = \frac{2\pi}{3}$, which lies in the second quadrant.

(This process of finding ***a coterminal angle or reference number*** results in a angle or number α, $0 \le \alpha < 2\pi$, so that we can determine in which quadrant the given angle or number lies.)

Therefore, $x = -\frac{10\pi}{3}$ lies in the second quadrant. Hence sin x and cosec x are +ve while the other four trigonometric functions *i.e.* cos x, tan x, cot x and sec x are –ve.

Example 2. *If* $\sin x = \frac{3}{5}$ *and x lies in second quadrant, find the values of other five trigonometric functions.* *(NCERT)*

Solution. Given $\sin x = \frac{3}{5}$ and x lies in the second quadrant.

$\therefore$ $\quad \text{cosec } x = \frac{1}{\sin x} = \frac{5}{3}$.

We know that $\sin^2 x + \cos^2 x = 1$

$\Rightarrow \quad \cos^2 x = 1 - \sin^2 x = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$

$\Rightarrow \quad \cos x = \pm \frac{4}{5}$.

But x lies in the second quadrant and cos x is –ve in the second quadrant, therefore, $\cos x = -\frac{4}{5}$.

$\therefore \quad \sec x = \frac{1}{\cos x} = -\frac{5}{4}$.

Further, $\tan x = \frac{\sin x}{\cos x} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4} \ \Rightarrow \cot x = \frac{1}{\tan x} = -\frac{4}{3}$.

Example 3. *If tan x =* $-\frac{5}{12}$ *and x lies in the second quadrant, find the values of other five trigonometric functions.* (NCERT)

Solution. Given $\tan x = -\frac{5}{12}$ and x lies in the second quadrant.

$\therefore \quad \cot x = \frac{1}{\tan x} = -\frac{12}{5}.$

We know that $\sec^2 x = 1 + \tan^2 x$

$\Rightarrow \quad \sec^2 x = 1 + \left(-\frac{5}{12}\right)^2 = 1 + \frac{25}{144} = \frac{169}{144} \Rightarrow \sec x = \pm\frac{13}{12}.$

But x lies in the second quadrant and $\sec x$ is –ve in the second quadrant, therefore, $\sec x = -\frac{13}{12}.$

$\therefore \quad \cos x = \frac{1}{\sec x} = -\frac{12}{13}.$

Further, $\sin x = \frac{\sin x}{\cos x} \cdot \cos x = \tan x \cos x = \left(-\frac{5}{12}\right)\times\left(-\frac{12}{13}\right) = \frac{5}{13}.$

$\therefore \quad \operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{5}.$

Example 4. *If sec x =* $\frac{13}{5}$ *and x lies in the fourth quadrant, find the values of other five trigonometric functions.* (NCERT)

Solution. Given $\sec x = \frac{13}{5}$ and x lies in the fourth quadrant.

$\therefore \quad \cos x = \frac{1}{\sec x} = \frac{5}{13}.$

We know that $\sin^2 x + \cos^2 x = 1$

$\Rightarrow \quad \sin^2 x = 1 - \cos^2 x = 1 - \left(\frac{5}{13}\right)^2 = 1 - \frac{25}{169} = \frac{144}{169}$

$\Rightarrow \quad \sin x = \pm\frac{12}{13}.$

But x lies in the fourth quadrant and $\sin x$ is –ve in the fourth quadrant, therefore, $\sin x = -\frac{12}{13}.$

$\therefore \quad \operatorname{cosec} x = \frac{1}{\sin x} = -\frac{13}{12}.$

Further, $\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{12}{13}}{\frac{5}{13}} = -\frac{12}{5} \quad \Rightarrow \cot x = \frac{1}{\tan x} = -\frac{5}{12}.$

Example 5. *If sin x =* $\frac{12}{13}$*, find the quadrant in which x can lie. Also find the values of remaining trigonometric functions of x.*

Solution. Given $\sin x = \frac{12}{13}$ which is +ve, therefore, x can lie in first or second quadrant.

We know that $\sin^2 x + \cos^2 x = 1$

$\Rightarrow \quad \cos^2 x = 1 - \sin^2 x = 1 - \left(\frac{12}{13}\right)^2 = 1 - \frac{144}{169} = \frac{25}{169}$

$\Rightarrow \quad \cos x = \pm\frac{5}{13}.$

Two cases arise :

Case I. When x lies in first quadrant, $\cos x$ is +ve.

$$\therefore \quad \cos x = \frac{5}{13},\ \tan x = \frac{\sin x}{\cos x} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5},\ \cot x = \frac{1}{\tan x} = \frac{5}{12},$$

$$\sec x = \frac{1}{\cos x} = \frac{13}{5},\ \operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}.$$

Case II. When x lies in second quadrant, $\cos x$ is negative.

$$\therefore \quad \cos x = -\frac{5}{13},\ \tan x = \frac{\sin x}{\cos x} = \frac{\frac{12}{13}}{-\frac{5}{13}} = -\frac{12}{5},$$

$$\cot x = \frac{1}{\tan x} = -\frac{5}{12},\ \sec x = \frac{1}{\cos x} = -\frac{13}{5},\ \operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}.$$

Example 6. *If $\tan \alpha = -2$, find the values of the remaining trigonometric functions of α.*

Solution. Given $\tan \alpha = -2$ which is –ve, therefore, α lies in second or fourth quadrant.

Also $\sec^2 \alpha = 1 + \tan^2 \alpha = 1 + (-2)^2 = 5 \Rightarrow \sec \alpha = \pm \sqrt{5}$.

Two cases arise :

Case I. When α lies in the second quadrant, $\sec \alpha$ is –ve.

$$\therefore \quad \sec \alpha = -\sqrt{5} \Rightarrow \cos \alpha = -\frac{1}{\sqrt{5}}.$$

$$\sin \alpha = \frac{\sin \alpha}{\cos \alpha}.\cos \alpha = \tan \alpha \cos \alpha = -2.\left(-\frac{1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}}$$

$$\Rightarrow \quad \operatorname{cosec} \alpha = \frac{\sqrt{5}}{2}.$$

Also $\tan \alpha = -2 \Rightarrow \cot \alpha = -\frac{1}{2}$.

Case II. When α lies in the fourth quadrant, $\sec \alpha$ is +ve.

$$\therefore \quad \sec \alpha = \sqrt{5} \Rightarrow \cos \alpha = \frac{1}{\sqrt{5}}.$$

$$\sin \alpha = \frac{\sin \alpha}{\cos \alpha}.\cos \alpha = \tan \alpha \cos \alpha = -2.\frac{1}{\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

$$\Rightarrow \quad \operatorname{cosec} \alpha = -\frac{\sqrt{5}}{2}.$$

Also $\tan \alpha = -2 \Rightarrow \cot \alpha = -\frac{1}{2}$.

Example 7. *If $\cos x = -\frac{2}{3}$ and $\pi < x < \frac{3\pi}{2}$, find the value of $4 \tan^2 x - 5 \operatorname{cosec}^2 x$.*

Solution. Given $\cos x = -\frac{2}{3}$ and $\pi < x < \frac{3\pi}{2}$ *i.e.* x lies in the third quadrant.

$$\therefore \quad \sec x = \frac{1}{\cos x} = -\frac{3}{2}.$$

We know that $\sin^2 x + \cos^2 x = 1$

$$\Rightarrow \quad \sin^2 x = 1 - \cos^2 x = 1 - \left(-\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\Rightarrow \quad \sin x = \pm \frac{\sqrt{5}}{3}.$$

But x lies in the third quadrant and sin x is –ve in the third quadrant, therefore, $\sin x = -\frac{\sqrt{5}}{3}$.

$$\therefore \quad \operatorname{cosec} x = \frac{1}{\sin x} = -\frac{3}{\sqrt{5}}.$$

Further, $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{-\frac{\sqrt{5}}{3}}{-\frac{2}{3}} = \dfrac{\sqrt{5}}{2}$.

$$\therefore \quad 4\tan^2 x - 5\operatorname{cosec}^2 x = 4\left(\frac{\sqrt{5}}{2}\right)^2 - 5\left(-\frac{3}{\sqrt{5}}\right)^2$$

$$= 4 \cdot \frac{5}{4} - 5 \cdot \frac{9}{5} = 5 - 9 = -4.$$

Example 8. *If x lies in the second quadrant, then show that*

$$\sqrt{\frac{1-\sin x}{1+\sin x}} + \sqrt{\frac{1+\sin x}{1-\sin x}} = -2 \sec x. \qquad \text{(NCERT Examplar Problems)}$$

Solution. $\text{L.H.S.} = \sqrt{\dfrac{1-\sin x}{1+\sin x}} + \sqrt{\dfrac{1+\sin x}{1-\sin x}}$

$$= \frac{1-\sin x}{\sqrt{1-\sin^2 x}} + \frac{1+\sin x}{\sqrt{1-\sin^2 x}} = \frac{2}{\sqrt{1-\sin^2 x}}$$

$$= \frac{2}{\sqrt{\cos^2 x}} = \frac{2}{|\cos x|} \qquad (\because \sqrt{x^2} = |x|, \text{ for all } x \in \mathbf{R})$$

(Given x lies in second quadrant, so cos x is –ve $\Rightarrow |\cos x| = -\cos x$)

$$= \frac{2}{-\cos x} = -2 \sec x = \text{R.H.S.}$$

Example 9. *(i) If sec x + tan x = p, obtain the values of sec x, tan x and sin x in terms of p.*

(ii) If p = 4 in above case, then find sin x and cos x. In which quadrant does x lie ?

Solution. (*i*) Given $\sec x + \tan x = p$ …(*i*)

We know that $\sec^2 x - \tan^2 x = 1$

$\Rightarrow \quad (\sec x + \tan x)(\sec x - \tan x) = 1$

$p(\sec x - \tan x) = 1$ (using (*i*))

$\Rightarrow \quad \sec x - \tan x = \dfrac{1}{p}$ …(*ii*)

From (*i*) and (*ii*), we get

$$2 \sec x = p + \frac{1}{p} \text{ and } 2 \tan x = p - \frac{1}{p}$$

$$\Rightarrow \quad \sec x = \frac{p^2+1}{2p} \text{ and } \tan x = \frac{p^2-1}{2p}.$$

Now $\dfrac{\tan x}{\sec x} = \dfrac{\sin x}{\cos x} \cdot \cos x = \sin x$

$$\therefore \quad \sin x = \frac{\tan x}{\sec x} = \frac{(p^2-1)/2p}{(p^2+1)/2p} = \frac{p^2-1}{p^2+1}.$$

(*ii*) If $p = 4$, we get $\sin x = \dfrac{p^2-1}{p^2+1} = \dfrac{4^2-1}{4^2+1} = \dfrac{15}{17}$.

$$\cos x = \frac{1}{\sec x} = \frac{2p}{p^2+1} = \frac{2.4}{4^2+1} = \frac{8}{17}.$$

As both sin x and cos x are +ve, x lies in the first quadrant.

Example 10. *If 5 sin x = 3, find the value of* $\dfrac{\sec x - \tan x}{\sec x + \tan x}$.

Solution. Given $5 \sin x = 3 \Rightarrow \sin x = \dfrac{3}{5}$.

Now $$\frac{\sec x - \tan x}{\sec x + \tan x} = \frac{\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}} = \frac{1-\sin x}{1+\sin x} = \frac{1-\dfrac{3}{5}}{1+\dfrac{3}{5}} = \frac{\dfrac{2}{5}}{\dfrac{8}{5}}$$

$$= \frac{2}{8} = \frac{1}{4}.$$

Example 11. *Find the value of* $\tan^2 \dfrac{\pi}{3} + 2\cos^2 \dfrac{\pi}{4} + 3\sec^2 \dfrac{\pi}{6} + 4\cos^2 \dfrac{\pi}{2}$.

Solution. $\tan^2 \dfrac{\pi}{3} + 2\cos^2 \dfrac{\pi}{4} + 3\sec^2 \dfrac{\pi}{6} + 4\cos^2 \dfrac{\pi}{2}$

$$= (\sqrt{3})^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 3\left(\frac{2}{\sqrt{3}}\right)^2 + 4(0)^2$$

$$= 3 + 2\,.\,\frac{1}{2} + 3\,.\,\frac{4}{3} + 0 = 3 + 1 + 4 = 8.$$

Example 12. *Find the values of the following :*

(*i*) $\tan \dfrac{19\pi}{3}$ (NCERT) (*ii*) $\sin\left(-\dfrac{11\pi}{3}\right)$ (NCERT)

(*iii*) $\cot\left(-\dfrac{15\pi}{4}\right)$ (NCERT) (*iv*) $\text{cosec}\left(-\dfrac{19\pi}{3}\right)$.

Solution. (*i*) $\tan \dfrac{19\pi}{3} = \tan\left(6\pi + \dfrac{\pi}{3}\right) = \tan\left(3 \times 2\pi + \dfrac{\pi}{3}\right)$

$= \tan \dfrac{\pi}{3}$ $(\because \tan(2n\pi + x) = \tan x)$

$= \sqrt{3}$.

(*ii*) $\sin\left(-\dfrac{11\pi}{3}\right) = \sin\left(-4\pi + \dfrac{\pi}{3}\right) = \sin\left((-2)2\pi + \dfrac{\pi}{3}\right)$

$= \sin \dfrac{\pi}{3}$ $(\because \sin(2n\pi + x) = \sin x)$

$= \dfrac{\sqrt{3}}{2}$.

(*iii*) $\cot\left(-\dfrac{15\pi}{4}\right) = \cot\left(-4\pi + \dfrac{\pi}{4}\right) = \cot\left((-2)2\pi + \dfrac{\pi}{4}\right)$

$= \cot \dfrac{\pi}{4}$ $(\because \cot(2n\pi + x) = \cot x)$

$= 1$.

(*iv*) $\operatorname{cosec}\left(-\frac{19\pi}{3}\right) = -\operatorname{cosec}\frac{19\pi}{3}$ $\qquad (\because \operatorname{cosec}(-x) = -\operatorname{cosec} x)$

$$= -\operatorname{cosec}\left(6\pi + \frac{\pi}{3}\right) = -\operatorname{cosec}\left(3 \times 2\pi + \frac{\pi}{3}\right)$$

$$= -\operatorname{cosec}\frac{\pi}{3} = -\frac{2}{\sqrt{3}}.$$

Example 13. *Find the values of the following :*

(i) cos (–1710°) (NCERT) (ii) cosec (–1410°) (NCERT)

Solution. (*i*) $\cos(-1710°) = \cos(-1800° + 90°) = \cos(-5 \times 360° + 90°)$

$$= \cos\left((-5)2\pi + \frac{\pi}{2}\right)$$

$$= \cos\frac{\pi}{2} \qquad (\because \cos(2n\pi + x) = \cos x)$$

$$= 0.$$

(*ii*) $\operatorname{cosec}(-1410°) = \operatorname{cosec}(-4 \times 360° + 30°) = \operatorname{cosec}\left((-4)2\pi + \frac{\pi}{6}\right)$

$$= \operatorname{cosec}\frac{\pi}{6} \qquad (\because \operatorname{cosec}(2n\pi + x) = \operatorname{cosec} x)$$

$$= 2.$$

Example 14. *Is the equation $2\sin^2 x - \cos x + 4 = 0$ possible?*

Solution. $2\sin^2 x - \cos x + 4 = 0$

$\Rightarrow \quad 2(1 - \cos^2 x) - \cos x + 4 = 0$

$\Rightarrow \quad -2\cos^2 x - \cos x + 6 = 0$

$\Rightarrow \quad 2\cos^2 x + \cos x - 6 = 0$

$\Rightarrow \quad (2\cos x - 3)(\cos x + 2) = 0$

$\Rightarrow \quad 2\cos x - 3 = 0$ or $\cos x + 2 = 0$

$\Rightarrow \quad \cos x = \frac{3}{2}$ or $\cos x = -2$, both of which are impossible as $-1 \le \cos x \le 1$.

Hence, the equation $2\sin^2 x - \cos x + 4 = 0$ is not possible.

Example 15. *For what real values of x is the equation $2\cos\theta = x + \frac{1}{x}$ possible?*

Solution. Given $2\cos\theta = x + \frac{1}{x}$

$\Rightarrow \quad x^2 - 2\cos\theta \,.\, x + 1 = 0$, which is a quadratic in x.

As x is real, discriminant ≥ 0

$\Rightarrow \quad (-2\cos\theta)^2 - 4.1.1 \ge 0$

$\Rightarrow \quad \cos^2\theta \ge 1$ but $\cos^2\theta \le 1$

$\Rightarrow \quad \cos^2\theta = 1 \;\Rightarrow \cos\theta = 1, -1.$

Case I. When $\cos\theta = 1$, we get $x^2 - 2x + 1 = 0 \;\Rightarrow x = 1.$

Case II. When $\cos\theta = -1$, we get $x^2 + 2x + 1 = 0 \;\Rightarrow x = -1.$

Hence, the values of x are 1 and -1.

Example 16. *If* $A = \cos^2 x + \sin^4 x$ *for all* x *in* $\mathbf{R}$*, then prove that* $\frac{3}{4} \le A \le 1$.

(*NCERT Examplar Problems*)

Solution. $A = \cos^2 x + \sin^4 x = \cos^2 x + \sin^2 x \, . \, \sin^2 x \le \cos^2 x + \sin^2 x$

($\because -1 \le \sin x \le 1$ for all x in $\mathbf{R} \Rightarrow 0 \le \sin^2 x \le 1$)

$\Rightarrow \quad A \le 1$ ($\because \cos^2 x + \sin^2 x = 1$, for all x in $\mathbf{R}$)

Also $\cos^2 x + \sin^4 x = 1 - \sin^2 x + \sin^4 x = \left(\sin^2 x - \frac{1}{2}\right)^2 + \frac{3}{4} \ge \frac{3}{4}$

$$\left(\because \left(\sin^2 x - \frac{1}{2}\right)^2 \ge 0 \text{ for all } x \text{ in } \mathbf{R}\right)$$

$\Rightarrow \quad A \ge \frac{3}{4}$.

Thus, $A \le 1$ and $A \ge \frac{3}{4} \Rightarrow A \le 1$ and $\frac{3}{4} \le A$

$\Rightarrow \quad \frac{3}{4} \le A \le 1.$

EXERCISE 3.2

Very short answer type questions (1 to 13) :

1. Write the domain of the following trigonometric functions :

(*i*) $\sin x$ (*ii*) $\cos x$ (*iii*) $\tan x$

(*iv*) $\cot x$ (*v*) $\sec x$ (*vi*) $\operatorname{cosec} x$.

2. Write the range of the following trigonometric functions :

(*i*) $\sin x$ (*ii*) $\cos x$ (*iii*) $\tan x$

(*iv*) $\cot x$ (*v*) $\sec x$ (*vi*) $\operatorname{cosec} x$.

3. What is the domain of the function f defined by $f(x) = \frac{1}{3 - 2\sin x}$?

4. Find the range of the following functions :

(*i*) $f(x) = 2 - 3 \cos x$ (*ii*) $f(x) = 2 + 5 \sin 3x$.

5. Which of the six trigonometric functions are positive for the angles

(*i*) $\frac{4\pi}{3}$ (*ii*) $-\frac{7\pi}{3}$?

6. In which quadrant does x lie if

(*i*) $\cos x$ is positive and $\tan x$ is negative (*ii*) both $\sin x$ and $\cos x$ are negative

(*iii*) $\sin x = \frac{4}{5}$ and $\cos x = -\frac{3}{5}$ (*iv*) $\sin x = \frac{2}{3}$ and $\cos x = -\frac{1}{3}$?

7. Find the values of the the following :

(*i*) $\tan \frac{25\pi}{4}$ (*ii*) $\sin \frac{31\pi}{3}$ (*NCERT*) (*iii*) $\sec \frac{5\pi}{3}$.

8. Find the values of the following :

(*i*) $\cot\left(-\frac{7\pi}{4}\right)$ (*ii*) $\sin\left(-\frac{17\pi}{3}\right)$ (*iii*) $\operatorname{cosec}\left(-\frac{25\pi}{3}\right)$.

9. Find the values of the following :

(*i*) $\sin 765°$ (*NCERT*) (*ii*) $\tan 1395°$ (*iii*) $\cos(-2070°)$.

10. If $\sin x = \frac{3}{5}$ and x lies in the second quadrant, find the value of $\cos x$.

11. If $\cos x = -\frac{2}{3}$ and x lies in the third quadrant, find the value of $\sin x$.

12. If $\tan x = -\frac{4}{3}$ and x lies in the fourth quadrant, find the value of $\cos x$.

13. If $\cot x = \frac{5}{12}$ and x lies in the third quadrant, find the value of $\sin x$.

14. Find the other five trigonometric functions if

(*i*) $\cos x = -\frac{1}{2}$ and x lies in the third quadrant (*NCERT*)

(*ii*) $\cos x = -\frac{3}{5}$ and x lies in the third quadrant (*NCERT*)

(*iii*) $\cot x = \frac{3}{4}$ and x lies in the third quadrant (*NCERT*)

(*iv*) $\cot x = -\frac{5}{12}$ and x lies in the second quadrant (*NCERT*)

(*v*) $\tan x = \frac{3}{4}$ and x does not lie in the first quadrant

(*vi*) $\operatorname{cosec} x = -\frac{13}{12}$ and x does not lie in the third quadrant.

15. If $\sin x = \frac{12}{13}$ and x lies in the second quadrant, show that $\sec x + \tan x = -5$.

16. If $\sin x \sec x = -1$ and x lies in the second quadrant, find $\sin x$ and $\sec x$.

17. If $\sin x : \cos x :: \sqrt{3} : 1$, find $\sin x$, $\cos x$.

18. If $\cos x = -\frac{3}{5}$ and $\pi < x < \frac{3\pi}{2}$, find the other t-ratios and hence evaluate $\frac{\operatorname{cosec} x + \cot x}{\sec x - \tan x}$.

19. If $\tan x = -\frac{4}{3}$, find the value of $9 \sec^2 x - 4 \cot x$.

20. If $\sec x = \sqrt{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find the value of $\frac{1 + \tan x + \operatorname{cosec} x}{1 + \cot x - \operatorname{cosec} x}$.

21. If $\sec x + \tan x = 1.5$, find the value of $\sec x$, $\tan x$, $\cos x$ and $\sin x$. In which quadrant does x lie ?

22. If $\operatorname{cosec} x - \cot x = \frac{3}{2}$, find $\cos x$. In which quadrant does x lie?

23. Show that

(*i*) $\sin \frac{\pi}{6} \cos 0 + \sin \frac{\pi}{4} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \cos \frac{\pi}{6} = \frac{7}{4}$

(*ii*) $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$

(*iii*) $4 \sin \frac{\pi}{6} \sin^2 \frac{\pi}{3} + 3 \cos \frac{\pi}{3} \tan \frac{\pi}{4} + \operatorname{cosec}^2 \frac{\pi}{2} = 2 \sec^2 \frac{\pi}{4}$.

24. Evaluate $\sec \frac{\pi}{6} \tan \frac{\pi}{3} + \sin \frac{\pi}{4} \operatorname{cosec} \frac{\pi}{4} + \cos \frac{\pi}{6} \cot \frac{\pi}{3}$.

ANSWERS

EXERCISE 3.1

1. (*i*) Third quadrant (*ii*) First quadrant (*iii*) 225°

2. (*i*) First quadrant (*ii*) Third or fourth quadrant (*iii*) Third quadrant

3. (*i*) $\frac{4\pi}{3}$ (*ii*) $-\frac{7\pi}{4}$ (*iii*) $\frac{19\pi}{6}$ **4.** (*i*) 300° (*ii*) 585° (*iii*) –864°

5. (*i*) $\frac{7\pi}{36}$ (*ii*) $\frac{26\pi}{9}$ (*iii*) $\frac{121\pi}{540}$ (*iv*) $-\frac{5\pi}{24}$

6. (*i*) 343° 38′ 11″ (*ii*) 42° 57′ 16″ (*iii*) –171° 49′ 5″

7. 12π **8.** (*i*) $\frac{2}{15}$ (*ii*) $\frac{1}{5}$ **9.** 238 cm **10.** $\frac{5\pi}{2}$ cm

11. $\left(\frac{20}{\pi}\right)^{\circ}$; Indian railways is an important means of transportation for both human beings and goods. Various goods such as coal, iron ore, heavy machinery etc. are transported through railways in India. Railways are playing a major role in the progress of the country.

12. 22 : 13 **13.** 44 cm **14.** 6.28 cm

15. (*i*) 11° 27′ 16″ (*ii*) 18° 19′ 38″ (*iii*) 29° 46′ 55″ **16.** 164.934 cm **17.** 132° 16′ 22″

18. 63°, 27° **19.** $\frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9}$ radians **20.** 847407.4 km

EXERCISE 3.2

1. (*i*) **R** (*ii*) **R** (*iii*) $\left\{x : x \in \mathbf{R}, x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{I}\right\}$

(*iv*) $\{x : x \in \mathbf{R}, x \neq n\pi, n \in \mathbf{I}\}$ (*v*) $\left\{x : x \in \mathbf{R}, x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{I}\right\}$

(*vi*) $\{x : x \in \mathbf{R}, x \neq 2n\pi, n \in \mathbf{I}\}$

2. (*i*) $[-1, 1]$ (*ii*) $[-1, 1]$ (*iii*) **R**

(*iv*) **R** (*v*) $(-\infty, -1] \cup [1, \infty)$ (*vi*) $(-\infty, -1] \cup [1, \infty)$

3. **R** **4.** (*i*) $[-1, 5]$ (*ii*) $[-3, 7]$

5. (*i*) tan, cot (*ii*) cos, sec.

6. (*i*) fourth (*ii*) third (*iii*) second (*iv*) not possible as we must have $\sin^2 x + \cos^2 x = 1$.

7. (*i*) 1 (*ii*) $\frac{\sqrt{3}}{2}$ (*iii*) 2 **8.** (*i*) 1 (*ii*) $\frac{\sqrt{3}}{2}$ (*iii*) $-\frac{2}{\sqrt{3}}$

9. (*i*) $\frac{1}{\sqrt{2}}$ (*ii*) –1 (*iii*) 0 **10.** $-\frac{4}{5}$ **11.** $-\frac{\sqrt{5}}{3}$

12. $\frac{3}{5}$ **13.** $-\frac{12}{13}$

14. (*i*) $\sin x = -\frac{\sqrt{3}}{2}$, $\tan x = \sqrt{3}$, $\cot x = \frac{1}{\sqrt{3}}$, $\sec x = -2$, $\operatorname{cosec} x = -\frac{2}{\sqrt{3}}$

(*ii*) $\sin x = -\frac{4}{5}$, $\tan x = \frac{4}{3}$, $\cot x = \frac{3}{4}$, $\sec x = -\frac{5}{3}$, $\operatorname{cosec} x = -\frac{5}{4}$

(*iii*) $\sin x = -\frac{4}{5}$, $\cos x = -\frac{3}{5}$, $\tan x = \frac{4}{3}$, $\sec x = -\frac{5}{3}$, $\operatorname{cosec} x = -\frac{5}{4}$

(*iv*) $\sin x = \frac{12}{13}$, $\cos x = -\frac{5}{13}$, $\tan x = -\frac{12}{5}$, $\sec x = -\frac{13}{5}$, $\operatorname{cosec} x = \frac{13}{12}$